Smooth solutions to a class of free boundary parabolic problems

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Abstract

We establish existence, uniqueness, and regularity results for solutions to a class of free boundary parabolic problems, including the free boundary heat equation which arises in the so called "focussing problem" in the mathematical theory of combustion. Such solutions are proved to be smooth with respect to time for positive t, if the data are smooth.

Key Words and Phrases: Heat equation, free boundary problems, fully non linear parabolic equations.

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1 Introduction

In this paper we consider the initial value problem for a class of free boundary parabolic equations,

$$\begin{cases} u_t(t,x) = \mathcal{L}u(t,x) + f(t,x,u(t,x),Du(t,x)), \ \ t \geq 0, \ x \in \Omega_t, \\ \\ u(t,x) = g_0(t,x), \ \ t \geq 0, \ x \in \partial\Omega_t, \\ \\ \dfrac{\partial u}{\partial n}(t,x) = g_1(t,x), \ \ t \geq 0, \ x \in \partial\Omega_t, \\ \\ u(0,x) = u_0(x), \ \ x \in \Omega_0, \end{cases} \tag{1.1}$$

where the family of open sets $\{\Omega_t : t \geq 0\} \subset \mathbb{R}^N$ and the function $u : \{(t,x) : t \geq 0, \ x \in \overline{\Omega}_t\} \mapsto \mathbb{R}$ are the unknowns. By $n = n(t,x)$ we denote the exterior unit normal vector to $\partial\Omega_t$ at $x \in \partial\Omega_t$.

The data are: the second order elliptic operator $\mathcal{L} = \sum_{i,j=1}^N a_{ij}(t,x)D_{ij}$, the functions $f : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, g_0, $g_1 : [0,T] \times \mathbb{R}^N \mapsto \mathbb{R}$, the (possibly unbounded) initial set $\Omega_0 \subset \mathbb{R}^N$, and the initial function $u_0 : \overline{\Omega}_0 \mapsto \mathbb{R}$. All of them are assumed to be regular enough. Moreover, u_0 satisfies the compatibility conditions

$$u_0(x) = g_0(x), \ \ \frac{\partial u_0}{\partial n}(x) = g_1(x), \ \ x \in \partial\Omega_0, \tag{1.2}$$

and g_0, g_1 satisfy the transversality condition

$$\frac{\partial g_0}{\partial n}(0,x) \neq g_1(0,x), \quad x \in \partial\Omega_0. \tag{1.3}$$

In space dimension $N = 1$ this problem has been well understood for a long time, see e.g. the review paper [16]. Here we are interested in the multidimensional case $N \geq 2$, where only a few results are available at present.

The simplest and most popular example of (1.1) is the free boundary heat equation,

$$\begin{cases} u_t = \Delta u \ \text{ in } \Omega_t, \quad u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial n} = C \ \text{ on } \partial\Omega_t, \\ \\ u(0,\cdot) = u_0 \ \text{ in } \Omega_0, \end{cases} \tag{1.4}$$

with $C \neq 0$, motivated by models in combustion theory ([6, 18]) for equidiffusional premixed flames. It can be seen as the high activation energy limit of the regularizing problems $u_t = \Delta u - \beta_\epsilon(u)$ in $[0,T] \times \mathbb{R}^N$, where $\beta_\epsilon(s) = \beta_1(s/\epsilon)/\epsilon$ has support in a small interval $[-\epsilon, 0]$. In [7], Caffarelli and Vazquez used this regularization to prove existence of global weak solutions to (1.4) for C^2 initial data (Ω_0, u_0), under suitable geometric assumptions on u_0. Such solutions may not be unique, and enjoy some regularity properties: the free boundary is locally Lipschitz continuous, and u is 1/2-Hölder continuous with respect to time, Lipschitz continuous with respect to the space variables. The same approach was used in [12] to prove uniqueness of the classical solution to (1.4) in a cylinder for initial data which are monotonic in the direction of the cylinder axis.

In this paper we follow a completely different approach, which leads to existence, uniqueness, and time smoothness for $t > 0$ of local regular solutions (with u bounded, in the case of unbounded domains) to problem (1.1). Concerning uniqueness, we prove uniqueness of the solution belonging to a suitable parabolic Hölder space and such that the transversality condition $\frac{\partial g_0}{\partial n}(t,x) \neq g_1(t,x)$, $x \in \partial\Omega_t$, is preserved throughout the interval of existence.

Our approach consists in transforming problem (1.1) into a fully nonlinear parabolic problem in the fixed domain Ω_0 for an auxiliary unknown w, for which the usual techniques of fully nonlinear parabolic problems in Hölder spaces (see e.g. [14, Ch. 8]) give a local existence and uniqueness result. Coming back to (1.1), for Ω_0 with $C^{3+\alpha+k}$ boundary and $u_0 \in C^{3+\alpha+k}(\overline{\Omega}_0)$ we obtain a local solution with $\partial\Omega_t \in C^{2+\alpha+k}$ and $u(t,\cdot)$ in $C^{2+\alpha+k}(\overline{\Omega}_t)$, $k = 0, 1$. So, we share with [7] the unpleasant fact that we lose one degree of space regularity for $t > 0$. Although this is natural for global weak solutions, as simple examples described in [7] show, it not wholly satisfactory for local classical solutions. One could expect that in a small time interval the solution remains at least as regular as the initial datum: this is what happens in other free boundary problems of parabolic type (see e.g. [8, 9] for boundary conditions of Stefan type), and in problem (1.1) for special initial data. Thus, our main results are uniqueness and time smoothness, and we consider our existence theorem just as a tool to prove the other results.

An noteworthy situation in which there is no loss of regularity, at least in small time intervals, is the case of initial data near special smooth solutions, such as stationary solutions, self-similar solutions, travelling waves. In particular, $C^{2+\alpha}$ initial data near any regular stationary solution (Ω, U) with bounded Ω were considered in the paper [3], where we studied stability of regular stationary solutions, establishing a linearized stability principle for (1.1) in the time independent case. Since self-similar solutions to (1.4) become stationary solutions to a problem of the type (1.1) after a suitable change of coordinates, we could also consider initial data for (1.4) near self-similar solutions. In the unbounded domain case, $C^{2+\alpha}$ initial data near a planar travelling wave solution of (1.4) have been considered in [5]. The wave turns out to be orbitally stable, but the discussion is not trivial, because in dimension $N \geq 2$ this is a very critical case of stability.

Another example without loss of regularity was considered in [1], where it was studied a two-phase version of (1.1) in a cylinder for $C^{2+\alpha}$ initial data far from special solutions but satisfying suitable monotonicity conditions: u_0 is assumed to be strictly monotonic in the direction orthogonal to the cylinder axis. This allows to take u as new independent variable for small t, an old trick already used by Meirmanov in the Stefan problem and also in a two-dimensional version of problem (1.1), see [15].

Together with loss of regularity, the other big question about problem (1.1) is uniqueness of the classical solution. Indeed, the uniqueness results available up to now concern only particular situations, such as radially symmetric solutions of (1.4), studied in [10], and solutions in cylinders, for initial data which are monotonic in the direction of the axis of the cylinder (see [15] in dimension $N = 2$ and [12] in any dimension), or in the direction orthogonal to the axis of the cylinder (see [1], for the two-phase case). Moreover the above mentioned papers [3, 5] give also uniqueness results in parabolic Hölder spaces, but only for solutions close to the special solutions considered.

Therefore, the problem of wellposedness for general initial data in dimension bigger than 1 still remains open, even in the simplest case (1.4).

Below we describe briefly our approach, which is similar to the approach of [3].

We look for $\partial\Omega_t$ as the range of the function $\xi \mapsto \xi + s(t,\xi)\nu(\xi)$, defined for $\xi \in \partial\Omega_0$, where $\nu(\xi)$ is the exterior unit normal vector to $\partial\Omega_0$ at ξ, and the unknown $s(t,\xi)$ is the signed distance of the point $x = \xi + s(t,\xi)\nu(\xi)$ from $\partial\Omega_t$. Then by a natural change of coordinates we transform Ω_t into Ω_0, at least for small time, and denoting by $\widetilde{u}$ the unknown u expressed in the new coordinates, we get a fixed boundary system for $(s, \widetilde{u})$, in the domain $[0,\delta] \times \overline{\Omega}_0$ with small δ. The change of coordinates is defined by $\xi \mapsto \xi + s(t,\xi)\nu(\xi)$ for $\xi \in \partial\Omega_0$, and it is extended smoothly to a $C^{2+\alpha}$ diffeomorphism $\xi \mapsto \xi + \Phi(t,\xi)$ in the whole of Ω_0.

The main step is now the introduction of a new unknown w defined by

$$\widetilde{u}(t,\xi) = u_0(\xi) + \langle Du_0(\xi), \Phi(t,\xi)\rangle + w(t,\xi), \tag{1.5}$$

which allows to decouple the system using the boundary condition $u = g_0$ at $\partial\Omega_t$. In the case $g_0 \equiv 0$, $g_1 \equiv C$ we get simply

$$s(t,\xi) = -\frac{w(t,\xi)}{C}, \quad t \geq 0,\ \xi \in \partial\Omega_0.$$

In the general case the formula is a bit more complicated. It comes out

$$s(t,\xi) = \frac{w(t,\xi)}{\frac{\partial g_0}{\partial \nu}(t,\xi) - g_1(0,\xi)} + R(t,\xi,w(t,\xi)),$$

where the remainder $R(t,\xi,u)$ vanishes at $(0,\xi,0)$ together with R_u. It is here that we need the transversality assumption (1.3).

