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OPTIMAL REGULARITY AND FREDHOLM PROPERTIES OF ABSTRACT PARABOLIC OPERATORS IN L^p SPACES ON THE REAL LINE

DAVIDE DI GIORGIO, ALESSANDRA LUNARDI AND ROLAND SCHNAUBELT

1. *Introduction*

This paper is devoted to the investigation of the operator

$$u \mapsto \mathcal{L}u := u' - A(\,\cdot\,)u$$

in the space $L^p(\mathbb{R};X)$, with $1 < p < \infty$, for generators $A(t) : D(A(t)) \subset X \to X$, with $t \in \mathbb{R}$, of analytic semigroups in a Banach space X. The natural domain of $\mathcal{L}$ is the space

$$D(\mathcal{L}) = \{u \in W^{1,p}(\mathbb{R};X) : u(t) \in D(A(t)) \text{ a.e., } A(\,\cdot\,)u(\,\cdot\,) \in L^p(\mathbb{R};X)\}.$$

Given any $f \in L^p(\mathbb{R};X)$, the problem

$$u'(t) - A(t)u(t) = f(t) \quad \text{with } t \in \mathbb{R}, \tag{1.1}$$

differs to a large extent from the Cauchy problem

$$\begin{cases} u'(t) - A(t)u(t) = f(t) & \text{for } a < t < b, \\ u(a) = x. \end{cases} \tag{1.2}$$

In treating (1.1), we encounter the same difficulties as in (1.2) as far as local regularity is concerned, but in addition we have to deal with the asymptotic behavior. Under mild regularity assumptions on $A(\,\cdot\,)$, the well-known Acquistapace–Terreni conditions, a parabolic evolution operator $G(t,s)$, with $t \geqslant s$, exists and the unique solution to (1.2) is represented by the familiar variation-of-constants formula

$$u(t) = G(t,a)x + \int_a^t G(t,s)f(s)\,ds \quad \text{for } a < t < b$$

(at least if f is, say, locally Hölder continuous in t). On the contrary, even in the autonomous case $A(t) \equiv A$ and for finite-dimensional X, problem (1.1) may have no solution, solutions may be not unique, and reasonable representation formulas for the solutions, when they do exist, are not available in general. The simplest situation occurs if the evolution operator has an exponential dichotomy with projections $\{P(s) : s \in \mathbb{R}\}$ in $\mathbb{R}$. Then problem (1.1) has a unique solution $u \in L^p(\mathbb{R};X)$ given by

$$u(t) = \int_{-\infty}^t G(t,s)(I - P(s))f(s)\,ds - \int_t^{+\infty} G(t,s)P(s)f(s)\,ds \quad \text{for } t \in \mathbb{R}.$$

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This happens, for instance, in the periodic case $A(t) = A(t+T)$ if the unit circle is contained in the resolvent set of $G(T,0)$. In the general case, one has an exponential dichotomy, for example, if the operators $A(t)$ are small perturbations of a fixed hyperbolic operator A. (See [**22**, **23**, **27**, **37**, **38**] and the references therein.) We recall that a closed operator is called hyperbolic if its spectrum does not intersect the imaginary axis.

In this paper we consider the asymptotically hyperbolic case, that is, we assume that $A(t) \to A_{\pm\infty}$ as $t \to \pm\infty$ for two hyperbolic sectorial operators $A_{-\infty}$ and $A_{+\infty}$. Since the domains of $A(t)$ may vary, the above limits have to be understood in the resolvent sense. Then exponential dichotomies exist in the half-lines $(-\infty, -T]$ and $[T, +\infty)$ for sufficiently large $T \geqslant 0$ by [**10**, **37**, **38**].

The difficulties concerning local regularity, shared by problems (1.1) and (1.2), are well understood. We say that the problem (1.2) has optimal (or maximal) regularity of type L^p if for each $f \in L^p((a,b);X)$ there is a unique solution u of (1.2) with $u(a) = 0$ such that $u \in W^{1,p}((a,b);X)$, $u(t) \in D(A(t))$ a.e., and $A(\cdot)u(\cdot) \in L^p((a,b);X)$. This property implies, in particular, that the operator $\mathcal{L}$ is closed on the above given domain; see Corollary 2.6. If $A(t)$ is equal to a fixed sectorial operator A and X is a UMD space, then optimal L^p regularity is equivalent to the $\mathcal{R}$-boundedness of $\{\xi R(i\xi, A - \omega I) : \xi \in \mathbb{R}\}$, thanks to a theorem by Weis, [**42**]. We use a non-autonomous version of this result proved by Štrkalj in [**39**]; cf. also [**32**]. (See § 2 and also [**17**] and [**25**] for details.) For instance, L^q spaces and (fractional) Sobolev spaces $W^{\alpha,q}$ with $1 < q < \infty$ are UMD spaces thanks to, for example, [**6**, Theorem III.4.5.2]. If Ω is an open set in $\mathbb{R}^d$ with smooth boundary and each $A(t)$ is the realization of an elliptic operator in $L^q(\Omega)$ with uniformly continuous coefficients and good boundary conditions, then the assumption of $\mathcal{R}$-boundedness holds; see [**17**] and § 5.

Concerning asymptotic behavior, we characterize the couples

$$(f,x) \in L^p((T,+\infty);X) \times X$$

such that the solution to

$$u'(t) = A(t)u(t) + f(t) \quad \text{for } t > T; \quad u(T) = x,$$

belongs to $W^{1,p}((T,+\infty);X)$ (Theorem 2.4), and the couples

$$(g,y) \in L^p((-\infty,-T);X) \times X$$

such that the backward problem

$$v'(t) = A(t)v(t) + g(t) \quad \text{for } t < -T; \quad v(-T) = y,$$

has a solution in $W^{1,p}((-\infty,-T);X)$ (Theorem 2.5). As in [**20**], where $\mathcal{L}$ was studied on $C^\alpha(\mathbb{R};X)$, such characterizations are the starting point to investigate the operator $\mathcal{L}$. We describe several properties of $\mathcal{L}$ in terms of the stable space $W^s(T)$ and the unstable subspace $W^u(T)$ at T. See Theorem 3.8, whose statement is similar to (and in fact, it was inspired by) the case of bounded operators $A(t)$ discussed in [**1**]. See also the papers [**11**] and [**30**] for earlier ODE results. As a corollary, we obtain the fact that if $P_{+\infty}(X)$ and $P_{-\infty}(X)$ are finite dimensional, then $\mathcal{L}$ is a Fredholm operator with index

$$\operatorname{ind} \mathcal{L} = \dim P_{-\infty}(X) - \dim P_{+\infty}(X).$$

Here $P_{+\infty}$ and $P_{-\infty}$ are the spectral projections with respect to the subsets of the spectra of $A_{+\infty}$ and of $A_{-\infty}$ with positive real part. The above formula coincides with the well-known spectral flow formula ('index $=-$spectral flow') in finite dimensions; cf. [**1**, **21**, **31**, **35**]. The spectral flow is an algebraic count of the eigenvalues of $A(t)$ that cross the imaginary axis as t runs from $-\infty$ to $+\infty$. Under suitable assumptions it is meaningful even in infinite dimensions, and it may or may not coincide with minus the index of $\mathcal{L}$. The first important infinite-dimensional example in which the spectral flow equals minus the index of $\mathcal{L}$ was given by Robbin and Salamon in the paper [**35**], for a path of self-adjoint operators with constant and compactly embedded domain in a Hilbert space, and $p = 2$. Their result was recently extended in [**33**] to any $p \in (1, +\infty)$ and to possibly non-symmetric operators in UMD spaces. In both papers, compactness plays an essential role, and the operators $A(t)$ need not be sectorial. On the other hand, interesting examples of smooth paths of bounded self-adjoint operators in a Hilbert space such that the formula 'index $= -$ spectral flow' does not hold for $p = 2$ are given in [**1**]. We point out that the operators $A(t)$ in [**35**, **33**] have common and compactly embedded domain, while we can consider non-constant domains and, more importantly, we have no compactness assumptions in our main results. However, we are restricted to sectorial operators $A(t)$ while less stringent spectral assumptions are made in [**35**, **33**]. This has important consequences for Cauchy problems: both forward and backward Cauchy problems are in general ill-posed under the assumptions of [**35**, **33**], so that they have no evolution operator, while in our case forward Cauchy problems are well-posed, and (as in [**26**]) we have a forward evolution operator $G(t, s)$, with $t \geqslant s$. Results like Theorems 2.4 and 2.5 are not meaningful in the setting of [**35**, **33**].

Fredholm properties of ill-posed problems on the line have further been considered in the work by Sandstede, Scheel and co-authors; see, for example, [**36**]. We refer to [**1**, **20**, **26**, **33**, **36**] for further references and comments.

Besides theorems on maximal L^p regularity and Fredholm properties for a given path of sectorial operators $A(t)$, we focus on perturbation theory for operators $B(t) : D(A(t)) \to X$ being of the same order as $A(t)$. Here again one has to use optimal L^p regularity, and in particular, the fact that the map $u \mapsto A(\cdot)u(\cdot)$ is bounded from $D(\mathcal{L})$ to $L^p(\mathbb{R}; X)$ under our assumptions. In § 4 we assume that $\mathcal{L}$ is a Fredholm operator, and we consider the operator $\widetilde{\mathcal{L}} : D(\mathcal{L}) \to L^p(\mathbb{R}; X)$ defined by $(\widetilde{\mathcal{L}}u)(t) := u'(t) - A(t)u(t) - B(t)u(t)$ for $t \in \mathbb{R}$. The theory developed in § 2 directly implies that $\widetilde{\mathcal{L}}$ is Fredholm provided that the $A(t)$-bounds of $B(t)$ are sufficiently small; see Theorem 4.2. In Theorems 4.8 and 4.9 we show similar results if $B(t) : D(A(t)) \to X$ is compact and it converges as $|t| \to \infty$. The case of bounded perturbations $B(t) : X \to X$ was treated in [**26**] in a more general setting. Moreover, if also $A(t) : X \to X$ are bounded and X is a Hilbert space, more precise and refined results can be found in [**1**].

In § 5 we establish the Fredholm property of $\mathcal{L}$ for parabolic boundary value systems of second order on bounded domains satisfying the Lopatinskii–Shapiro conditions. Second, we study Ornstein–Uhlenbeck type operators perturbed by potentials in L^q spaces on $\mathbb{R}^d$ with respect to suitable weighted measures.