In any case, we can go through and we get a final problem for the only unknown w,

$$\begin{cases} w_t = \mathcal{A}w + F(t,w), \quad t \geq 0,\ \xi \in \overline{\Omega}_0, \\ \\ \mathcal{B}w = G(t,w), \quad t \geq 0,\ \xi \in \partial\Omega_0, \\ \\ w(0,\cdot) = 0, \quad \xi \in \overline{\Omega}_0, \end{cases} \tag{1.6}$$

where $\mathcal{A}$, $\mathcal{B}$ are linear differential operators and F, G are nonlinear functions. Problem (1.6) is fully nonlinear because F depends on w and on its space derivatives up to the second order, and G depends on w and on its first order space derivatives. Moreover, F is nonlocal in the second order derivatives of w. However, both F and G are smooth enough, and F_w, G_w vanish at $t = 0$, $w = 0$. The boundary operator $\mathcal{B}$ is obtained from

the Neumann boundary condition $\frac{\partial u}{\partial n} = g_1$ after the change of coordinates and unknowns; in the case $g_0 \equiv 0$, $g_1 \equiv C$ we get

$$\mathcal{B}v = \frac{\partial v}{\partial \nu} - \frac{1}{C}\frac{\partial^2 u_0}{\partial \nu^2}\, v.$$

The second order linear operator $\mathcal{A}$ is the sum of $\mathcal{L}$ plus a nonlocal operator, acting only on w and on the first order space derivatives of w at $t = 0$. This is crucial for our analysis: since the principal part of $\mathcal{A}$ coincides with the principal part of $\mathcal{L}$ at $t = 0$, the linearized problem near $t = 0$, $w_0 = 0$ is a good linear parabolic problem, for which optimal Hölder regularity results and estimates are available; we need such estimates to solve the nonlinear problem in a standard way, using the Contraction Theorem in a ball of the parabolic Hölder space $C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}_0)$, with $\delta > 0$ suitably small, or else in a ball of $C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega}_0)$ if the data are more regular.

It is clear now why we lose regularity: not only because of the change of coordinates, but also because w has the same space regularity of Du_0, due to the splitting (1.5). This problem may be avoided for initial data near smooth (at least $C^{3+\alpha}$) stationary solutions, where it is possible to linearize around the stationary solution itself; see [3].

However, it is easy to see that uniqueness of w implies uniqueness of $(s, \widetilde{u})$, and hence uniqueness of the regular solution to the original problem (1.1). Moreover, using standard techniques of fully nonlinear problems, it is possible to prove that $t \mapsto w(t,\cdot)$ is analytic with values in $C^2(\overline{\Omega}_0)$ in the interval $(0,\delta)$ if the data are analytic. It follows that both the free boundary and u are analytic with respect to time for t in $(0,\delta)$.

By a covering argument we get eventually that the free boundary of any sufficiently smooth solution defined in an interval $[0,a]$ is time analytic in $(0,a]$, provided $\frac{\partial g_0}{\partial n}(t,x) \neq g_1(t,x)$ for $0 \le t \le a$, $x \in \partial\Omega_t$.

2 Transformation to a fully nonlinear fixed boundary problem

We write below the precise regularity assumption on the data. We fix once and for all a number $\alpha \in (0,1)$.

(a) *The initial data.* The initial data are couples (Ω_0, u_0) where $\Omega_0 \subset \mathbb{R}^N$ is a nonempty open set with $C^{3+\alpha}$ boundary $\partial\Omega_0$. If Ω_0 is unbounded we need that the boundary is uniformly $C^{3+\alpha}$, in the sense that the diffeomorphisms which locally straighten the boundary and their inverses have $C^{3+\alpha}$ norms bounded by a constant. Moreover, $u_0 \in C^{3+\alpha}(\overline{\Omega}_0)$. In particular, u_0 and all its partial derivatives up to the third order are bounded.

(b) *The other data.* We have

$$\mathcal{L}v = a_{ij}(t,x)D_{ij}v$$

where the coefficients a_{ij} and their first order space derivatives belong to $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^N)$, and satisfy a uniform ellipticity condition: there is $\nu > 0$ such that $a_{ij}(t,x)\xi_i\xi_j \ge \nu|\xi|^2$ for all t, x, ξ.

The function $f = f(t,x,u,p)$ is once differentiable with respect to x, twice differentiable with respect to (u,p). f and its derivatives are continuous in (t,x,u,p), and $C^{\alpha/2,\alpha}$ in (t,x), locally uniformly with respect to (u,p).

Moreover, g_0 and g_1 are differentiable with respect to the space variables, and g_0, $D_i g_0 \in C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^N)$, g_1, $D_i g_1 \in C^{1/2+\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^N)$ for $i = 1,\dots,N$.

(c) *Compatibility and transversality conditions.* Since we deal with regular solutions up to $t = 0$ we have to assume that (1.2) holds. We assume furthermore that (1.3) holds. This is a condition on the structure of the problem, it says that ours is not an obstacle-like problem. However, it is necessary for our approach to work.

2.1 The change of coordinates

For every $\xi \in \partial\Omega_0$ let $\nu(\xi) = n(0,\xi)$ be the unit exterior normal vector to $\partial\Omega_0$. There is a neighborhood $\mathcal{N}$ of $\partial\Omega_0$ such that each $x \in \mathcal{N}$ may be written in a unique way as $x = \xi + s\nu(\xi)$ with $\xi \in \partial\Omega_0$ and $s \in \mathbb{R}$. The number s is called the signed distance of x from $\partial\Omega_0$ and it is a regular function of x. As we consider local regular solutions, for t small enough $\partial\Omega_t \subset \mathcal{N}$; we shall represent $\partial\Omega_t$ as

$$\partial\Omega_t = \{\xi + s(t,\xi)\nu(\xi); \xi \in \partial\Omega_0\} \tag{2.1}$$

where $s(t,\xi)$ is an unknown regular function. The function $\xi \mapsto \xi + s(t,\xi)\nu(\xi)$ is a diffeomorphism from $\partial\Omega_0$ to $\partial\Omega_t$, and it may be extended in a standard way to a diffeomorphism from Ω_0 to Ω_t, as follows. First we extend the normal field: for every $\xi \in \mathcal{N}$ we denote by ξ' the point of $\partial\Omega_0$ closest to ξ, then for every $\xi \in \mathbb{R}^N$ we set

$$\widetilde{\nu}(\xi) = \begin{cases} \theta(\xi)\nu(\xi') & \text{if } \xi \in \mathcal{N}, \\ \\ 0 \quad \text{otherwise,} \end{cases} \tag{2.2}$$

where θ is a C^∞ function with support contained in $\mathcal{N}$, such that $\theta \equiv 1$ in a neighborhood of $\partial\Omega_0$, $\theta \equiv 0$ outside a bigger neighborhood of $\partial\Omega_0$.

Then we set

$$\Phi(t,\xi) = \begin{cases} \theta(\xi)s(t,\xi')\nu(\xi') & \text{if } \xi \in \mathcal{N}, \\ \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.3}$$

To simplify formulas, with a slight abuse of notation we shall write

$$\Phi(t,\xi) = s(t,\xi')\widetilde{\nu}(\xi).$$

More generally, if a is a function defined in $[0,T] \times \mathcal{N} \cap \overline{\Omega}_0$ and b is a function defined in $\overline{\Omega}_0$, with support contained in $\mathcal{N} \cap \overline{\Omega}_0$, we shall write $a \cdot b$ to denote the function $(t,\xi) \mapsto a(t,\xi)b(\xi)$ for $\xi \in \mathcal{N} \cap \overline{\Omega}_0$, $(t,\xi) \mapsto 0$ if $\xi \in \overline{\Omega}_0 \setminus \mathcal{N}$.

For t small, the mapping

$$\xi \mapsto x(t,\xi) = \xi + \Phi(t,\xi), \quad \xi \in \mathbb{R}^N, \tag{2.4}$$

is then a $C^{2+\alpha}$ diffeomorphism from $\mathbb{R}^N$ to itself, which maps Ω_0 onto Ω_t.

Let $\widetilde{u}$ denote the unknown u in the variables (t,ξ), i.e.

$$\widetilde{u}(t,\xi) = u(t,\xi + \Phi(t,\xi)), \quad t \geq 0,\ \xi \in \overline{\Omega}_0. \tag{2.5}$$

Similarly, we set $\widetilde{a}_{ij}(t,\xi) = a_{ij}(t,\xi + \Phi(t,\xi))$.

With the change of variables $(t,x) \mapsto (t,\xi)$, the equation $u_t = \mathcal{L}u + f$ is transformed to

$$\widetilde{u}_t - \langle D\widetilde{u}, (I + {}^tD\Phi)^{-1}\Phi_t\rangle = \widetilde{\mathcal{L}}\widetilde{u} + f(t,\xi+\Phi(t,\xi),\widetilde{u},(I+{}^tD\Phi)^{-1}D\widetilde{u}), \quad t \geq 0,\ \xi \in \overline{\Omega}_0, \tag{2.6}$$

where $[D\Phi]_{ij} = \frac{\partial\Phi_i}{\partial\xi_j}(t,\xi)$, the superscript t denotes the transposed matrix, and $\widetilde{\mathcal{L}}$ is the new expression of the operator $\mathcal{L}$, i.e.

$$\widetilde{\mathcal{L}} = \sum_{i,j,h,k=1}^{N} \widetilde{a}_{ik}\frac{\partial\xi_j}{\partial x_i}\frac{\partial\xi_h}{\partial x_k}D_{jh} + \sum_{i,j,k=1}^{N} \widetilde{a}_{ik}\frac{\partial^2\xi_j}{\partial x_i\partial x_k}D_j.$$

Here for each t the function $\xi(t,x)$ is the inverse of (2.4), so that the matrix with components $\frac{\partial\xi_h}{\partial x_k}$ is nothing but $(I + D\Phi(t,\xi))^{-1}$.

2.2 The splitting

To decouple problem (2.6) we introduce a new unknown w defined by

$$\widetilde{u}(t,\xi) = u_0(\xi) + \langle Du_0(\xi), \Phi(t,\xi)\rangle + w(t,\xi), \quad t \geq 0,\ \xi \in \overline{\Omega}_0. \tag{2.7}$$

Accordingly, we rewrite (2.6) as

$$\langle Du_0, \Phi_t\rangle + w_t - \langle D(u_0 + \langle Du_0, \Phi\rangle + w), (I + {}^tD\Phi)^{-1}\Phi_t\rangle = \widetilde{\mathcal{L}}(u_0 + \langle Du_0, \Phi\rangle + w)$$

$$+ f(t, \xi + \Phi, u_0 + \langle Du_0, \Phi\rangle + w, (I + {}^tD\Phi)^{-1}D(u_0 + \langle Du_0, \Phi\rangle + w)),\ \ t \geq 0,\ \xi \in \overline{\Omega}_0, \tag{2.8}$$

The decomposition (2.7) may look a bit strange; it is similar to the decompositions used in the papers [3, 4, 5] and in others to study stability problems. It is crucial in our analysis because it lets us decouple system (2.8), expressing s in terms of w. Indeed, the boundary condition $u - g_0 = 0$ on $\partial\Omega_t$ is rewritten as

$$g_0(0,\xi) + s(t,\xi)g_1(0,\xi) + w(t,\xi) - g_0(t, \xi + s(t,\xi)\nu(\xi)) = 0, \quad \xi \in \partial\Omega_0, \tag{2.9}$$

that is $F(t,\xi,s,w) = 0$, with $F_s(t,\xi,s,w) = g_1(0,\xi) - \frac{\partial g_0}{\partial \nu}(t,\xi) \neq 0$ for t small, due to the transversality assumption (1.3). By the Implicit Function Theorem we get s as a function of (t,ξ,w) for small w; more precisely

$$s(t,\xi) = \mathcal{F}_0(t,\xi,w(t,\xi)),\ \ t \geq 0,\ \xi \in \partial\Omega_0, \tag{2.10}$$

so that

$$\Phi(t,\xi) = \mathcal{F}_0(t,\xi',w(t,\xi'))\widetilde{\nu}(\xi),\ \ t \geq 0,\ \xi \in \overline{\Omega}_0. \tag{2.11}$$

Note that in the free boundary heat equation (1.4) formula (2.10) reduces to

$$s(t,\xi) = -\frac{w(t,\xi)}{C},\ \ t \geq 0,\ \xi \in \overline{\Omega}_0.$$

It is clear now why we need the transversality assumption (c); it is used to get the unknown s in terms of w, decoupling the system.

For future use we remark here that for each $\xi \in \partial\Omega_0$ we have

$$\mathcal{F}_0(t,\xi,w(t,\xi)) = \frac{w(t,\xi) - g_0(t,\xi) + g_0(0,\xi)}{\frac{\partial g_0}{\partial \nu}(t,\xi) - g_1(0,\xi)} + Q(t,\xi,w(t,\xi) - g_0(t,\xi) + g_0(0,\xi)), \tag{2.12}$$

where $Q(t,\xi,0) = Q_u(t,\xi,0) = 0$.

Replacing (2.11) in (2.8) we get

$$w_t = \mathcal{F}_1(t,\xi,w,Dw,D^2w) + \mathcal{F}_2(t,\xi,w,Dw)s_t,\ \ t \geq 0,\ \xi \in \overline{\Omega}_0, \tag{2.13}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ are obtained respectively from

$$\widetilde{\mathcal{L}}(u_0+\langle Du_0,\Phi\rangle+w)+f(t,\xi+\Phi,u_0+\langle Du_0,\Phi\rangle+w,(I+{}^tD\Phi)^{-1}D(u_0+\langle Du_0,\Phi\rangle+w)) \tag{2.14}$$

and from

$$-\langle Du_0 - (I + D\Phi)^{-1}D(u_0 + \langle Du_0, \Phi\rangle + w), \widetilde{\nu}\rangle \tag{2.15}$$

replacing $\Phi = \mathcal{F}_0(t,\xi',w(t,\xi'))\widetilde{\nu}(\xi)$.

Equation (2.13) still contains s_t; to eliminate it we use again the identity (2.9) which gives, after differentiation with respect to time,

$$w_t = \frac{\partial g_0}{\partial t}(t,\xi + s\nu) + (\langle Dg_0(t,\xi + s\nu),\nu\rangle - g_1(0,\xi))s_t,\ \ t \geq 0,\ \xi \in \partial\Omega_0.$$

Replacing in (2.13) for $\xi \in \partial\Omega_0$ we get

$$s_t(\langle Dg_0(t,\xi+\mathcal{F}_0(t,\xi,w)\nu),\nu\rangle - g_1(0,\xi) - \mathcal{F}_2(t,\xi,w,Dw)) + \frac{\partial g_0}{\partial t}(t,\xi+\mathcal{F}_0(t,\xi,w)\nu)$$

$$= \mathcal{F}_1(t,\xi,w,Dw,D^2w), \quad t \geq 0,\ \xi \in \partial\Omega_0,$$

so that, at least for t small, it is possible to get s_t as a function of w,

$$s_t = \mathcal{F}_3(t,\xi,w,Dw,D^2w) = \frac{\mathcal{F}_1 - \frac{\partial g_0}{\partial t}(t,\xi+\mathcal{F}_0\nu)}{\langle Dg_0(t,\xi+\mathcal{F}_0\nu),\nu\rangle - g_1(0,\xi) - \mathcal{F}_2} \tag{2.16}$$

which, replaced in (2.13), gives the final equation for w,

$$w_t = \mathcal{F}(t,w)(\xi), \quad t \geq 0,\ \xi \in \overline{\Omega}_0, \tag{2.17}$$

where

$$\mathcal{F}(t,w)(\xi) = \mathcal{F}_1(t,\xi,w,Dw,D^2w) + \mathcal{F}_2(t,\xi,w,Dw)\mathcal{F}_3(t,\xi,w,Dw,D^2w). \tag{2.18}$$

Although the explicit expression of $\mathcal{F}$ is rather complicated, we may note immediately that for ξ in $\Omega_0 \setminus \mathcal{N}$, that is far from the boundary, we have

$$\mathcal{F}(t,w)(\xi) = \mathcal{L}w + \mathcal{L}u_0 + f(t,\xi,w+u_0,Dw+Du_0).$$

This is due to the fact that our change of coordinates reduces to the identity far from the boundary. Moreover, even near the boundary we have

$$\mathcal{F}(0,0)(\xi) = \sum_{i,j=1}^{N} a_{ij}(0,\xi)D_{ij}u_0(\xi) + f(0,\xi,u_0(\xi),Du_0(\xi)). \tag{2.19}$$

For ξ near the boundary, the function $\mathcal{F}(t,v)$ is defined for t small and for $v \in C^2(\overline{\Omega}_0)$ with small C^1 norm. More precisely, let $\delta_0 \in (0,T]$ be such that

$$\frac{\partial g_0}{\partial \nu}(t,\xi) - g_1(0,\xi) \neq 0, \quad 0 \leq t \leq \delta_0,\ \xi \in \partial\Omega_0.$$

The function $\mathcal{F}(t,v)$ is defined for $t \in [0,\delta_0]$, and for $v \in C^2(\overline{\Omega}_0)$ with $\|v\|_{C^1} \leq r$, in such a way that both $\mathcal{F}_0(t,\xi,v(\xi))$ and $\mathcal{F}_3(t,\xi,v(\xi),Dv(\xi),D^2v(\xi))$ are well defined for $\xi \in \partial\Omega_0$ and for $\xi \in \overline{\Omega}_0$, respectively.

From formulas (2.10), (2.14), (2.15), (2.16) we see that $\mathcal{F}(t,v)(\xi)$ depends on v, Dv, D^2v and their traces at the boundary through functions which are at least $C^{\alpha/2,\alpha}$ with respect to (t,ξ) and at least $C^{2+\alpha}$ with respect to v and to its space derivatives of the first and of the second order. Therefore, for $0 \leq t \leq \delta_0$, the function $v \mapsto \mathcal{F}(t,v)$ is continuously differentiable from $\{v \in C^2(\overline{\Omega}_0) : \|v\|_{C^1} \leq r\}$ to $C(\overline{\Omega}_0)$, and from $\{v \in C^{2+\alpha}(\overline{\Omega}_0) : \|v\|_{C^1} \leq r\}$ to $C^\alpha(\overline{\Omega}_0)$.