The definition of $\mathcal{L}$ can be extended to the case where one only has an exponentially bounded, strongly continuous evolution operator $G(t, s)$, based on the variation-of-constants formula. In this setting Latushkin and Tomilov characterized in the very recent paper [**26**] the Fredholm property of $\mathcal{L}$ in terms of the exponential dichotomy of $G(t,s)$ and of a condition connecting the projections at $-T$ and T, precisely the Fredholmity of the operator $\mathcal{N}: P(-T)(X) \mapsto P(T)(X)$, where $\mathcal{N}x = P(T)G(T,-T)x$. See also [**8**] and [**9**] for related investigations. In Proposition 3.9 we recover one implication of this result, as a consequence of (a part of) Theorem 3.8. We remark that the assumptions of [**26**] are weaker than ours, but our proofs are simpler and more direct, and our results are more specific: in the situation of [**26**] one cannot determine the domain of $\mathcal{L}$ in a reasonable way, and optimal regularity results like Theorems 2.4 and 2.5 are out of reach.

The last due comparison is with [**20**], which is the counterpart of this paper in the Hölder space setting. In fact, we followed the approach of [**20**] as far as possible. The present assumptions on the operators $A(t)$ are more general, because we allow for non-constant domains; moreover, we develop a perturbation theory that is not considered in [**20**] and we give much more general examples. A generalization of the results of [**20**] to the case of operators with non-constant domains satisfying the Acquistapace–Terreni conditions may be found in the thesis [**19**].

2. *Notation, assumptions and preliminaries*

We are given a family of sectorial operators $\{A(t) : t \in \mathbb{R}\}$ satisfying the Acquistapace–Terreni conditions, [**4**, **2**]: there are $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and $K > 0$ such that

$$\begin{aligned} \rho(A(t)) \supset \Sigma_{\omega,\theta} &:= \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| \leqslant \theta\} \cup \{\omega\}, \\ \|R(\lambda, A(t))\| &\leqslant \frac{K}{1 + |\lambda - \omega|} \end{aligned} \tag{2.1}$$

for all $t \in \mathbb{R}$ and $\lambda \in \Sigma_{\omega,\theta}$, and there are α_i and β_i for $i = 1, \dots, k$, with $0 \leqslant \beta_i < \alpha_i \leqslant 2$, such that $\delta = \min\{\alpha_i - \beta_i : i = 1, \dots, k\} \in (0,1)$ and

$$\|A(t)R(\lambda, A(t))\,[R(\omega, A(t)) - R(\omega, A(s))]\| \leqslant K \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda - \omega|^{\beta_i - 1} \tag{2.2}$$

for all $t \in \mathbb{R}$ and $\lambda \in \Sigma_{\omega,\theta} \setminus \{\omega\}$. These conditions imply that the family $\{A(t) : t \in \mathbb{R}\}$ generates an evolution operator $G(t,s)$, with $t \geqslant s \in \mathbb{R}$, which is strongly continuous for $t > s$. In other words, for each $s \in \mathbb{R}$ the Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t) & \text{for } t > s, \\ u(s) = x \end{cases} \tag{2.3}$$

(with $x \in \overline{D(A(s))}$) is well-posed. Its unique solution

$$u \in C([s, +\infty); X) \cap C^1((s, +\infty); X)$$

is given by

$$u(t) = G(t,s)x = e^{(t-s)A(s)}x + \int_s^t Z(r,s)x\,dr, \tag{2.4}$$

where $Z(t,s)$ is the operator given by formula (2.7) of [**2**]. In Theorem 2.3 and Lemma 2.2 of [**2**] it is proved that there exist constants $M_0, M_1 > 0$ such that

$$\|G(t,s)\|_{L(X)} \leqslant M_0, \quad \|A(t)G(t,s)\|_{L(X)} \leqslant M_1\,(t-s)^{-1} \quad \text{for } s < t < s+2, \tag{2.5}$$

and that there exist constants $c, c_{\nu,p} > 0$ such that

$$\|Z(r,s)\|_{L(X)} \leqslant c\,(r-s)^{\delta-1}, \quad \|Z(r,s)\|_{L((X,D(A(s)))_{\nu,p},X)} \leqslant c_{\nu,p}\,(r-s)^{\delta+\nu-1} \tag{2.6}$$

for $p \in [1,+\infty]$, $\nu \in [0,1)$, and $s < r \leqslant s+2$. We recall that, if $A : D(A) \to X$ is a sectorial operator, then for $\nu \in (0,1)$, $(X, D(A))_{\nu,p}$ is the real interpolation space between X and $D(A)$ defined by

$$\begin{cases} (X,D(A))_{\nu,p} = \{x \in X : t \mapsto g(t) = \|t^{1-\nu} A e^{tA} x\| \text{ belongs to } L^p((0,1), dt/t)\}, \\ \|x\|_{(X,D(A))_{\nu,p}} = \|x\| + \|g\|_{L^p((0,1),dt/t)}, \end{cases}$$

and for $\nu = 0$ we set $(X, D(A))_{0,p} = X$. From now on we denote by $\|\cdot\|_{\nu,p,s}$ the norm in $(X, D(A(s)))_{\nu,p}$. We refer to [**4**, **2**, **6**, **27**] for the construction of the evolution operator and to [**27**, **40**] for interpolation spaces.

We shall assume that the path $t \mapsto A(t)$ is asymptotically hyperbolic, that is, there are two operators $A_{-\infty} : D(A_{-\infty}) \to X$ and $A_{+\infty} : D(A_{+\infty}) \to X$ satisfying (2.1) and

$$\begin{aligned} &\lim_{t\to-\infty} R(\omega, A(t)) = R(\omega, A_{-\infty}), \\ &\lim_{t\to+\infty} R(\omega, A(t)) = R(\omega, A_{+\infty}) \quad (\text{in } L(X)); \end{aligned} \tag{2.7}$$

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{-\infty}) \cap i\mathbb{R} = \emptyset. \tag{2.8}$$

Finally, to have local maximal L^p regularity, we assume that the operators $A(t)$ are uniformly $\mathcal{R}$-sectorial. More precisely, we suppose that

$$X \text{ is a UMD space} \quad \text{and} \quad \sup\big\{\mathcal{R}\{\xi R(i\xi, A(t) - \omega I) : \xi \in \mathbb{R}\} : t \in \mathbb{R}\big\} < \infty. \tag{2.9}$$

A Banach space X is a UMD space (that is, X has the 'unconditional martingale sequences property') if and only if the Hilbert transform is bounded on $L^2(\mathbb{R}, X)$. It is known that in this case X is reflexive; see the references given in [**6**, § III.4.4]. Hence (2.1) and (2.9) imply that the operators $A(t)$ are densely defined, and thus the evolution operator is strongly continuous at $t = s$ by [**2**, Theorem 2.3]. The $\mathcal{R}$-bound $\mathcal{R}(\mathcal{T})$ of a family $\mathcal{T}$ of bounded linear operators is the infimum of all constants $C > 0$ such that

$$\Big\|\sum_{j=0}^{n} \varepsilon_j T_j x_j\Big\|_{L^2([0,1],X)} \leqslant C \Big\|\sum_{j=0}^{n} \varepsilon_j x_j\Big\|_{L^2([0,1],X)}$$

for all $n \in \mathbb{N} \cup \{0\}$, $T_0, \ldots, T_n \in \mathcal{T}$, $x_0, \ldots, x_n \in X$, where $\varepsilon_j(t) = \operatorname{sign}\sin(2^j \pi t)$ for $j \in \mathbb{N} \cup \{0\}$, are the Rademacher functions on $[0,1]$; see [**17**, **25**, **42**], and the references therein. Observe that the $\mathcal{R}$-boundedness of $\mathcal{T}$ implies its boundedness and that the converse holds for Hilbert spaces X, due to Plancherel's theorem. In particular, condition (2.9) follows from (2.1) if X is a Hilbert space. We observe that (2.9) implies that

$$\sup\big\{\mathcal{R}\{\lambda R(\lambda, A(t) - \omega I) : |\arg\lambda| \leqslant \phi\} : t \in \mathbb{R}\big\} =: R < \infty \tag{2.10}$$

for some $\phi \in (\pi/2, \theta)$ by (the proof of) Theorem 4.2 of [**42**]. Moreover, if we replace ω in (2.10) by a larger real number, then (2.10) remains valid with the same ϕ and R due to Proposition 2.8 in [**42**].

We shall consider the operator $\mathcal{L}$ defined by

$$\begin{cases} D(\mathcal{L}) := \{u \in W^{1,p}(\mathbb{R};X) : u(t) \in D(A(t)) \text{ a.e., } A(\cdot)u(\cdot) \in L^p(\mathbb{R};X)\}, \\ \mathcal{L} : D(\mathcal{L}) \to L^p(\mathbb{R};X), \quad (\mathcal{L}u)(t) = u'(t) - A(t)u(t) \end{cases} \tag{2.11}$$

for $1 < p < \infty$. In this context we introduce the space of maximal regularity

$$\mathcal{E}(I) = \{u \in W^{1,p}(I;X) : u(t) \in D(A(t))\text{a.e., } A(\cdot)u(\cdot) \in L^p(I;X)\} \tag{2.12}$$

for an interval $I \subset \mathbb{R}$, endowed with its natural norm $\|u'\|_{L^p(I;X)} + \|A(\cdot)u(\cdot)\|_{L^p(I;X)}$.

The main tool in our study will be exponential dichotomies. We recall that an evolution operator $G(t,s)$ is said to have an exponential dichotomy in an interval $I \subset \mathbb{R}$ if there exist a family of projections $P(t) \in L(X)$, with $t \in I$, being strongly continuous with respect to t, and numbers $\beta, N > 0$ such that for all $s,t \in I$ with $s \leqslant t$ we have

$$\begin{aligned} &\text{(a) } G(t,s)P(s) = P(t)G(t,s), \\ &\text{(b) } G(t,s) : P(s)(X) \to P(t)(X) \text{ is invertible with inverse } \widetilde{G}(s,t), \\ &\text{(c) } \|G(t,s)(I - P(s))\| \leqslant Ne^{-\beta(t-s)}, \\ &\text{(d) } \|\widetilde{G}(s,t)P(t)\| \leqslant Ne^{-\beta(t-s)}. \end{aligned} \tag{2.13}$$

Since $A(t)P(t) = A(t)G(t,s)\widetilde{G}(s,t)P(t)$ for $t > s$ with $t,s \in I$, the 'unstable projection' $P(t)$ maps X continuously into $D(A(t))$ for every $t \in I \setminus \inf I$, and $A(t)P(t)$ is uniformly bounded for $t \geqslant \eta + \inf I$ with $\eta > 0$, and for all $t \in I$ if I is unbounded below. Hence $P(t) : X \to (X, D(A(t)))_{\nu,p}$ is bounded as well, and we denote its norm by

$$P_{\nu,p,t} := \|P(t)\|_{L(X,(X,D(A(t)))_{\nu,p})}. \tag{2.14}$$

For more details on exponential dichotomies see [**13**, **23**, **27**, **38**] and the references therein.