In the next lemma we describe the linear part of $\mathcal{F}$ with respect to v near $v = 0$. We recall that for every $\xi \in \mathcal{N}$, ξ' denotes the point of $\partial\Omega_0$ closest to ξ.

Lemma 2.1 *For every $t \in [0,\delta_0]$ we have $\mathcal{F}_v(t,0) = \mathcal{A}(t,\partial)$, where*

$$(\mathcal{A}(t,\partial)v)(\xi) = \sum_{i,j=1}^{n} (\alpha_{ij}(t,\xi)D_{ij}v(\xi) + \alpha'_{ij}(t,\xi)D_{ij}v(\xi'))$$

$$+\sum_{i=1}^{n}(\beta_i(t,\xi)D_iv(\xi) + \beta'_i(t,\xi)D_iv(\xi')) + \gamma(t,\xi)v(\xi) + \gamma'(t,\xi)v(\xi'),\ 0 \leq t \leq T,\ \xi \in \overline{\Omega}_0. \tag{2.20}$$

The coefficients α_{ij}, β_i, γ, α'_{ij}, β'_i, γ' *are in* $C^{\alpha/2,\alpha}([0,\delta_0]\times\overline{\Omega}_0)$. α'_{ij}, β'_i, γ' *vanish in* $\Omega_0\setminus\mathcal{N}$, *and*

$$\alpha_{ij}(0,\xi)=a_{ij}(0,\xi),\ \ \alpha'_{ij}(0,\xi)=0,\ \ \xi\in\overline{\Omega}_0.$$

Proof — It is clear from the expression of $\mathcal{F}$ that the linear part of $v\mapsto\mathcal{F}(t,v)$ near $v=0$ is a nonlocal second order linear differential operator $\mathcal{A}(t,\partial)$ of the type (2.20). To identify its principal part at $t=0$, it is sufficient to consider $\mathcal{F}_1$. Indeed, the product $\mathcal{F}_2(0,\xi,v,Dv)\mathcal{F}_3(0,\xi,v,Dv,D^2v)$ does not contribute to the principal part, because $\mathcal{F}_2$ does not depend on the second order derivatives of v, and it vanishes at $t=0$, $v=0$.

Using (2.14) we see that only the term $\widetilde{\mathcal{L}}(u_0+\langle Du_0,\Phi\rangle+w)$ in $\mathcal{F}_1$ contributes to the principal part of $\mathcal{A}(0,\partial)$. Its explicit expression is

$$\sum_{i,k,h,j=1}^{N} a_{ik}(0,\xi+\Phi)\frac{\partial\xi_j}{\partial x_i}(\xi+\Phi)\,\frac{\partial\xi_h}{\partial x_k}(\xi+\Phi)\,D_{jh}(u_0+\langle Du_0,\Phi\rangle+v)$$

$$+\sum_{i,k,j=1}^{N} a_{ik}(0,\xi+\Phi)\frac{\partial^2\xi_j}{\partial x_i\partial x_k}(\xi+\Phi)\,D_j(u_0+\langle Du_0,\Phi\rangle+v)$$

$$=\mathcal{L}_0(v,\Phi)+\mathcal{L}_1(v,\Phi),$$

where the matrix M with entries $m_{hk}=\partial\xi_h/\partial x_k$ is equal to $(I+D\Phi)^{-1}$. We recall that Φ depends on v through (2.10) and (2.3), i.e. $\Phi(t,\xi,v)=\mathcal{F}_0(t,\xi',v(\xi'))\tilde{\nu}(\xi)$ and $\Phi(0,\xi,0)=0$.

At $t=0$, $v=0$, $\Phi=0$, the linear part of $\mathcal{L}_0$ with respect to (v,Φ) is

$$\sum_{i,k=1}^{N} a_{ik}(0,\xi)(D_{ik}v+D_{ik}\langle Du_0,\Phi\rangle)+\text{ lower order terms.}$$

From the developement

$$(I+D\Phi)^{-1}=I-D\Phi+(D\Phi)^2(I+D\Phi)^{-1} \tag{2.21}$$

we get that the linear part of $\mathcal{L}_1$ with respect to (v,Φ) is

$$-\sum_{i,k,j=1}^{N} a_{ik}(0,\xi)\frac{\partial^2\Phi_j}{\partial x_i\partial x_k}\,D_ju_0+\text{ lower order terms}$$

Therefore the second order derivatives of Φ cancel in the sum; what remains is

$$\mathcal{A}(0,\partial)=\sum_{i,k=1}^{N} a_{ik}(0,\xi)D_{ik}+\text{ lower order terms,}$$

and the statement follows. ∎

Lemma 2.1 implies that equation (2.17) is parabolic near $t=0$.

Equation (2.17) has to be supported with an initial and a boundary condition. The initial condition comes from the splitting (2.7), it is simply

$$w(0,\xi)=0,\ \ \xi\in\overline{\Omega}_0.$$

The boundary condition arises from $\partial u/\partial n=g_1$. Since

$$n(t,\xi+\Phi)=\frac{(I+{}^tD\Phi)^{-1}\nu(\xi)}{|(I+{}^tD\Phi)^{-1}\nu(\xi)|}$$

and

$$Du(t,\xi+\Phi) = (I + {}^tD\Phi)^{-1} D\widetilde{u}(t,\xi),$$

from $\partial u/\partial n = g_1$ we get

$$\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1} D(u_0 + \langle Du_0, \Phi\rangle + w)\rangle - g_1(t,\xi+\Phi)|(I + {}^tD\Phi)^{-1}\nu| = 0, \quad (2.22)$$

which gives a new boundary condition for w,

$$\mathcal{G}(t,\xi,w(t,\xi),Dw(t,\xi)) = 0, \ \ t \geq 0, \ \xi \in \partial\Omega_0, \quad (2.23)$$

as soon as we replace $\Phi = \mathcal{F}_0(t,\xi',w(t,\xi'))\widetilde{\nu}(\xi)$ in (2.22). The function $\mathcal{G}(t,\xi,u,p)$ and its first and second order derivatives with respect to u and p_i, $i = 1,\dots,N$, are continuous in (t,ξ,u,p) and $C^{1/2+\alpha/2,1+\alpha}$ in (t,ξ). Moreover $\mathcal{G}(0,\xi,0,0) = 0$.

In the next lemma we identify the linear part of $\mathcal{G}$ with respect to (u,p) at $(0,\xi,0,0)$.

Lemma 2.2 *We have*

$$\mathcal{G}(t,\xi,w,Dw) = \mathcal{B}w - G(t,\xi,w,Dw)(\xi)$$

where $\mathcal{B}$ is the linear differential operator defined by

$$\mathcal{B}v = \frac{\partial v}{\partial \nu} + v\left(\frac{\frac{\partial^2 u_0}{\partial \nu^2}(\xi) - \frac{\partial g_1}{\partial \nu}(0,\xi)}{\frac{\partial g_0}{\partial \nu}(0,\xi) - g_1(0,\xi)}\right) - \langle D^{tang}\left(\frac{v}{\frac{\partial g_0}{\partial \nu}(0,\xi) - g_1(0,\xi)}\right), D^{tang} g_0(0,\xi)\rangle, \quad (2.24)$$

and for every $\xi \in \partial\Omega_0$,

$$\begin{aligned} &G(0,\xi,0,0) = G_u(0,\xi,0,0) = G_{p_i}(0,\xi,0,0), \ \ i = 1,\dots,N, \\ &G_t(0,\xi,0,0) = \langle D^{tang}\left(\frac{D_t g_0(0,\xi)}{\frac{\partial g_0}{\partial \nu}(0,\xi) - g_1(0,\xi)}\right), D^{tang} g_0(0,\xi)\rangle \\ &-D_t g_0(0,\xi)\left(\frac{\frac{\partial^2 u_0}{\partial \nu^2}(\xi) - \frac{\partial g_1}{\partial \nu}(0,\xi)}{\frac{\partial g_0}{\partial \nu}(0,\xi) - g_1(0,\xi)}\right) + D_t g_1(0,\xi) \end{aligned} \quad (2.25)$$

Proof — Taking (2.21) into account, we rewrite the first addendum of (2.22) as

$$\begin{aligned} &\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1} D(u_0 + \langle Du_0, \Phi\rangle + w)\rangle \\ &= \frac{\partial u_0}{\partial \nu} + \frac{\partial}{\partial \nu}\langle Du_0, \Phi\rangle + \frac{\partial w}{\partial \nu} - \langle ({}^tD\Phi)\nu, Du_0\rangle - \langle \nu, ({}^tD\Phi)Du_0\rangle + R(\xi, Dw, \Phi, D\Phi) \end{aligned}$$

where $R(\xi,p,q,r)$ and its derivatives with respect to p_i, q_i, r_{ij}, $i,j = 1,\dots,N$, vanish at $(\xi,0,0,0)$.

In the above sum we have

$$\frac{\partial}{\partial \nu}\langle Du_0, \Phi\rangle = \frac{\partial^2 u_0}{\partial \nu^2}\, s, \ \ ({}^tD\Phi)\nu = D^{tang} s, \ \ \langle \nu, ({}^tD\Phi)Du_0\rangle = 0,$$

where the last equality follows from $(D\Phi)\nu = 0$. Replacing $s = \mathcal{F}_0(t,\xi,w)$ and taking (2.12) into account, we get

$$\begin{aligned} &\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1} D(u_0 + \langle Du_0, \Phi\rangle + w)\rangle \\ &= \frac{\partial u_0}{\partial \nu} + \frac{\partial w}{\partial \nu} + \frac{\partial^2 u_0}{\partial \nu^2}\left(\frac{w}{\frac{\partial g_0(t,\xi)}{\partial \nu} - g_1(0,\xi)}\right) - \langle D^{tang}\left(\frac{w}{\frac{\partial g_0(t,\xi)}{\partial \nu} - g_1(0,\xi)}\right), D^{tang} u_0\rangle \\ &+ \frac{\partial^2 u_0}{\partial \nu^2}\left(\frac{g_0(t,\xi) - g_0(0,\xi)}{\frac{\partial g_0(t,\xi)}{\partial \nu} - g_1(0,\xi)}\right) - \langle D^{tang}\left(\frac{g_0(t,\xi) - g_0(0,\xi)}{\frac{\partial g_0(t,\xi)}{\partial \nu} - g_1(0,\xi)}\right), D^{tang} u_0\rangle + R_1(t,\xi,w,Dw) \end{aligned}$$

where $R_1(t,\xi,u,p)$ vanishes at $(0,\xi,0,0)$ as well as its derivatives with respect to t, u, p_i, $i=1,\dots,N$.

On the other hand, since $\langle({}^tD\Phi)\nu,\nu\rangle=0$, we have

$$g_1(t,\xi+\Phi)|(I+{}^tD\Phi)^{-1}\nu|=g_1(t,\xi)+\langle Dg_1(t,\xi),s\nu(\xi)\rangle+R_3(t,\xi,\Phi,D\Phi)$$

where $R_3(t,\xi,p,q)$ and its derivatives with respect to t, p_i, q_{ij}, $i,j=1,\dots,N$, vanish at $(t,\xi,0,0)$. Replacing s and Φ and taking again (2.12) into account, we get

$$g_1(t,\xi+\Phi)|(I+{}^tD\Phi)^{-1}\nu|=g_1(t,\xi)+\frac{\partial g_1(t,\xi)}{\partial\nu}\left(\frac{w+g_0(t,\xi)-g_0(0,\xi)}{\frac{\partial g_0(t,\xi)}{\partial\nu}-g_1(0,\xi)}\right)+R_4(t,\xi,w,Dw)$$

where $R_4(t,\xi,u,p)$ and its derivatives with respect to (t,u,p_i) vanish at $(0,\xi,0,0)$. Summing up, the statement follows. ∎

2.3 The problem for w

The final system for w is therefore

$$\begin{cases} w_t=\mathcal{A}(t,\partial)w+F(t,w(t,\cdot)),\ \ t\geq 0,\ \xi\in\overline{\Omega}_0,\\ \\ \mathcal{B}w=G(t,w(t,\cdot)),\ \ t\geq 0,\ \xi\in\partial\Omega_0,\\ \\ w(0,\xi)=0,\ \ \xi\in\overline{\Omega}_0, \end{cases} \tag{2.26}$$

where we have set

$$F(t,v)(\xi)=\mathcal{F}(t,v)(\xi)-\mathcal{A}(t,\partial)v(\xi),$$
$$G(t,v)(\xi)=G(t,\xi,v(\xi),Dv(\xi)),\ \ \xi\in\partial\Omega_0.$$

Problem (2.26) will be solved by a fixed point theorem in a ball of the parabolic Hölder space $C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}_0)$, with δ small. The main tool is the next theorem about Hölder regularity and estimates for solutions of linear problems.

Theorem 2.3 *Let $\alpha\in(0,1)$ and $\delta_0>0$. Let Ω be an open set in R^N with uniformly $C^{2+\alpha}$ boundary, and let α_{ij}, β_i, γ, α'_{ij}, β'_i, $\gamma'\in C^{\alpha/2,\alpha}([0,\delta_0]\times\overline{\Omega})$ be such that*

$$\sum_{i,j=1}^N\alpha_{ij}(t,\xi)\eta_i\eta_j\geq\nu|\eta|^2,\ \ \xi\in\overline{\Omega},\ 0\leq t\leq\delta_0,$$

for some $\nu>0$, and

$$\alpha'_{ij}(0,\xi)=0,\ \ \xi\in\overline{\Omega}.$$

Moreover, let ϕ_i, $\psi\in C^{1/2+\alpha/2,1+\alpha}([0,\delta_0]\times\partial\Omega)$ be such that

$$\sum_{i=1}^N\phi_i(t,\xi)\nu_i(\xi)\neq 0,\ \ \xi\in\partial\Omega,\ 0\leq t\leq\delta_0.$$

For every $v\in C^2(\overline{\Omega})$ set

$$(\mathcal{A}(t,\partial)v)(\xi)=\sum_{i,j=1}^N(\alpha_{ij}(t,\xi)D_{ij}v(\xi)+\alpha'_{ij}(t,\xi)D_{ij}v(\xi'))$$

$$+\sum_{i=1}^N(\beta_i(t,\xi)D_iv(\xi)+\beta'_i(t,\xi)D_iv(\xi'))+\gamma(t,\xi)v(\xi)+\gamma'(t,\xi)v(\xi'),\ \ 0\leq t\leq\delta_0,\ \xi\in\overline{\Omega},$$

and

$$(\mathcal{B}(t,\partial)v)(\xi) = \psi(t,\xi)v(\xi) + \sum_{i=1}^{N} \phi_i(t,\xi) D_i v(\xi), \quad 0 \le t \le \delta_0,\ \xi \in \partial\Omega.$$

Then there is $\delta_1 \in (0, \delta_0]$ *such that for every* $f_1 \in C^{\alpha/2,\alpha}([0,\delta]\times\overline{\Omega})$, $g_1 \in C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\partial\Omega)$ *with* $0 < \delta \le \delta_1$, *satisfying the compatibility condition*

$$g_1(0,\cdot) = 0 \ \text{ in } \partial\Omega, \tag{2.27}$$

the problem

$$\begin{cases} v_t = \mathcal{A}(t,\partial)v + f_1, \quad 0 \le t \le \delta,\ \xi \in \overline{\Omega}, \\ \\ \mathcal{B}(t,\partial)v = g_1, \quad t \ge 0,\ \xi \in \partial\Omega, \\ \\ v(0,x) = 0, \quad \xi \in \overline{\Omega}, \end{cases} \tag{2.28}$$

has a unique solution $v \in C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega})$. *Moreover there exists* $C > 0$ *independent of* δ *such that*

$$\|v\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega})} \le C\Big(\|f_1\|_{C^{\alpha/2,\alpha}([0,\delta]\times\overline{\Omega})} + \|g_1\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\partial\Omega)}\Big). \tag{2.29}$$

If in addition Ω *has uniformly* $C^{3+\alpha}$ *boundary, and* α_{ij}, β_i, γ, α'_{ij}, β'_i, γ' *belong to* $C^{1/2+\alpha/2,1+\alpha}([0,\delta_0]\times\overline{\Omega})$, ϕ_i, $\psi \in C^{1+\alpha/2,2+\alpha}([0,\delta_0]\times\partial\Omega)$, *then there is* $\delta_1 \le \delta_0$ *such that if* $f_1 \in C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\overline{\Omega})$, $g_1 \in C^{1+\alpha/2,2+\alpha}([0,\delta]\times\partial\Omega)$ *with* $0 < \delta \le \delta_1$ *satisfy the compatibility condition (2.27) and the further compatibility condition*

$$\mathcal{B}(0,\partial)f_1(0,\cdot) = \frac{\partial g_1}{\partial t}(0,\cdot) \ \text{ in } \partial\Omega, \tag{2.30}$$

the solution v *of problem (2.28) is in* $C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega})$. *Moreover there exists* $C > 0$ *independent of* δ *such that*

$$\|v\|_{C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega})} \le C\Big(\|f_1\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\overline{\Omega})} + \|g_1\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\partial\Omega)}\Big). \tag{2.31}$$

Proof — If the coefficients α'_{ij}, β'_i, γ' vanish, the result is well known and we can take $\delta_1 = \delta_0$: see e.g. [11] for bounded Ω, [17] for unbounded Ω. In the general case the result follows by a (rather standard) perturbation argument, which we sketch below.