Under assumptions (2.1), (2.2), (2.7) and (2.8), there exists $T \geqslant 0$ such that $G(t,s)$ has exponential dichotomies in $(-\infty, -T]$ and in $[T, +\infty)$. For $k = 1$ in (2.2) and the interval $[T, +\infty)$, this has been shown in Theorem 4.3 of [**38**]. The proofs given there may be extended in an obvious way to the general condition (2.2) and the interval $(-\infty, -T]$. The case of dense domains was treated before in [**10**] and, for a slightly stronger version of (2.7), in [**37**]. Moreover, we have

$$\dim P(t)(X) = \begin{cases} \dim P_{+\infty}(X) & \text{for } t \geqslant T, \\ \dim P_{-\infty}(X) & \text{for } t \leqslant -T, \end{cases}$$

where $P_{\pm\infty}$ are the projections for $A_{\pm\infty}$. Finally, in the proof of [**38**, Theorem 4.3], the projections $P(t)$ (for $t \geqslant T$ and $t \leqslant -T$, respectively) are obtained as the restriction of projections for a parabolic evolution operator having an exponential dichotomy on $I = \mathbb{R}$. Thus the constants $P_{\nu,p,t}$ introduced above are in fact uniformly bounded for $|t| \geqslant T$ in our situation.

We have to establish some results about forward and backward Cauchy problems in the L^p setting, which are known in C^α spaces; see [**27**, Chapter 6]. The starting point is local maximal L^p regularity of the evolution operator.

Lemma 2.1. *Assume that* (2.1) *and* (2.2) *hold. Let* $a < b \in \mathbb{R}$ *and* $p \in (1, +\infty)$. *Then for each* $x \in X$, *the function* $t \mapsto G(t,a)x$ *belongs to*

$W^{1,p}((a,b);X)$ if and only if $x \in (X, D(A(a)))_{1-1/p,p}$. In this case there is $C = C(p, b-a)$ such that $\|G(\cdot,a)x\|_{W^{1,p}((a,b);X)} \leqslant C\|x\|_{1-1/p,p,a}$.

Proof. Formula (2.4) shows that

$$A(t)G(t,a)x = \frac{d}{dt}G(t,a)x = A(a)e^{(t-a)A(a)}x + Z(t,a)x \quad \text{for } a < t \leqslant b. \tag{2.15}$$

Recall that x belongs to $(X, D(A(a)))_{1-1/p,p}$ if and only if the map

$$t \mapsto A(a)e^{(t-a)A(a)}x$$

is contained in $L^p((a,b);X)$. By estimate (2.6), the function $Z(\cdot,a)x$ belongs to $L^q((a,b);X)$ for every $x \in X$ and q such that $q(1-\delta) < 1$. If $x \in (X, D(A(a)))_{\nu,p}$, then $Z(\cdot,a)x \in L^q((a,b);X)$ for every q such that $q(1-\delta-\nu) < 1$. In particular, if $x \in (X, D(A(a)))_{1-1/p,p}$ then $Z(\cdot,a)x$ belongs to $L^p((a,b);X)$. Therefore, if $x \in (X, D(A(a)))_{1-1/p,p}$ then $u \in W^{1,p}((a,b);X)$, and the asserted estimate holds.

To prove the converse, let $u \in W^{1,p}((a,b);X)$. If $p(1-\delta) < 1$, then $Z(\cdot,a)x \in L^p((a,b);X)$ and hence

$$A(a)e^{(\cdot-a)A(a)}x \in L^p((a,b);X),$$

so that $x \in (X, D(A(a)))_{1-1/p,p}$. If $p(1-\delta) \geqslant 1$, set $q_1 = (1-\frac{1}{2}\delta)^{-1}$. Since $q_1(1-\delta) < 1$, we have $Z(\cdot,a)x \in L^{q_1}((a,b);X)$ and thus $x \in (X, D(A(a)))_{1-1/q_1,q_1}$. It follows that $Z(\cdot,a)x \in L^q((a,b);X)$ for each q such that $q(1-\frac{3}{2}\delta) < 1$. If $p(1-\frac{3}{2}\delta) < 1$ we have finished, otherwise we proceed in this way, and after n steps (with $p(1-(\frac{1}{2}n+1)\delta) < 1$) we obtain $x \in (X, D(A(a)))_{1-1/p,p}$. □

THEOREM 2.2. *Assume that (2.1), (2.2) and (2.9) hold. Let $a < b \in \mathbb{R}$ and $1 < p < +\infty$, and let $f \in L^p((a,b);X)$ and $x \in (X, D(A(a)))_{1-1/p,p}$. Then the problem*

$$\begin{cases} u'(t) = A(t)u(t) + f(t) & \text{for } a < t < b, \\ u(a) = x, \end{cases} \tag{2.16}$$

has a unique solution $u \in \mathcal{E}((a,b))$, given by

$$u(t) = G(t,a)u(a) + \int_a^t G(t,\tau)f(\tau)\,d\tau \quad \text{for } t \geqslant a. \tag{2.17}$$

There is a constant $C_{p,b-a}$ (independent of f and x) such that

$$\|u\|_{W^{1,p}((a,b);X)} + \|A(\cdot)u(\cdot)\|_{L^p((a,b);X)} \leqslant C_{p,b-a}\,(\|f\|_{L^p((a,b);X)} + \|x\|_{1-1/p,p,a}). \tag{2.18}$$

If $x \in X$, then equation (2.17) gives the unique solution in the class $C([a,b];X) \cap W^{1,p}_{\mathrm{loc}}((a,b];X)$ with $u(t) \in D(A(t))$ a.e.

Proof. For $x = 0$ the existence of a solution $u \in \mathcal{E}((a,b))$ was shown in Satz 4.2.6 of [**39**] for the case $k = 1$ in (2.2). The proof also works for the general case, and it can be seen that the constant $C_{p,b-a}$ only depends on the length of the interval, but not on the initial time itself. Alternatively, one can use Theorem 1 of [**32**]. Now Lemma 2.1 and [**2**, Theorem 2.3] yield the existence for the general case $x \neq 0$, since u is the sum of the solution to (2.16) with $x = 0$ plus $G(\cdot,a)x$.

Uniqueness and formula (2.17) are shown in the usual way. Let

$$u \in C([a,b];X) \cap W^{1,p}_{\mathrm{loc}}((a,b];X)$$

be a solution of (2.16). Fix $t \geqslant 0$ and $\varepsilon > 0$; set $v(s) = G(t,s)u(s)$ for $s \in [a+\varepsilon, t]$. Due to [**2**, Theorem 2.3], we obtain

$$v'(s) = -G(t,s)A(s)u(s) + G(t,s)A(s)u(s) + G(t,s)f(s) = G(t,s)f(s)$$

for a.e. $s \in [a+\varepsilon, t]$. Integrating from $a+\varepsilon$ to t and using the continuity of v and G, we deduce that u satisfies (2.17). □

In the next corollary we show a crucial embedding of the space $\mathcal{E}((a,b))$, defined in (2.12).

Corollary 2.3. *Assume that* (2.1), (2.2) *and* (2.9) *hold. Let* $a < b \in \mathbb{R}$ *and* $p \in (1, +\infty)$. *If* $u \in \mathcal{E}((a,b))$ *then* $u(t_0) \in (X, D(A(t_0)))_{1-1/p,p}$ *for all* $t_0 \in [a,b]$. *Moreover, for every* $t_0 \in [a,b]$ *there exists a positive constant* $\widetilde{C} = \widetilde{C}_{p,b-a}$ *such that*

$$\|u(t_0)\|_{1-1/p,\,p,t_0} \leqslant \widetilde{C}\,(\|u\|_{W^{1,p}((a,b);X)} + \|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p((a,b);X)}). \tag{2.19}$$

Proof. First, we assume that $t_0 \leqslant b-1$. Set $f(t) := u'(t) - A(t)u(t)$ for $t_0 \leqslant t \leqslant t_0+1$. The restriction of u to $[t_0, t_0+1]$ is the sum of $G(t,t_0)u(t_0)$ plus the solution of the Cauchy problem

$$\begin{cases} v'(t) - A(t)v(t) = f(t) & \text{for } t_0 < t \leqslant t_0+1, \\ v(t_0) = 0, \end{cases}$$

which belongs to $W^{1,p}((t_0,t_0+1);X)$ by Theorem 2.2. Therefore, $t \mapsto G(t,t_0)u(t_0)$ is in $W^{1,p}((t_0,t_0+1);X)$, and so $u(t_0) \in (X, D(A(t_0)))_{1-1/p,p}$ by Lemma 2.1. Moreover, the definition of the interpolation space, (2.15), (2.6), and Theorem 2.2 imply that

$$\begin{aligned}
&\|u(t_0)\|_{1-1/p,\,p,t_0} \\
&\quad = \|u(t_0)\| + \|A(t_0)e^{\cdot A(t_0)}u(t_0)\|_{L^p([t_0,t_0+1];X)} \\
&\quad \leqslant \|u(t_0)\| + \|A(\,\cdot\,)G(\,\cdot\,,t_0)u(t_0)\|_{L^p([t_0,t_0+1];X)} + \|Z(\,\cdot\,,t_0)u(t_0)\|_{L^p([t_0,t_0+1];X)} \\
&\quad \leqslant c\,(\|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p([t_0,t_0+1];X)} + \|A(\,\cdot\,)v(\,\cdot\,)\|_{L^p([t_0,t_0+1];X)} + \|u(t_0)\|_{\nu,\,p,t_0}) \\
&\quad \leqslant c\,(\|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p([t_0,t_0+1];X)} + \|u'(\,\cdot\,)\|_{L^p([t_0,t_0+1];X)} + \|u(t_0)\|_{\nu,\,p,t_0}),
\end{aligned}$$

where $\nu = \max\{0, 1-1/p-\frac{1}{2}\delta\}$ and the constants c only depend on the given constants. As in the proof of Lemma 2.1, we can iterate this procedure until $\nu = 0$. Then the asserted estimate follows from the embedding

$$W^{1,p}((t_0,t_0+1);X) \subseteq C((t_0,t_0+1);X)).$$

If $t_0 > b-1$, then we extend u and A to $[b, 2b-a]$ by defining $u(t) := u(2b-t)$ and $A(t) := A(2b-t)$ for $b \leqslant t \leqslant 2b-a$. Set $f(t) := u'(t) - A(t)u(t)$ for $a \leqslant t \leqslant 2b-a$. If $2b-a-t_0 \geqslant 1$, we can conclude as above. Otherwise, we repeat the extension until we obtain a time interval that is longer than 1, so that we can derive the asserted estimate as before. □

Once local optimal L^p regularity results are established, we may study optimal L^p regularity in half-lines. This is done in Theorem 2.4 for right half-lines and in Theorem 2.5 for left half-lines, using the well-known formulas (2.22) and (2.31); see for example, [**23**, §§ 5.1, 5.2]. Under the assumptions of Theorem 2.4 it may happen that the constant $P_{\nu,p,t}$ defined in (2.14) blows up as $t \to a$. In §§ 3 and 4 we shall assume that (2.1), (2.2), (2.7), (2.8) and (2.9) hold. As observed after formula (2.13), in this case $T \geqslant 0$ is fixed in advance and we may take $a = -\infty$.