We write $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, where

$$(\mathcal{A}_0(t,\partial)v)(\xi) = \textstyle\sum_{i,j=1}^{n} \alpha_{ij}(t,\xi) D_{ij} v(\xi) + \sum_{i=1}^{n} \beta_i(t,\xi) D_i v(\xi) + \gamma(t,\xi) v(\xi),$$

$$(\mathcal{A}_1(t,\partial)v)(\xi) = \textstyle\sum_{i,j=1}^{n} \alpha'_{ij}(t,\xi) D_{ij} v(\xi') + \sum_{i=1}^{n} \beta'_i(t,\xi) D_i v(\xi') + \gamma'(t,\xi) v(\xi'),$$

and we consider the operator $\mathcal{A}$ as a perturbation of $\mathcal{A}_0$. So, we look for a solution of (2.28) as a fixed point of the operator $\Lambda : X = \{u \in C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}) : u(0,\cdot) = 0\} \mapsto X$, defined by $\Lambda u = v$, v being the solution of

$$\begin{cases} v_t = \mathcal{A}_0(t,\partial)v + \mathcal{A}_1(t,\partial)u + f_1, \quad 0 \le t \le \delta,\ x \in \overline{\Omega}, \\ \\ \mathcal{B}(t,\partial)v = g_1, \quad t \ge 0,\ x \in \partial\Omega, \\ \\ v(0,x) = 0, \quad x \in \overline{\Omega}. \end{cases}$$

We shall show that if δ is small enough, then Λ is a $1/2$-contraction in X. If u_1, $u_2 \in X$, $\Lambda u_1 - \Lambda u_2$ is the solution v of

$$\begin{cases} v_t = \mathcal{A}_0(t,\partial)v + \mathcal{A}_1(t,\partial)(u_1 - u_2), \quad 0 \le t \le \delta,\ x \in \overline{\Omega}, \\ \\ \mathcal{B}(t,\partial)v = 0, \quad t \ge 0,\ x \in \partial\Omega, \\ \\ v(0,x) = 0, \quad x \in \overline{\Omega}, \end{cases}$$

so that $\|\Lambda u_1 - \Lambda u_2\|_X \le C\|\mathcal{A}_1(t,\partial)(u_1-u_2)\|_{C^{\alpha/2,\alpha}([0,\delta]\times\overline{\Omega})}$. It is easy to see that there is $C>0$ such that

$$\left\| \sum_{i=1}^n \beta_i' D_i(u_1-u_2) + \gamma'(u_1-u_2) \right\| \le C\delta^{\alpha/2}\|(u_1-u_2)\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega})}.$$

Moreover, since $\alpha_{ij}'(0,\cdot)=0$ and $D_{ij}(u_1-u_2)(0,\cdot)=0$, we have $\|\alpha_{ij}'\|_\infty \le \delta^{\alpha/2}[\alpha_{ij}']_{C^{\alpha/2,\alpha}}$, $\|D_{ij}(u_1-u_2)\|_\infty \le \delta^{\alpha/2}[u_1-u_2]_{C^{1+\alpha/2,2+\alpha}}$. Therefore,

$$\|\alpha_{ij}' D_{ij}(u_1-u_2)\|_\infty \le C\delta^{\alpha/2}\|(u_1-u_2)\|_{C^{1+\alpha/2,2+\alpha}},$$

and

$$[\alpha_{ij}' D_{ij}(u_1-u_2)]_{C^{\alpha/2,\alpha}} \le \|\alpha_{ij}'\|_\infty [D_{ij}(u_1-u_2)]_{C^{\alpha/2,\alpha}} + [\alpha_{ij}']_{C^{\alpha/2,\alpha}}\|D_{ij}(u_1-u_2)\|_\infty$$

$$\le C\delta^{\alpha/2}\|(u_1-u_2)\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega})}.$$

Therefore for δ small enough Λ is a $1/2$-contraction, and the first part of the statement follows. The proof of the second part is similar. ∎

Now we are ready to solve problem (2.26).

Theorem 2.4 *There is $R_0>0$ such that for every $R\ge R_0$ and for every sufficiently small $\delta>0$ problem (2.26) has a unique solution in the ball $B(0,R)\subset C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}_0)$.*

Proof — Let δ_0, $r>0$ be such that $\mathcal{F}(t,v)$ is defined for $t\in[0,\delta_0]$ and for $v\in C^2(\overline{\Omega}_0)$ with $\|v\|_{C^1}\le r$. Fix $R>0$ and define Y to be the space

$$Y=\{v\in C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}_0),\ v(0,\cdot)=0,\ \|v\|_{C^{1+\alpha/2,2+\alpha}}\le R\}$$

where $\delta\in(0,\delta_0]$ is so small that $\|v\|_{C^{1+\alpha/2,2+\alpha}}\le R$ implies $\|v(t,\cdot)\|_{C^1}\le r$ for every $t\in[0,\delta]$. Therefore $F(t,v(t,\cdot))$ is well defined for all $t\in[0,\delta]$ and $v\in Y$.

Fixed any $v\in Y$, we introduce the linear problem

$$\begin{cases} w_t = \mathcal{A}(t,\partial)w + F(t,v(t,\cdot)),\ \ 0\le t\le\delta,\ x\in\overline{\Omega}_0, \\ \mathcal{B}w = G(t,v(t,\cdot)),\ \ 0\le t\le\delta,\ x\in\partial\Omega_0, \\ w(0,x)=0,\ \ x\in\overline{\Omega}_0, \end{cases} \tag{2.32}$$

to which we apply theorem 2.3. Since the compatibility condition $G(0,v(0))=0$ is satisfied, if $\delta\le\delta_1$ (2.32) has a unique solution $w\in C^{1+\alpha/2,2+\alpha}([0,\delta]\times\overline{\Omega}_0)$.

We define $\Gamma(v)=w$, w being the solution of (2.32). Our aim is to find a local solution to (2.26) as a fixed point of Γ.

We shall prove that for suitable δ and R (roughly: R large and δ small) Γ maps Y into itself and it is a contraction.

For u, $v\in Y$ the function $w=\Gamma(v)-\Gamma(u)$ is the solution of

$$\begin{cases} w_t = \mathcal{A}w + F(t,v(t,\cdot)) - F(t,u(t,\cdot)),\ \ 0\le t\le\delta,\ \xi\in\overline{\Omega}_0, \\ \mathcal{B}w = G(t,v(t,\cdot)) - G(t,u(t,\cdot)),\ \ 0\le t\le\delta,\ \xi\in\partial\Omega_0, \\ w(0,\xi)=0,\ \ \xi\in\overline{\Omega}_0. \end{cases}$$

We have therefore the estimate

$$\|w\|_{C^{1+\alpha/2,2+\alpha}} \le C(\|F(t,v)-F(t,u)\|_{C^{\alpha/2,\alpha}} + \|G(t,v)-G(t,u)\|_{C^{1/2+\alpha/2,1+\alpha}})$$

with C independent of δ. We claim now that

$$\|F(t,v)-F(t,u)\|_{C^{\alpha/2,\alpha}}+\|G(t,v)-G(t,u)\|_{C^{1/2+\alpha/2,1+\alpha}}\le K(R)\delta^{\alpha/2}\|u-v\|_{C^{1+\alpha/2,2+\alpha}}, \tag{2.33}$$

where K is an increasing function. The proof of (2.33) is straightforward although lenghty, it may be found in [14, Ch. 8] in the case of a local F.

Once (2.33) is established, we get immediately that Γ is a 1/2-contraction provided

$$C\,K(R)\delta^{\alpha/2}\le\frac{1}{2}.$$

In this case, for each $u\in Y$ we have

$$\|\Gamma(u)\|_{C^{1+\alpha/2,2+\alpha}}\le\|\Gamma(0)\|_{C^{1+\alpha/2,2+\alpha}}+\frac{R}{2},$$

where $\Gamma(0)$ is the solution of

$$\begin{cases} w_t=\mathcal{A}(t,\partial)w+F(t,0), \quad 0\le t\le\delta,\ x\in\overline{\Omega}_0,\\ \\ \mathcal{B}w=G(t,0), \quad 0\le t\le\delta,\ x\in\partial\Omega_0,\\ \\ w(0,x)=0, \quad x\in\overline{\Omega}_0, \end{cases}$$

so that

$$\|\Gamma(0)\|_{C^{1+\alpha/2,2+\alpha}}\le C(\|F(t,0)\|_{C^{\alpha/2,\alpha}}+\|G(t,0)\|_{C^{1/2+\alpha/2,1+\alpha}}).$$

If R is so large that

$$C(\|F(t,0)\|_{C^{\alpha/2,\alpha}([0,\delta_0]\times\overline{\Omega}_0)}+\|G(t,0)\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta_0]\times\partial\Omega_0)})\le\frac{R}{2}$$

then for every $u\in Y$ we have $\|\Gamma(u)\|_{C^{1+\alpha/2,2+\alpha}}\le R$, and Γ maps Y into itself. Therefore Γ has a unique fixed point in Y. ∎

Since the problem for w is parabolic, it is natural to have further regularity results, depending on the regularity of the data. Here we prove two further regularity theorems: a time-space regularity theorem up to $t=0$, which will be used later, and a smoothness theorem for $t>0$.

The Hölder time-space regularity theorem needs the following regularity and compatibility assumptions:

(d) *Further regularity assumptions.* $\Omega_0\subset R^N$ is a nonempty open set with uniformly $C^{4+\alpha}$ boundary $\partial\Omega_0$, and $u_0\in C^{4+\alpha}(\overline{\Omega}_0)$. The coefficients a_{ij} and their first order space derivatives belong to $C^{1/2+\alpha/2,1+\alpha}([0,T]\times R^N)$. The function $f=f(t,x,u,p)$ is differentiable with respect to x, twice differentiable with respect to (u,p); f and its derivatives are continuous in (t,x,u,p), and $C^{1/2+\alpha/2,1+\alpha}$ in (t,x), locally uniformly with respect to (u,p). Moreover, g_0 and g_1 are differentiable with respect to the space variables, and g_0, $D_ig_0\in C^{3/2+\alpha/2,3+\alpha}([0,T]\times R^N)$, g_1, $D_ig_1\in C^{1+\alpha/2,2+\alpha}([0,T]\times R^N)$ for $i=1,\dots,N$.