THEOREM 2.4. *Assume that* (2.1), (2.2) *and* (2.9) *hold, and that* $G(t,s)$ *has an exponential dichotomy on an interval* $(a,+\infty)$. *Fix* $T > a$. *For each* $t_0 \geqslant T$, $1 < p < +\infty$, $f \in L^p((t_0,+\infty);X)$, *and* $x \in X$, *let* u *be the solution of*

$$u'(t) = A(t)u(t) + f(t) \quad \text{for } t > t_0; \quad u(t_0) = x. \tag{2.20}$$

Then u *belongs to* $L^p((t_0,+\infty);X)$ *if and only if*

$$P(t_0)x = -\int_{t_0}^{+\infty} \widetilde{G}(t_0,s)P(s)f(s)\,ds, \tag{2.21}$$

in which case it is given by

$$\begin{aligned} u(t) &= G(t,t_0)(I - P(t_0))x \\ &\quad + \int_{t_0}^{t} G(t,s)(I - P(s))f(s)\,ds - \int_{t}^{+\infty} \widetilde{G}(t,s)P(s)f(s)\,ds. \end{aligned} \tag{2.22}$$

If, in addition, $x \in (X, D(A(t_0)))_{1-1/p,p}$ *then* $u \in \mathcal{E}((t_0,+\infty))$, *and* $u(t) \in (X, D(A(t)))_{1-1/p,p}$ *for each* $t \geqslant t_0$. *Moreover, there is* $C_1 = C_1(T) > 0$ *independent of* x, f *and* t_0, *such that*

$$\begin{aligned} \|u\|_{W^{1,p}((t_0,+\infty);X)} + \|A(\cdot)u(\cdot)\|_{L^p((t_0,+\infty);X)} + \sup_{t \geqslant t_0} \|u(t)\|_{1-1/p,p,t} \\ \leqslant C_1 \left(\|x\|_{1-1/p,p,t_0} + \|f\|_{L^p((t_0,+\infty);X)} \right). \end{aligned} \tag{2.23}$$

Proof. By Theorem 2.2 the solution of (2.20) is given by the variation-of-constants formula

$$u(t) = G(t,t_0)x + \int_{t_0}^{t} G(t,s)f(s)\,ds \quad \text{for } t_0 < t.$$

We can thus split $u(t)$ into the sum $u_1(t) + u_2(t)$ where

$$\begin{aligned} u_1(t) &:= G(t,t_0)(I - P(t_0))x \\ &\qquad + \int_{t_0}^{t} G(t,s)(I - P(s))f(s)\,ds - \int_{t}^{+\infty} \widetilde{G}(t,s)P(s)f(s)\,ds, \\ u_2(t) &:= G(t,t_0)\left(P(t_0)x + \int_{t_0}^{+\infty} \widetilde{G}(t_0,s)P(s)f(s)\,ds \right). \end{aligned}$$

Using estimates (2.13)(c),(d) and Young's inequality, we obtain

$$\begin{aligned}\int_{t_0}^{+\infty} &\|G(t,t_0)(I-P(t_0))x\|^p\,dt \\ &\leqslant \int_{t_0}^{+\infty} N^p e^{-p\beta(t-t_0)}\|x\|^p\,dt = \frac{N^p}{p\beta}\|x\|^p,\end{aligned}$$

$$\begin{aligned}\int_{t_0}^{+\infty} &\left\|\int_{t_0}^{t} G(t,s)(I-P(s))f(s)\,ds\right\|^p dt \\ &\leqslant \left\|\left(\chi_{[0,+\infty)}\, Ne^{-\beta(\cdot)} * \chi_{[t_0,+\infty)}\, \|f(\cdot)\|_X\right)\right\|^p_{L^p(\mathbb{R};X)} \leqslant N^p\beta^{-p}\|f\|^p_{L^p((t_0,+\infty);X)},\end{aligned}$$

$$\begin{aligned}\int_{t_0}^{+\infty} &\left\|\int_{t}^{+\infty} \widetilde{G}(t,s)P(s)f(s)\,ds\right\|^p dt \\ &\leqslant \left\|\left(\chi_{(-\infty,0]}\, Ne^{\beta(\cdot)} * \chi_{[t_0,+\infty)}\, \|f(\cdot)\|_X\right)\right\|^p_{L^p(\mathbb{R};X)} \leqslant N^p\beta^{-p}\|f\|^p_{L^p((t_0,+\infty);X)},\end{aligned}$$

where χ_E is the characteristic function of the set E. Hence,

$$\|u_1\|_{L^p((t_0,+\infty),X)} \leqslant N(p\beta)^{-1/p}\|x\| + 2N\beta^{-1}\|f\|_{L^p((t_0,+\infty);X)}. \tag{2.24}$$

Moreover,

$$\left\|P(t_0)x + \int_{t_0}^{+\infty} \widetilde{G}(t_0,s)P(s)f(s)\,ds\right\| = \|\widetilde{G}(t_0,t)u_2(t)\| \leqslant Ne^{-\beta(t-t_0)}\|u_2(t)\|,$$

and

$$\begin{aligned}\|u_2(t)\| &\geqslant N^{-1}e^{\beta(t-t_0)}\left\|P(t_0)x + \int_{t_0}^{+\infty} \widetilde{G}(t_0,s)P(s)f(s)\,ds\right\| \\ &= N^{-1}e^{\beta(t-t_0)}\|u_2(t_0)\|.\end{aligned}$$

Consequently, we have $u_2 \notin L^p((t_0,+\infty),X)$ unless $u_2(t_0)=0$. Therefore, $u \in L^p((t_0,+\infty),X)$ if and only if $u_2(t_0)=0$; that is, (2.21) holds.

Now assume that (2.21) holds, and let $x \in (X, D(A(t_0)))_{1-1/p,\,p}$. Then the solution u is given by (2.22). We will prove that $A(\cdot)u(\cdot) \in L^p((t_0,+\infty);X)$. Using estimates (2.13)(c)(d) and (2.15), (2.5) and (2.6) again, we first get

$$\begin{aligned}\int_{t_0}^{+\infty} &\|A(t)G(t,t_0)(I-P(t_0))x\|^p\,dt \\ &\leqslant 2^p\int_{t_0}^{t_0+1} (\|A(t_0)e^{(t-t_0)A(t_0)}(I-P(t_0))x\|^p + \|Z(t,t_0)(I-P(t_0))x\|^p)\,dt \\ &\quad + \int_{t_0+1}^{+\infty} \|A(t)G(t,t-1)\|^p\ \|G(t-1,t_0)(I-P(t_0))x\|^p\,dt \\ &\leqslant 2^p(1+P_{1-1/p,\,p,t_0})^p\|x\|^p_{1-1/p,\,p,t_0} + (2c_{1-1/p,\,p}(1+P_{1-1/p,\,p,t_0})\|x\|_{1-1/p,\,p,t_0})^p \\ &\quad \cdot \int_{t_0}^{t_0+1} (t-t_0)^{\delta p-1}\,dt + (M_1N\|x\|)^p\int_{t_0}^{+\infty} e^{-p\beta(t-t_0-1)}\,dt.\end{aligned}$$

(We recall that $P_{1-1/p,\,p,t_0} = \|P(t_0)\|_{L(X,(X,D(A(t_0)))_{1-1/p,p})}$). Hence there is $K_1>0$ such that

$$\|A(\cdot)G(\cdot,t_0)(I-P(t_0))x\|_{L^p((t_0,+\infty);X)} \leqslant K_1\|x\|_{1-1/p,\,p,t_0}. \tag{2.25}$$

Second, we have

$$\begin{aligned}\int_{t_0}^{+\infty}\Big\|A(t)\int_{t_0}^{t}G(t,s)(I-P(s))f(s)\,ds\Big\|^p\,dt\\ \leqslant \int_{t_0}^{t_0+1}\Big\|A(t)\int_{t_0}^{t}G(t,s)(I-P(s))f(s)\,ds\Big\|^p\,dt\\ +\sum_{k=1}^{+\infty}\int_{t_0+k}^{t_0+k+1}2^p\Big\|A(t)\int_{t_0+k-1}^{t}G(t,s)(I-P(s))f(s)\,ds\Big\|^p\,dt\\ +\sum_{k=1}^{+\infty}\int_{t_0+k}^{t_0+k+1}2^p\Big\|A(t)G(t,t-1)\int_{t_0}^{t_0+k-1}G(t-1,s)(I-P(s))f(s)\,ds\Big\|^p\,dt.\end{aligned}$$

Then the inequalities (2.18), (2.5) and (2.13)(c) imply that

$$\begin{aligned}\int_{t_0}^{+\infty}\Big\|A(t)\int_{t_0}^{t}G(t,s)(I-P(s))f(s)\,ds\Big\|^p\,dt\\ \leqslant C_{p,1}^p\,\|f\|^p_{L^p((t_0,t_0+1);X)}+\sum_{k=1}^{+\infty}2^pC_{p,2}^p\,\|f\|^p_{L^p((t_0+k-1,t_0+k+1);X)}\\ +\sum_{k=1}^{+\infty}2^pM_1^pN^p\int_{t_0+k}^{t_0+k+1}\left(\int_{t_0}^{t_0+k-1}e^{-\beta(t-s-1)}\|f(s)\|\,ds\right)^p dt\\ \leqslant 3\cdot 2^pC_{p,2}^p\|f\|^p_{L^p((t_0,+\infty);X)}+2^pM_1^pN^p\int_{t_0}^{+\infty}\left(\int_{t_0}^{t}e^{-\beta(t-s-1)}\|f(s)\|\,ds\right)^p dt\\ \leqslant \big(3\cdot 2^pC_{p,2}^p+(2M_1N\beta^{-1}e^{\beta})^p\big)\|f\|^p_{L^p((t_0,+\infty);X)}.\end{aligned}$$

Hence there is $K_2>0$ such that

$$\|A(\cdot)\int_{t_0}^{\cdot}G(\cdot,s)(I-P(s))f(s)\,ds\|_{L^p((t_0,+\infty);X)}\leqslant K_2\|f\|_{L^p((t_0,+\infty);X)}. \tag{2.26}$$

Similarly, we estimate the third summand in (2.22) by

$$\begin{aligned}\int_{t_0}^{+\infty}\Big\|A(t)\int_{t}^{+\infty}\widetilde{G}(t,s)P(s)f(s)\,ds\Big\|^p\,dt\\ =\int_{t_0}^{+\infty}\Big\|A(t)G(t,t-1)\int_{t}^{+\infty}\widetilde{G}(t-1,s)P(s)f(s)\,ds\Big\|^p\,dt\\ \leqslant\int_{t_0}^{+\infty}N^pM_1^p\left(\int_{t}^{+\infty}e^{-\beta(s-t+1)}\|f(s)\|\,ds\right)^p dt\\ \leqslant N^pM_1^p\beta^{-1}e^{-\beta}\|f\|^p_{L^p((t_0,+\infty);X)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big\|A(\cdot)\int_{\cdot}^{+\infty}\widetilde{G}(\cdot,s)P(s)f(s)\,ds\Big\|_{L^p((t_0,+\infty);X)}\\ \leqslant NM_1\beta^{-1/p}e^{-\beta/p}\|f\|_{L^p((t_0,+\infty);X)}.\end{aligned} \tag{2.27}$$

Since u is given by (2.22) and solves (2.20), the estimate

$$\|u\|_{W^{1,p}((t_0,+\infty);X)} + \|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p((t_0,+\infty);X)} \leqslant C_1' \left(\|x\|_{1-1/p,\,p,t_0} + \|f\|_{L^p((t_0,+\infty);X)}\right) \tag{2.28}$$

follows from the inequalities (2.25), (2.26) and (2.27). In order to estimate $\|u(t)\|_{1-1/p,\,p,t}$, we apply Corollary 2.3 to the restriction of u to $[t,t+1]$, and then we use (2.28). So we have shown (2.23). □

THEOREM 2.5. *Assume that* (2.1), (2.2) *and* (2.9) *hold, and that* $G(t,s)$ *has an exponential dichotomy on an interval* $(-\infty,-T]$. *Let* $t_0 \leqslant -T$, $y\in X$, $1<p<+\infty$, *and* $g\in L^p((-\infty,t_0);X)$. *Then the problem*

$$v'(t) = A(t)v(t) + g(t) \quad \text{for } t<t_0; \quad v(t_0)=y, \tag{2.29}$$

has a solution v *in* $L^p((-\infty,t_0);X)$ *if and only if*

$$(I-P(t_0))y = \int_{-\infty}^{t_0} G(t_0,s)(I-P(s))g(s)\,ds, \tag{2.30}$$

in which case v *is given by*

$$v(t) = \widetilde{G}(t,t_0)P(t_0)y + \int_{t_0}^{t} \widetilde{G}(t,s)P(s)g(s)\,ds + \int_{-\infty}^{t} G(t,s)(I-P(s))g(s)\,ds. \tag{2.31}$$

Moreover, $v\in W^{1,p}((-\infty,t_0);X)$, *and for each* $t\leqslant t_0$, $v(t)\in D(A(t))$ a.e. *and* $v(t)\in (X,D(A(t)))_{1-1/p,\,p}$. *There exists* $C_2>0$ (*independent of* y, f *and* t_0) *such that*

$$\|v\|_{W^{1,p}((-\infty,t_0);X)} + \|A(\,\cdot\,)v(\,\cdot\,)\|_{L^p((-\infty,t_0);X)} + \sup_{t\leqslant t_0}\|v(t)\|_{1-1/p,\,p,t} \leqslant C_2\left(\|y\|_X + \|g\|_{L^p((-\infty,t_0);X)}\right). \tag{2.32}$$

Proof. Let v be a solution of (2.29). For every $a\leqslant t_0$ the variation-of-constants formula (2.17) gives

$$v(t) = G(t,a)v(a) + \int_a^t G(t,s)g(s)\,ds \quad \text{for } a\leqslant t\leqslant t_0, \tag{2.33}$$

so that

$$(I-P(t))v(t) = G(t,a)(I-P(a))v(a) - \int_{-\infty}^{a} G(t,s)(I-P(s))g(s)\,ds + \int_{-\infty}^{t} G(t,s)(I-P(s))g(s)\,ds.$$

Suppose now that $v\in L^p((-\infty,t_0);X)$. Since v is continuous, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ going to $-\infty$ such that $\lim_{n\to+\infty} v(a_n)=0$. Taking $a=a_n$ in the above expression, we obtain

$$\begin{aligned}(I-P(t))v(t) &- \int_{-\infty}^{t} G(t,s)(I-P(s))g(s)\,ds \\ &= G(t,a_n)(I-P(a_n))v(a_n) - \int_{-\infty}^{a_n} G(t,s)(I-P(s))g(s)\,ds \qquad (2.34)\end{aligned}$$

for every $n\in\mathbb{N}$ and $t\geqslant a_n$. Estimate (2.13)(c) yields

$$\begin{aligned}&\left\| G(t,a_n)(I-P(a_n))v(a_n) - \int_{-\infty}^{a_n} G(t,s)(I-P(s))g(s)\,ds \right\| \\ &\qquad\leqslant Ne^{-\beta(t-a_n)}\|v(a_n)\| + \int_{-\infty}^{a_n} Ne^{-\beta(t-s)}\|g(s)\|\,ds.\end{aligned}$$

Therefore, letting $n\to+\infty$ in (2.34), we deduce that

$$(I-P(t))v(t) = \int_{-\infty}^{t} G(t,s)(I-P(s))g(s)\,ds \qquad (2.35)$$

for every $t\leqslant t_0$. If we take $t=t_0$, identity (2.30) follows.

Suppose now that (2.30) holds. Then the function v defined by (2.31) satisfies (2.33), as is easy to check, and $v(t_0)=y$, so that v solves (2.29). Let us verify that $v\in L^p((-\infty,t_0);X)$. Using (2.13)(c) and Young's inequality as in Theorem 2.4, we obtain

$$\left\| \int_{-\infty}^{\cdot} G(\,\cdot\,,s)(I-P(s))g(s)\,ds \right\|_{L^p((-\infty,t_0);X)} \leqslant N\beta^{-1}\|g\|_{L^p((-\infty,t_0);X)}, \qquad (2.36)$$

and hence $(I-P(\,\cdot\,))v(\,\cdot\,)\in L^p((-\infty,t_0);X)$. In order to estimate $P(t)v(t)$ for $t\leqslant t_0$, we first apply $P(t)$ to both sides of (2.33) and then use the inverse of $G(t_0,t)_{|P(t)(X)}$:

$$\begin{aligned}P(t)v(t) &= P(t)G(t,a)v(a) + P(t)\int_a^t G(t,s)g(s)\,ds \\ &= \widetilde{G}(t,t_0)\left(P(t_0)G(t_0,a)v(a) + \int_a^{t_0} G(t_0,s)P(s)g(s)\,ds\right) \\ &\quad + \int_{t_0}^{t} \widetilde{G}(t,s)P(s)g(s)\,ds \\ &= \widetilde{G}(t,t_0)P(t_0)y + \int_{t_0}^{t} \widetilde{G}(t,s)P(s)g(s)\,ds\end{aligned}$$

where we have again employed (2.33) for $t=t_0$. As in the proof of Theorem 2.4, estimate (2.13)(d) and Young's inequality yield

$$\|P(\,\cdot\,)v(\,\cdot\,)\|_{L^p((-\infty,t_0);X)} \leqslant N(p\beta)^{-1/p}\|y\| + N\beta^{-1}\|g\|_{L^p((-\infty,t_0);X)}. \qquad (2.37)$$

Estimates (2.36) and (2.37) now imply that $v\in L^p((-\infty,t_0);X)$.

In order to prove the second part of the statement we have to control $\|A(\cdot)v(\cdot)\|_{L^p((-\infty,t_0);X)}$. Arguing as in Theorem 2.4, we derive

$$\left.\begin{aligned}&\int_{-\infty}^{t_0}\|A(t)\widetilde{G}(t,t_0)P(t_0)y\|^p\,dt\\&\qquad\leqslant\int_{-\infty}^{t_0}\|A(t)G(t,t-1)\|^p\,\|\widetilde{G}(t-1,t_0)P(t_0)y\|^p\,dt\\&\qquad\leqslant\int_{-\infty}^{t_0}M_1^pN^pe^{-p\beta(t_0-t+1)}\|y\|^p\,dt,\\&\|A(\cdot)\widetilde{G}(\cdot,t_0)P(t_0)y\|_{L^p((-\infty,t_0);X)}\leqslant M_1N(p\beta)^{-1/p}e^{\beta}\|y\|.\end{aligned}\right\}\tag{2.38}$$

Similarly, we obtain

$$\left.\begin{aligned}&\int_{-\infty}^{t_0}\left\|A(t)\int_{t_0}^{t}\widetilde{G}(t,s)P(s)g(s)\,ds\right\|^p dt\\&\qquad\leqslant M_1^pN^p\|\chi_{\mathbb{R}_-}e^{\beta(\cdot-1)}*\chi_{(-\infty,t_0]}\|g(\cdot)\|\,\|^p_{L^p((-\infty,t_0);X)}\\&\qquad\leqslant(M_1N\beta^{-1}e^{-\beta}\|g\|_{L^p((-\infty,t_0);X)})^p,\\&\left\|A(\cdot)\int_{t_0}^{\cdot}\widetilde{G}(\cdot,s)P(s)g(s)\,ds\right\|_{L^p((-\infty,t_0);X)}\\&\qquad\leqslant M_1N\beta^{-1}e^{-\beta}\|g\|_{L^p((-\infty,t_0);X)}.\end{aligned}\right\}\tag{2.39}$$