(e) *Further compatibility conditions.* Let $\mathcal{B}$ be the operator defined in (2.24). We assume that for every $\xi\in\partial\Omega_0$

$$\mathcal{B}\Big(\mathcal{L}(0,\partial)u_0+f(0,\cdot,u_0,Du_0)\Big)=\mathcal{B}\frac{\partial g_0}{\partial t}(0,\cdot)-\frac{\partial}{\partial\nu}\Big(\frac{\partial g_0}{\partial t}(0,\cdot)\Big)+\frac{\partial g_1}{\partial t}(0,\cdot).$$

This condition comes out from $\frac{\partial u}{\partial n}-g_1=0$ on $\partial\Omega_t$, differentiating with respect to time and taking the trace on $\partial\Omega_0$ at $t=0$. It is necessary for $\frac{\partial}{\partial t}\frac{\partial u}{\partial n}$ be continuous up to $t=0$.

Theorem 2.5 *Let assumptions (a), (b), (c), (d), (e) hold, and let* $w : [0,\delta] \times \overline{\Omega}_0 \to R$ *be the solution to problem (2.26) given by theorem 2.4. Then there exists* $\delta' \in (0,\delta)$ *such that* w *belongs to* $C^{1/2+\alpha/2,3+\alpha}([0,\delta'] \times \overline{\Omega}_0)$.

Sketch of the proof — Let us come back to the proof of theorem 2.4. Now we look for a fixed point of Γ in the space

$$Y' = \{v \in C^{3/2+\alpha/2,3+\alpha}([0,\delta] \times \overline{\Omega}_0),\ v(0,\cdot) = 0,\ \|v(t,\cdot)\|_{C^{3/2+\alpha/2,3+\alpha}} \le R'\}$$

which is closed in $C^{1+\alpha/2,2+\alpha}([0,\delta] \times \overline{\Omega}_0)$. If we prove that Γ maps Y' into itself for R' large and δ small we are done, because from the proof of theorem 2.4 we already know that Γ is a 1/2-contraction with respect to the $C^{1+\alpha/2,2+\alpha}$ distance. To this aim we use the last part of theorem 2.3. It yields that $\Gamma v \in C^{3/2+\alpha/2,3+\alpha}([0,\delta] \times \overline{\Omega}_0)$ provided $\varphi(t,\xi) = F(t,v(t,\cdot))(\xi) \in C^{1/2+\alpha/2,1+\alpha}([0,\delta] \times \overline{\Omega}_0)$, $\psi(t,\xi) = G(t,v(t,\cdot))(\xi) \in C^{1+\alpha/2,2+\alpha}([0,\delta] \times \partial\Omega_0)$, and that the compatibility condition

$$\mathcal{B}F(0,0) = \frac{d}{dt}G(t,v(t))_{|t=0} \tag{2.34}$$

holds. In this case, it gives also the estimate

$$\|\Gamma v\|_{C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega}_0)} \le C(\|\varphi\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\overline{\Omega}_0)} + \|\psi\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\partial\Omega_0)}).$$

The compatibility condition (2.34) holds thanks to assumption (e). From the regularity assumptions (d) we get easily that $\varphi \in C^{1/2+\alpha/2,1+\alpha}([0,\delta] \times \overline{\Omega}_0)$, and that $\psi \in C^{1+\alpha/2,2+\alpha}([0,\delta] \times \partial\Omega_0)$. A lenghty but elementary estimation shows that

$$\|(t,\xi) \mapsto \varphi(t,\xi) - \varphi(t,0)\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\overline{\Omega}_0)} \le C_1\delta^{\alpha/2}\|v\|_{C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega}_0)},$$

$$\|(t,\xi) \mapsto \psi(t,\xi) - \psi(t,0)\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\partial\Omega_0)} \le C_2\delta^{\alpha/2}\|v\|_{C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega}_0)},$$

for every $v \in Y'$. It follows that

$$\|\Gamma v\|_{C^{3/2+\alpha/2,3+\alpha}([0,\delta]\times\overline{\Omega}_0)} \le$$

$$\le C(\|F(t,0)\|_{C^{1/2+\alpha/2,1+\alpha}([0,\delta]\times\overline{\Omega}_0)} + \|G(t,0)\|_{C^{1+\alpha/2,2+\alpha}([0,\delta]\times\partial\Omega_0)} + (C_1+C_2)R')$$

so that Γ maps Y' into itself if R' is large enough and δ is small enough. ∎

Now we prove local time smoothness of w. Of course we need some smoothness assumptions on the data. Together with assumptions (a), (b), (c), we shall consider

(f) *Smoothness assumptions.* The coefficients a_{ij}, b_i, c, and the functions f, g_0, g_1 are C^∞ for $t > 0$.

Theorem 2.6 *Let (a), (b), (c), (f) hold, and let* $w : [0,\delta] \times \overline{\Omega}_0 \to R$ *be the solution to problem (2.26) given by theorem 2.4. Then there exists* $\delta' \in (0,\delta)$ *such that the function* $t \mapsto w(t,\cdot)$ *is in* $C^\infty((0,\delta'); C^2(\overline{\Omega}_0))$.

Proof — Let $\delta' \in (0,\delta)$. For every η sufficiently close to 1 the function $w_\eta(t,\cdot) = w(t\eta,\cdot)$ is well defined for $t \in [0,\delta]$, and it is a solution of

$$\Psi(\eta,v)(t) = (v_t - \eta\mathcal{A}(t,\partial)v - \eta F(t,v(t,\cdot))\,;\, \mathcal{B}v - G(t,v(t,\cdot))) = (0,0).$$

Ψ may be seen as a function from $[1-r,1+r] \times \{v \in C^{1+\beta/2,2+\beta}([0,\delta'] \times \overline{\Omega}_0) :\ v(0,\cdot) = 0,\ \|v\| \le \rho\}$ to $C^{\beta/2,\beta}([0,\delta'] \times \overline{\Omega}) \times C^{1/2+\beta/2,1+\beta}([0,\delta'] \times \partial\Omega_0)$, for every $\beta \in (0,\alpha]$, and for every small $\rho > 0$.

At $(\eta, v) = (1, w)$ we have of course $\Psi(1, w) = (0, 0)$. Ψ is smooth, and at $(1, w)$ the derivative of Ψ with respect to v is given by

$$(\Psi_v(1, w)v)(t) = (v_t - \mathcal{A}(t, \partial)v - F_v(t, w(t, \cdot))v \, ; \, \mathcal{B}v - G_v(t, w(t, \cdot))v).$$

By theorem 2.3, it is an isomorphism between the space $\{v \in C^{1+\beta/2,2+\beta}([0, \delta'] \times \overline{\Omega}_0) : v(0, \cdot) = 0\}$ and the space $C^{\beta/2,\beta}([0, \delta'] \times \overline{\Omega}_0) \times C^{1/2+\beta/2,1+\beta}([0, \delta'] \times \partial\Omega_0)$, provided δ' is small enough.

Therefore by the Implicit Function theorem there are r_1, $r_2 > 0$ and a smooth function

$$\psi : (1 - r_1, 1 + r_1) \mapsto B(w, r_2) \subset \{v \in C^{1+\beta/2,2+\beta}([0, \delta'] \times \overline{\Omega}_0) : \ v(0, \cdot) = 0\}$$

such that for $\eta \in (1 - r_1, 1 + r_1)$ and $v \in B(w, r_2)$ with $\Psi(\eta, v) = (0, 0)$ we have $v = \psi(\eta)$.

Fix $\beta \in (0, \alpha)$. Then we have

$$\lim_{\eta \to 1} w_\eta = w \ \text{ in } C^{1+\beta/2,2+\beta}([0, \delta'] \times \overline{\Omega}_0),$$

so that w_η belongs to $B(w, r_2) \subset C^{1+\beta/2,2+\beta}([0, \delta'] \times \overline{\Omega}_0)$ if η is sufficiently close to 1. Since $\Psi(\eta, w_\eta) = (0, 0)$, then $w_\eta = \psi(\eta)$. It follows that for η close to 1, the function $\eta \mapsto w(t\eta, \xi)$ is analytic with values in $C^{1,2}([0, \delta'] \times \overline{\Omega}_0)$. This implies easily that w and its first and second order space derivatives are smooth with respect to time for $0 < t < \delta'$. ∎

The idea of introducing the parameter η to prove time regularity of the solution is not new; see the paper [2].

Remark 2.7 Throughout the section we have considered only problems with initial time $t_0 = 0$, but it is clear that the results of theorems 2.4, 2.5 and 2.6 still hold for problem

$$\begin{cases} w_t = \mathcal{A}(t, \partial)w + F_0(t, w(t, \cdot)), \ \ t \geq t_0, \ \xi \in \overline{\Omega}_0, \\ \\ \mathcal{B}_0 w = G_0(t, w(t, \cdot)), \ \ t \geq t_0, \ \xi \in \partial\Omega_0, \\ \\ w(t_0, \xi) = 0, \ \ \xi \in \overline{\Omega}_0, \end{cases} \tag{2.35}$$

with initial time $t_0 \in (0, T)$, provided the due compatibility and transversality conditions are satisfied at $t = t_0$. Here $\mathcal{B}_0$, F_0 and G_0 are defined as $\mathcal{B}$, F and G replacing $g_0(0, \xi)$ and $g_1(0, \xi)$ by $g_0(t_0, \xi)$ and $g_1(t_0, \xi)$.