Finally, to estimate the L^p norm of $A(\cdot)\int_{-\infty}^{\cdot}G(\cdot,s)(I-P(s))g(s)\,ds$, we have again to split it into a series and use Theorem 2.2 on local L^p-maximal regularity. So we get

$$\begin{aligned}&\int_{-\infty}^{t_0}\left\|A(t)\int_{-\infty}^{t}G(t,s)(I-P(s))g(s)\,ds\right\|^p dt\\&\leqslant\sum_{k=0}^{+\infty}\int_{t_0-k-1}^{t_0-k}2^p\left\|A(t)\int_{-\infty}^{t_0-k-2}G(t,s)(I-P(s))g(s)\,ds\right\|^p dt\\&\quad+\sum_{k=0}^{+\infty}\int_{t_0-k-1}^{t_0-k}2^p\left\|A(t)\int_{t_0-k-2}^{t}G(t,s)(I-P(s))g(s)\,ds\right\|^p dt\\&\leqslant\sum_{k=0}^{+\infty}\int_{t_0-k-1}^{t_0-k}2^pM_1^p\left\|\int_{-\infty}^{t_0-k-2}G(t-1,s)(I-P(s))g(s)\,ds\right\|^p dt\\&\quad+\sum_{k=0}^{+\infty}C_{p,2}^p2^p\|g\|^p_{L^p((t_0-k-2,t_0-k);X)}\\&\leqslant(2M_1N)^p\,\|\chi_{\mathbb{R}_+}e^{-\beta(\cdot-1)}*\chi_{(-\infty,t_0]}\|g(\cdot)\|\,\|^p_{L^p(\mathbb{R};X)}+2C_{p,2}^p2^p\,\|g\|^p_{L^p((-\infty,t_0);X)}\\&\leqslant(2M_1N)^p\beta^{-p}e^{p\beta}\|g\|^p_{L^p((-\infty,t_0);X)}+2^{p+1}C_{p,2}^p\|g\|^p_{L^p((-\infty,t_0);X)}.\end{aligned}$$

Thus there is $K_3>0$ such that

$$\left\|A(\cdot)\int_{-\infty}^{\cdot}G(\cdot,s)(I-P(s))g(s)\,ds\right\|_{L^p((-\infty,t_0);X)}\leqslant K_3\|g\|_{L^p((-\infty,t_0);X)}.\tag{2.40}$$

Since v is given by (2.31) and solves (2.29), the inequalities (2.38), (2.39), (2.40), (2.19) and Corollary 2.3 imply (2.32), as in the proof of the previous theorem. □

Corollary 2.6. *Assume that (2.1), (2.2) and (2.9) hold and let* $1 < p < +\infty$. *Then there is a real number* γ *such that the equation* $\gamma u + \mathcal{L}u = f$ *has a unique solution* $u \in D(\mathcal{L})$ *for each* $f \in L^p(\mathbb{R}; X)$, *given by*

$$u(t) = \int_{-\infty}^{t} e^{-\gamma(t-s)} G(t,s) f(s)\, ds \quad \text{for } t \in \mathbb{R}.$$

Moreover, there is a constant C_3 *such that, for* $u \in D(\mathcal{L})$,

$$\|u\|_{W^{1,p}(\mathbb{R},X)} + \|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p(\mathbb{R};X)} \leqslant C_3 \,\|f\|_{L^p(\mathbb{R};X)}.$$

Proof. By (2.5) there is a $\gamma \geqslant 0$ such that $e^{-\gamma(t-s)}G(t,s)$ is exponentially stable. Theorem 2.5 for $A(t) - \gamma I$ and $P(t) \equiv 0$ then easily implies the assertions. □

3. *Properties of the operator* $\mathcal{L}$

Throughout this section $\{A(t) : t \in \mathbb{R}\}$ is a family of operators satisfying assumptions (2.1), (2.2), (2.7), (2.8) and (2.9), $G(t,s)$ is the associated evolution operator, and $\mathcal{L}$ is the operator defined in (2.11) with $p \in (1, +\infty)$. In particular, $G(t,s)$ has exponential dichotomies on $(-\infty, -T]$ and $[T, +\infty)$ with projections $P(t)$, for some $T \geqslant 0$. The stable and unstable subspaces are defined as usual; cf. [**1**, **20**].

Definition 3.1. Let $t_0 \in \mathbb{R}$. We define the *stable space* at t_0 by

$$W^s(t_0) := \Big\{ x \in X : \lim_{t \to +\infty} G(t, t_0)x = 0 \Big\},$$

and the *unstable space* at t_0 by

$$\begin{aligned} W^u(t_0) := \Big\{ x \in X : \exists\, u \in C^1((-\infty, t_0]; X) \text{ such that } u(t) \in D(A(t)),\ t \leqslant t_0, \\ u'(t) = A(t)u(t),\ A(\,\cdot\,)u(\,\cdot\,) \in C((-\infty, t_0]; X), \\ u(t_0) = x,\ \lim_{t \to -\infty} u(t) = 0 \Big\}. \end{aligned}$$

Lemma 3.2. *The following statements hold true:*

(i) *for each* $t_0 \geqslant T$, $W^s(t_0) = (I - P(t_0))(X)$; *for each* $t_0 \leqslant -T$, $W^u(t_0) = P(t_0)(X)$;

(ii) *for each* $t_0 \geqslant T$, $W^s(t_0) = \{x \in X : G(\,\cdot, t_0)x \in L^p((t_0, +\infty); X)\}$;

(iii) *for each* $t_0 \leqslant -T$, $W^u(t_0) = \{x \in X : \exists\, u \in W^{1,p}((-\infty, t_0); X)$ *with* $u(t) \in D(A(t))$ *and* $u'(t) = A(t)u(t)$ *a.e.,* $u(t_0) = x\}$

(iv) *for each* $t, t_0 \in \mathbb{R}$ *with* $t \geqslant t_0$, $G(t, t_0)W^s(t_0) \subseteq W^s(t)$;

(v) *for each* $t, t_0 \in \mathbb{R}$ *with* $t \geqslant t_0$, $G(t, t_0)W^u(t_0) = W^u(t)$;

(vi) *for each* $t_0 \in \mathbb{R}$, $W^s(t_0)$ *is closed.*

Proof. The first assertion in (i) follows directly from the exponential dichotomy on $[T, +\infty)$. For the second one, let $x = u(t_0) \in W^u(t_0)$ with u as in Definition 3.1. Then $u(t_0) = G(t_0, t)u(t)$ and

$$\|(I - P(t_0))u(t_0)\| = \|G(t_0, t)(I - P(t))u(t)\| \leqslant N e^{-\beta(t_0 - t)} \|u\|_\infty$$

for all $t \leqslant t_0 \leqslant -T$. Letting $t \to -\infty$ we see that $x \in P(t_0)(X)$. The other inclusion is clear. The remaining assertions can be shown exactly as Proposition 3.2 in [**20**], now using Theorems 2.4 and 2.5. □

To study the operator $\mathcal{L}$, it is useful to introduce the realizations of the operator $u \mapsto u' - A(\cdot)u$ in spaces on half-lines; that is,

$$\begin{aligned} \mathcal{L}^+ : D(\mathcal{L}^+) = \mathcal{E}((T,+\infty)) \to L^p((T,+\infty);X); \\ (\mathcal{L}^+u)(t) = u'(t) - A(t)u(t) \quad \text{for } t > T. \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathcal{L}^- : D(\mathcal{L}^-) = \mathcal{E}((-\infty,T)) \to L^p((-\infty,T);X); \\ (\mathcal{L}^-u)(t) = u'(t) - A(t)u(t) \quad \text{for } t < T. \end{aligned} \tag{3.2}$$

Theorems 2.4 and 2.5 allow us to introduce right inverses R^+ on $L^p((T,+\infty);X)$ and R^- on $L^p((-\infty,T);X)$ for $\mathcal{L}^+$ and $\mathcal{L}^-$, respectively:

$$(R^+h)(t) = -\int_t^{+\infty} \widetilde{G}(t,s)P(s)h(s)\,ds + \int_T^t G(t,s)(I-P(s))h(s)\,ds, \qquad \text{for } t \geqslant T; \tag{3.3}$$

$$(R^-h)(t) = \begin{cases} \displaystyle\int_{-\infty}^t G(t,s)(I-P(s))h(s)\,ds + \int_{-T}^t \widetilde{G}(t,s)P(s)h(s)\,ds, & \text{for } t \leqslant -T, \\ \displaystyle\int_{-\infty}^{-T} G(t,s)(I-P(s))h(s)\,ds + \int_{-T}^t G(t,s)h(s)\,ds, & \text{for } -T \leqslant t \leqslant T. \end{cases} \tag{3.4}$$

PROPOSITION 3.3. *The following statements hold:*

(i) *R^+ is a bounded operator from $L^p((T,+\infty);X)$ to $D(\mathcal{L}^+)$, and we have $\mathcal{L}^+R^+h = h$ for each $h \in L^p((T,+\infty);X)$;*

(ii) *R^- is a bounded operator from $L^p((-\infty,T);X)$ to $D(\mathcal{L}^-)$, and we have $\mathcal{L}^-R^-h = h$ for each $h \in L^p((-\infty,T);X)$.*

Proof. Statement (i) is an immediate consequence of Theorem 2.4, since R^+h coincides with the solution u of (2.20) with $t_0 = T$, $x = -\int_T^{+\infty} \widetilde{G}(T,s)P(s)h(s)\,ds$, and $f = h$, given by formula (2.22).

Concerning statement (ii), let $h \in L^p((-\infty,T);X)$. By Theorem 2.5, the restriction of R^-h to $(-\infty,-T)$ belongs to $\mathcal{E}((-\infty,-T))$, its norm in this space is less than $C\|h\|_{L^p((-\infty,-T);X)}$, and $(R^-h)'(t) = A(t)R^-h(t) + h(t)$ for almost all $t < -T$. So Corollary 2.3 yields $(R^-h)(-T) \in (X, D(A(-T)))_{1-1/p,p}$ and

$$\|(R^-h)(-T)\|_{(X,D(A(-T)))_{1-1/p,p}} \leqslant C\|h\|_{L^p((-\infty,-T);X)}.$$

Theorem 2.2 thus implies that the restriction of R^-h to $[-T,T]$ is contained in $\mathcal{E}([-T,T])$, that its norm in $\mathcal{E}([-T,T])$ is bounded by

$$C(\|h\|_{L^p((-T,T);X)} + \|(R^-h)(-T)\|_{(X,D(A(-T)))_{1-1/p,p}}),$$

and that $(R^-h)'(t) = A(t)R^-h(t) + h(t)$ for almost all $t \in (-T,T)$. The assertion follows once the restrictions of R^-h to $(-\infty,-T]$ and to $[-T,T]$ have been patched together. □

The following trace lemma is taken from [**20**], where it was used in the C^α setting. The same construction works in the L^p setting.

LEMMA 3.4. *For every* $w_0 \in P(T)(X)$ *there exists* $u_0 \in D(\mathcal{L})$ *such that* $(R^+u_0)(T) = w_0$, $(R^-u_0)(T) = 0$, *and* $\|u_0\|_{D(\mathcal{L})} \leqslant K\|w_0\|$, *where* $K \geqslant 0$ *is a constant independent of* w_0.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that

$$\|\varphi\|_\infty \leqslant 1, \qquad \varphi(t) = 0 \quad \text{for } t \leqslant T, \qquad \int_T^{+\infty} \varphi(s)\, ds = -1,$$

and set

$$u_0(t) := \varphi(t)G(t,T)w_0 \quad \text{for } t \geqslant T, \qquad u_0(t) := 0 \quad \text{for } t \leqslant T.$$

Then $u_0 \in D(\mathcal{L})$ and there exists a constant $K \geqslant 0$ such that $\|u_0\|_{D(\mathcal{L})} \leqslant K\|w_0\|$. Finally, $R^+u_0(T) = w_0$ and $R^-u_0(T) = 0$. □

At this point, we have all the tools to extend the results of [**20**] to our situation.

PROPOSITION 3.5. (i) *We have*

$$\operatorname{Ker}\mathcal{L}^+ = \{u : u(t) = G(t,T)x \quad \text{for } t \geqslant T;\ x \in (I - P(T))(X) \cap (X, D(A(T)))_{1-1/p,p}\}.$$

(ii) *We have*

$$\operatorname{Ker}\mathcal{L}^- = \{u : u(t) = G(t,-T)x \text{ for } -T \leqslant t \leqslant T;\ u(t) = \widetilde{G}(t,-T)x \text{ for } t \leqslant -T;\ x \in P(-T)(X)\}.$$

(iii) *The kernel of* $\mathcal{L}$ *is the set of the functions* $u : \mathbb{R} \mapsto X$ *that may be represented as*

$$u(t) = \begin{cases} \widetilde{G}(t,-T)x & \text{for } t \leqslant -T, \\ u(t) = G(t,-T)x & \text{for } t \geqslant -T, \end{cases}$$

where $x \in P(-T)(X)$ *is such that* $G(T,-T)x \in (I - P(T))(X)$. *Consequently, it is isomorphic to* $\{x \in P(-T)(X) : G(T,-T)x \in (I - P(T))(X)\} := Z$, *with isomorphism* $u \mapsto u(-T)$.

(iv) $\operatorname{Range}\mathcal{L} = \{h \in L^p(\mathbb{R};X) : R^+h(T) - R^-h(T) \in W^s(T) + W^u(T)\}$.

(v) $\overline{\operatorname{Range}\mathcal{L}} = \{h \in L^p(\mathbb{R};X) : R^+h(T) - R^-h(T) \in \overline{W^s(T) + W^u(T)}\}$.

Proof. Assertions (i) and (ii) are consequences of Lemma 3.2 and Theorems 2.4 and 2.5.

Part (iii) follows from (i) and (ii): the restrictions to $[T,+\infty)$ and to $(-\infty,T]$ of any $u \in \operatorname{Ker}\mathcal{L}$ belong to $\operatorname{Ker}\mathcal{L}^+$ and to $\operatorname{Ker}\mathcal{L}^-$, respectively. Therefore $u(T) = G(T,-T)u(-T) \in (I - P(T))(X)$ and $u(-T) \in P(-T)(X)$, that is, $u(-T) \in Z$, and u has the asserted representation. Conversely, each $x \in Z$ allows one to define a unique element $u \in \operatorname{Ker}\mathcal{L}$ with $u(-T) = x$ as in the claim.

To prove (iv), let $h = \mathcal{L}u$ for some $u \in D(\mathcal{L})$. Restricting this equation to half-lines, we deduce from Proposition 3.3 that

$$u = \begin{cases} R^+h + v_+ & \text{on } [T, +\infty), \\ R^-h + v_- & \text{on } (-\infty, T], \end{cases}$$

for some $v_\pm$ in the kernel of $\mathcal{L}^\pm$. Thus

$$(R^+h)(T) - (R^-h)(T) = v_-(T) - v_+(T) \in W^u(T) + W^s(T)$$

by (i) and (ii). Conversely, let $h \in L^p(\mathbb{R}; X)$ with

$$(R^+h)(T) - (R^-h)(T) = x_s + x_u \in W^s(T) + W^u(T).$$

Corollary 2.3 yields $(R^+h)(T), (R^-h)(T) \in (X, D(A(T)))_{1-1/p,p}$. Since $x_u \in D(A(T))$ by Lemma 3.2(iv), we have $x_s \in (X, D(A(T)))_{1-1/p,p}$. We now define

$$u(t) = \begin{cases} -G(t,T)x_s + (R^+h)(t) & \text{for } t \geqslant T, \\ \widetilde{u}(t) + (R^-h)(t) & \text{for } t \leqslant T, \end{cases}$$

where $\widetilde{u}(T) = x_u$ for a function $\widetilde{u}$ as in Definition 3.1. It is easy to see that $u \in D(\mathcal{L})$ and $\mathcal{L}u = h$. Assertion (v) follows from (iv) and Lemma 3.4 as in Proposition 3.7 of [**20**]. □

We recall the definitions of semi-Fredholm and Fredholm operators, and of semi-Fredholm and Fredholm couples of subspaces.

Definition 3.6. Let E and F be Banach spaces. We say that a closed linear operator $A : D(A) \subseteq E \to F$ is a *semi-Fredholm operator* if Range A is closed and if at least one of the dimensions dim Ker A and codim Range A is finite. If both dimensions are finite, we say that A is a *Fredholm operator*. The index of a semi-Fredholm operator A is defined by

$$\operatorname{ind} A := \dim \operatorname{Ker} A - \operatorname{codim} \operatorname{Range} A.$$

Definition 3.7. Let V and W be subspaces of a Banach space E. We say that (V, W) is a *semi-Fredholm couple* if $V + W$ is closed and if at least one of the dimensions $\dim(V \cap W)$ and $\operatorname{codim}(V + W)$ is finite. If both dimensions are finite, we say that (V, W) is a *Fredholm couple*. The index of a semi-Fredholm couple (V, W) is defined by

$$\operatorname{ind}(V, W) := \dim(V \cap W) - \operatorname{codim}(V + W).$$

Now we are able to describe the properties of $\mathcal{L}$ in terms of properties of the subspaces $W^s(T)$ and $W^u(T)$, arguing exactly as in Theorem 3.10 in [**20**] and its corollaries and using the above results.

Theorem 3.8. *Assume that* (2.1), (2.2), (2.7), (2.8) *and* (2.9) *are satisfied. Then the following assertions hold.*

(i) *Range $\mathcal{L}$ is closed if and only if $W^s(T) + W^u(T)$ is closed.*

(ii) *The operator $\mathcal{L}$ is surjective if and only if $W^s(T) + W^u(T) = X$.*

(iii) *We have* $\dim(\mathrm{Ker}\,\mathcal{L}) = \dim(W^s(T) \cap W^u(T)) + \dim(\mathrm{Ker}\,G(T,-T)_{|P(-T)(X)})$. *Consequently, if* $\mathcal{L}$ *is one-to-one, then*

$$W^s(T) \cap W^u(T) = \mathrm{Ker}\,G(T,-T)_{|P(-T)(X)} = \{0\}.$$

If $G(T,-T)_{|P(-T)(X)}$ *is one-to-one and* $W^s(T) \cap W^u(T) = \{0\}$*, then* $\mathcal{L}$ *is one-to-one.*

(iv) *If* $\mathcal{L}$ *is invertible, then* $W^s(T) \oplus W^u(T) = X$*. If* $G(T,-T)_{|P(-T)(X)}$ *is one-to-one and* $W^s(T) \oplus W^u(T) = X$*, then* $\mathcal{L}$ *is invertible.*

(v) *If* $\mathcal{L}$ *is a semi-Fredholm operator, then* $(W^s(T), W^u(T))$ *is a semi-Fredholm couple and*

$$\mathrm{codim}(W^s(T) + W^u(T)) = \mathrm{codim}(\mathrm{Range}\,\mathcal{L}), \quad \mathrm{ind}(W^s(T), W^u(T)) \leqslant \mathrm{ind}\,\mathcal{L}. \tag{3.5}$$

If, in addition, the kernel of $G(T,-T)_{|P(-T)(X)}$ *is finite dimensional, then*

$$\mathrm{ind}(W^s(T), W^u(T)) = \mathrm{ind}\,\mathcal{L} - \dim \mathrm{Ker}\,G(T,-T)_{|P(-T)(X)}. \tag{3.6}$$

Conversely: if $(W^s(T), W^u(T))$ *is a semi-Fredholm couple and the kernel of* $G(T,-T)_{|P(-T)(X)}$ *is finite dimensional, then* $\mathcal{L}$ *is a semi-Fredholm operator and* (3.6) *holds. If* $(W^s(T), W^u(T))$ *is a Fredholm couple, then* $\mathcal{L}$ *is semi-Fredholm and* (3.5) *holds; if in addition the kernel of* $G(T,-T)_{|P(-T)(X)}$ *is finite dimensional, then* $\mathcal{L}$ *is a Fredholm operator and* (3.6) *holds.*

Concerning the kernel of $G(T,-T)_{|P(-T)(X)}$, we remark that, in general, a para˙bolic evolution operator $G(t,s)$ is not one-to-one. See for example [**29**]. Sufficient conditions for backward uniqueness are known: see [**7**, **12**] for abstract evolution operators in Hilbert spaces, and [**41**] for evolution operators associated to specific parabolic partial differential operators. But a satisfactory description of the kernel of $G(t,s)$ (or of some restriction of $G(t,s)$) under general assumptions does not exist in the literature, and it constitutes an important open problem.