Once we have locally solved (2.26) we come back to the original problem (1.1) using (2.11) and (2.1) to define $\partial\Omega_t$. Note that s has the same regularity of w, i.e. it is in $C^{1+\alpha/2,2+\alpha}([0, \delta] \times \partial\Omega_0)$ under assumptions (a), (b), (c); it is in $C^{3/2+\alpha/2,3+\alpha}([0, \delta] \times \partial\Omega_0)$ under assumptions (a), (b), (c), (d), (e); it is smooth with respect to time for $t > 0$ under assumptions (a), (b), (c), (f). Then we define $\widetilde{u}$ through (2.7), where Φ is given by (2.3). Again, $\widetilde{u}$ has the same regularity of w. As a last step we define u through the change of coordinates, $u(t, x) = \widetilde{u}(t, \xi)$ where $x = \xi + \Phi(t, \xi)$. We have so proved the following existence and regularity result.

Theorem 2.8 *Let (a), (b), (c) hold. Then there is $\delta > 0$ such that problem (1.1) has a solution (Ω_t, u) such that the $N + 1$-dimensional hypersurface $\mathcal{S} = \{(t, x) : \ 0 \leq t \leq \delta, \ x \in \partial\Omega_t\}$ and each $\partial\Omega_t$ are of class $C^{1+\alpha/2,2+\alpha}$, and the function $u : \{(t, x); \ 0 \leq t \leq \delta, \ x \in \overline{\Omega}_t\} \mapsto R$ is of class $C^{1+\alpha/2,2+\alpha}$.*

If also (d), (e) hold, then $\mathcal{S}$ and each $\partial\Omega_t$ are of class $C^{3/2+\alpha/2,3+\alpha}$, and the function u is of class $C^{3/2+\alpha/2,3+\alpha}$.

If (a), (b), (c), (f) hold, then the free boundary is C^∞ with respect to time for $t > 0$, as well as u and its first and second order space derivatives.

3 Uniqueness and time smoothness

We say that a family of open sets $\{\Omega_t : a \le t \le b\} \subset R^N$ is of class $C^{1+\alpha/2,2+\alpha}$ in the interval $[a,b]$ if each boundary $\partial\Omega_t$ and the $(N+1)$-dimensional hypersurface $\{(t,x) : a \le t \le b, x \in \partial\Omega_t\}$ are of class $C^{1+\alpha/2,2+\alpha}$.

As a consequence of the uniqueness part of theorem 2.4 we get uniqueness of regular solutions to problems of the type (1.1).

Corollary 3.1 *Let assumptions (a), (b), (c) hold, and let* $(\Omega_t^{(1)}, u^{(1)}(t,x))$, $(\Omega_t^{(2)}, u^{(2)}(t,x))$ *be two solutions to (1.1) of class* $C^{1+\alpha/2,2+\alpha}$ *in an interval* $[0,a]$, *with space derivatives of class* $C^{1+\alpha/2,2+\alpha}$[1]. *Assume in addition that* $\partial g_0/\partial n(t,x) \neq g_1(t,x)$ *for every* $t \in [0,a]$, $x \in \partial\Omega_t^{(1)}$, *where* $n(t,x)$ *is the exterior unit normal vector to* $\partial\Omega_t^{(1)}$ *at the point* x. *Then* $(\Omega_t^{(1)}, u^{(1)}(t,\cdot)) = (\Omega_t^{(2)}, u^{(2)}(t,\cdot))$ *for every* $t \in [0,a]$.

Let in addition assumption (f) hold, and let $(\Omega_t, u(t,x))$ *be any solution to (1.1) of class* $C^{1+\alpha/2,2+\alpha}$ *in an interval* $[0,a]$, *with space derivatives of class* $C^{1+\alpha/2,2+\alpha}$, *and such that* $\partial g_0/\partial n(t,x) \neq g_1(t,x)$ *for every* $t \in [0,a]$. *Then* $\partial\Omega_t$ *and* u *are* C^∞ *with respect to* t *in* $(0,a)$.

Proof — Let us prove uniqueness. Let t_0 be the supremum of the numbers $t \in [0,a]$ such that $\Omega_s^{(1)} = \Omega_s^{(2)}$ for $0 \le s \le t$ and $u^{(1)}(s,x) = u^{(2)}(s,x)$ for $0 \le s \le t$, $x \in \Omega_s^{(1)} = \Omega_s^{(1)}$. Then we have also $\Omega_{t_0}^{(1)} = \Omega_{t_0}^{(2)}$ and $u^{(1)}(t_0,x) = u^{(2)}(t_0,x)$.

Assume by contradiction that $t_0 < a$. Consider the initial value problem

$$\begin{cases} u_t(t,x) = \mathcal{L}u(t,x) + f(t,x), \quad t \ge t_0, \ x \in \Omega_t, \\ u(t,x) = g_0(t,x), \quad t \ge t_0, \ x \in \partial\Omega_t, \\ \dfrac{\partial u}{\partial n}(t,x) = g_1(t,x), \quad t \ge t_0, \ x \in \partial\Omega_t, \\ u(t_0,x) = u^{(1)}(t_0,x), \quad x \in \Omega_{t_0}^{(1)}, \end{cases}$$

which can be obviously written as a problem with initial time 0, with data satisfying all the assumptions (a), (b), (c) at the beginning of section 2. Following the construction of section 2, we arrive at two solutions $w^{(1)}$, $w^{(2)}$ of (2.35) in the interval $[t_0,a]$, belonging to $C^{1+\alpha/2,2+\alpha}([t_0,a] \times \overline{\Omega}_{t_0}^{(1)})$ and with the property that for each $\delta \in (0, a-t_0]$ the restrictions of $w^{(1)}$ and $w^{(2)}$ to $[t_0, t_0+\delta] \times \overline{\Omega}_{t_0}^{(1)}$ do not coincide. Let $R = \max\{R_0, \|w^{(1)}\|_{C^{1+\alpha/2,2+\alpha}([t_0,a]\times\overline{\Omega}_{t_0}^{(1)})}, \|w^{(2)}\|_{C^{1+\alpha/2,2+\alpha}([t_0,a]\times\overline{\Omega}_{t_0}^{(1)})}\}$, where R_0 is given by theorem 2.4. From theorem 2.4 we know that if δ is sufficiently small, then (2.26) has a unique solution in the ball $B(0,R) \subset C^{1+\alpha/2,2+\alpha}([t_0,t_0+\delta] \times \overline{\Omega}_{t_0}^{(1)})$, and this is a contradiction.

Let us prove time smoothness. By theorem 2.6, for every $\overline{t} \in [0,a]$ there are $r = r(\overline{t}) > 0$ and $\delta = \delta(\overline{t}) > 0$ such the solution to problem (2.35) is in $C^\infty((t_0,t_0+\delta); C^2(\overline{\Omega}_0))$ for any $t_0 \in [0,a] \cap [\overline{t}-r, \overline{t}+r]$. Fix now any $\varepsilon \in (0,a)$. The union of the intervals $(\overline{t}, \overline{t}+\delta(\overline{t})) \cap [0,a]$ contains $[\varepsilon,a]$, and there are $\overline{t}_1, \dots, \overline{t}_n$ such that the union $\cup_{i=1}^n (\overline{t}_i, \overline{t}_i + \delta(\overline{t}_i)) \cap [0,a]$ still contains $[\varepsilon,a]$. In each of these intervals, time smoothness of w implies time smoothness of s (hence, time smoothness of $\partial\Omega_t$) and time smoothness of u and of its first and second order space derivatives. The statement follows. ∎

[1] Thanks to theorem 2.8, at least one solution with such regularity properties exists in a small interval if also assumptions (d), (e) are satisfied.

Remark 3.2 Things may be considerably simplified if $\partial\Omega_0$ is the graph of a regular function, say $\Omega_0 = \{\xi \in R^N : \xi_N \leq \phi(\xi_1, \ldots, \xi_{n-1})\}$ with $\phi \in C^{3+\alpha}(R^{N-1})$. In this case we may look for Ω_t of the form $\Omega_t = \{\xi \in R^N : \xi_N \leq \phi(\xi_1, \ldots, \xi_{n-1}) + s(t, \xi_1, \ldots, \xi_{n-1})\}$ where the unknown s is not in general the signed distance from $\partial\Omega_0$, but it works as well. The change of coordinates which transforms Ω_t into Ω_0 is now trivial,

$$\xi = (x_1, \ldots, x_{n-1}, x_N - s(t, x_1, \ldots, x_{n-1})), \quad x \in \Omega_t.$$

In addition to some simplifications in notation and formulas, the main advantage of this change of coordinates is that the procedure of theorem 2.6 gives also time analyticity (in the case of analytic data, of course) for $t > 0$. See e.g. [13], who used this method to get time analyticity in a system arising as a model in combustion theory, for initial data close to a travelling wave solution. Note that the map $\mathbf{\Psi}$ used in theorem 2.6 is smooth but not analytic in general, because of the change of coordinates (2.4) which contains the cutoff function θ introduced in §2.1.

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