As a consequence of Theorem 3.8, we recover a characterization of the Fredholm property of $\mathcal{L}$ given in [**26**]. We further give simple sufficient conditions for $\mathcal{L}$ to be a Fredholm operator; cf. [**20**].

Proposition 3.9. *Under the assumptions of Theorem* 3.8*, define the operator*

$$\mathcal{N} : P(-T)(X) \mapsto P(T)(X), \quad \mathcal{N}x := P(T)G(T,-T)x.$$

Then $\mathcal{L}$ *is a semi-Fredholm (respectively, Fredholm) operator if and only if* $\mathcal{N}$ *is a semi-Fredholm (respectively, Fredholm) operator. If this is the case, we have* $\dim \mathrm{Ker}\,\mathcal{L} = \dim \mathrm{Ker}\,\mathcal{N}$ *and* $\mathrm{codim}\,\mathrm{Range}\,\mathcal{L} = \mathrm{codim}\,\mathrm{Range}\,\mathcal{N}$*, so that* $\mathcal{L}$ *and* $\mathcal{N}$ *have the same index.*

Proof. Statement (iii) of Proposition 3.5 implies that the kernel of $\mathcal{L}$ is isomorphic to the kernel of $\mathcal{N}$, an isomorphism being $u \mapsto u(-T)$.

Now we prove that the range of $\mathcal{L}$ is closed if and only if the range of $\mathcal{N}$ is closed. By Theorem 3.8, it is enough to prove that $W^s(T) + W^u(T)$ is closed if and only if the range of $\mathcal{N}$ is closed.

Let $x_n = P(T)G(T,-T)y_n$, with $y_n \in P(-T)(X)$, converge to $x \in P(T)(X)$ as $n \to +\infty$. Then

$$x_n = G(T,-T)y_n - (I - P(T))G(T,-T)y_n,$$

where $G(T,-T)y_n \in W^u(T)$ by Lemma 3.2(i)+(v), and $(I - P(T))G(T,-T)y_n \in W^s(T)$ by Lemma 3.2(i). Therefore $x_n \in W^s(T) + W^u(T)$. If $W^s(T) + W^u(T)$ is closed, then $x \in W^s(T) + W^u(T)$, and again by Lemma 3.2 we obtain $x = (I - P(T))z + G(T,-T)P(-T)y$ for some $z, y \in X$. From $x \in P(T)(X)$ we deduce that $x = P(T)G(T,-T)P(-T)y$, so that $x \in \operatorname{Range} \mathcal{N}$.

The converse is similar: if $x_n \in W^s(T) + W^u(T)$ converges to $x \in X$, by Lemma 3.2, $x_n = (I - P(T))z_n + G(T,-T)P(-T)y_n$ for some $z_n, y_n \in X$, and $P(T)x_n = P(T)G(T,-T)P(-T)y_n \in \operatorname{Range} \mathcal{N}$ converges to $P(T)x$ as $n \to +\infty$. If the range of $\mathcal{N}$ is closed, then $P(T)x = P(T)G(T,-T)w$ for some $w \in P(-T)(X)$, so that $x = (I - P(T))(x - G(T,-T)w) + G(T,-T)w \in W^s(T) + W^u(T)$ by Lemma 3.2.

Similar arguments show that the mapping $[x] \mapsto [P(T)x]$, from the quotient space $X/(W^s(T) + W^u(T))$ to the quotient space $P(T)(X)/\mathcal{N}(P(-T)(X))$, is an isomorphism. □

Corollary 3.10. *If $\dim P_{+\infty}(X) < \infty$ and $\dim P_{-\infty}(X) < \infty$ then $\mathcal{L}$ is Fredholm with index*

$$\operatorname{ind} \mathcal{L} = \dim P_{-\infty}(X) - \dim P_{+\infty}(X).$$

Corollary 3.11. *If the embeddings $D(A_{+\infty}) \hookrightarrow X$ and $D(A_{-\infty}) \hookrightarrow X$ are compact, then $\mathcal{L}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{L} = \dim P_{-\infty}(X) - \dim P_{+\infty}(X).$$

Proposition 3.5 provides us with a convenient description of the kernel of $\mathcal{L}$. For many applications, for example, in the proof of Proposition 4.4 below, it is important to determine the range of $\mathcal{L}$ in a similar way via duality. This task is simplified by the fact that $L^p(\mathbb{R}; X)$ is reflexive and has the dual $L^q(\mathbb{R}, X^*)$ with $q = p/(p-1)$. (Recall that (2.9) implies that X is reflexive.) If $\mathcal{L}$ has closed range, then

$$\operatorname{Range} \mathcal{L} = (\operatorname{Ker} \mathcal{L}^*)^{\perp} := \left\{ h \in L^p(\mathbb{R}; X) : \int_{\mathbb{R}} hv \, dx = 0 \ \forall v \in \operatorname{Ker} \mathcal{L}^* \right\} \tag{3.7}$$

by formulas (III.5.10) and (III.1.24) in [**24**]. In order to determine $\mathcal{L}^*$, we introduce the so-called evolution semigroup

$$(T(t)f)(s) = G(s, s-t)f(s-t), \quad \text{where } s \in \mathbb{R},\ f \in L^p(\mathbb{R}; X),\ t \geqslant 0,$$

on $L^p(\mathbb{R}; X)$; cf. [**13**, **26**, **37**, **38**]. By (2.5), there is a number $\gamma \geqslant 0$ such that $e^{-\gamma(t-s)}G(t,s)$ is exponentially stable. Then it is easy to verify that $T(\cdot)$ is a C_0-semigroup.

Proposition 3.12. *Let $\mathcal{G}$ be the infinitesimal generator of the semigroup $T(t)$ defined above. Then $\mathcal{G} = -\mathcal{L}$.*

Proof. The resolvent $(\gamma - \mathcal{G})^{-1}$ is given by

$$((\gamma - \mathcal{G})^{-1} f)(t) = \int_0^{+\infty} e^{-\gamma r} (T(r)f)(t)\, dr = \int_{-\infty}^{t} e^{-\gamma(t-s)} G(t,s) f(s)\, ds$$

for $f \in L^p(\mathbb{R}, X)$ and a.e. $t \in \mathbb{R}$. Combining this equality with Corollary 2.6, we see that the statement follows. □

As a result, $-\mathcal{L}^*$ generates the adjoint C_0-semigroup $T(\,\cdot\,)^*$ on $L^q(\mathbb{R}, X^*)$. Hence, the kernel of $\mathcal{L}^*$ is the space of functions $g \in L^q(\mathbb{R}, X^*)$ such that $T(\,\cdot\,)^* g = g$ for each $t > 0$. Since

$$(T(t)^* g)(s) = G(s+t, s)^* g(s+t) \quad \text{for } s \in \mathbb{R},\ g \in L^q(\mathbb{R}, X^*),\ t \geqslant 0,$$

we deduce that

$$\operatorname{Ker} \mathcal{L}^* = \{v \in L^q(\mathbb{R}, X^*) : v(s) = G(t,s)^* v(t)\ \forall\, t \geqslant s\}. \tag{3.8}$$

So we have shown the following result.

Proposition 3.13. *Assume that* (2.1), (2.2) *and* (2.9) *hold and that* $\mathcal{L}$ *has a closed range* (*for example, if* $\mathcal{L}$ *is semi-Fredholm*). *Then the range of* $\mathcal{L}$ *is equal to the space*

$$\left\{ h \in L^p(\mathbb{R}; X) : \int_{\mathbb{R}} hv\, dx = 0\ \forall\, v \in L^q(\mathbb{R}, X^*) \text{ with } v(s) = G(t,s)^* v(t)\ \forall\, t \geqslant s \right\}.$$

One can see that the so-called 'complete adjoint trajectories' v (that is, the functions satisfying $v(s) = G(t,s)^* v(t)$ for $t \geqslant s$) solve the dual evolution equation

$$-v'(s) = A(s)^* v(s) \quad \text{for } s \in \mathbb{R}, \tag{3.9}$$

in a weak sense. The function v is a classical solution of (3.9) if, in addition, the adjoint operators $A(t)^*$ satisfy the Acquistapace–Terreni conditions (2.1) and (2.2); cf. [**3**, Proposition 2.9].

4. *Perturbations*

Let $A(\,\cdot\,)$ satisfy assumptions (2.1), (2.2), (2.9), and let $B(t) : D(B(t)) \subset X \to X$ be a family of operators such that $D(A(t)) \subset D(B(t))$ and

$$\|B(t)x\| \leqslant a\|A(t)x\| + b\|x\|, \quad \text{with } x \in D(A(t)) \text{ and } t \in \mathbb{R}, \tag{4.1}$$

for some constants $a, b \geqslant 0$. We introduce the operator

$$\widetilde{\mathcal{L}} : D(\mathcal{L}) \to L^p(\mathbb{R}, X); \qquad (\widetilde{\mathcal{L}} u)(t) := u'(t) - A(t)u(t) - B(t)u(t) \quad \text{for } t \in \mathbb{R}.$$

Suppose that $\mathcal{L}$ is a Fredholm operator. Then the question arises under which assumptions on $B(t)$ the operator $\widetilde{\mathcal{L}}$ is Fredholm as well. We give three answers, one for small perturbations $B(t)$ and two more in the case of relatively compact perturbations.

4.1. *Small $A(t)$-bounded perturbations*

We first provide conditions on $B(t)$ such that the operators $A(t)+B(t)$ inherit (2.1), (2.2) and (2.9) from $A(t)$.

LEMMA 4.1. *Assume that* (2.1), (2.2) *and* (4.1) *hold with* $a<(1+K)^{-1}$. *Let the mapping* $\mathbb{R}\ni t\mapsto B(t)R(\omega,A(t))\in L(X)$ *be uniformly Hölder continuous. Then the operators* $A(t)+B(t)$ *with domain* $D(A(t))$, *where* $t\in\mathbb{R}$, *satisfy* (2.1) *and* (2.2) (*possibly with different constants*). *If, in addition,* (2.9) *holds and* $a<((1+K)(1+R))^{-1}$ (*with* R *from* (2.10)), *then* $A(t)+B(t)$ *satisfies* (2.9) (*possibly with different constants*).

Proof. Fix $\eta\in(a(1+K),1)$. It is well known that for sufficiently large $\gamma\geqslant\omega$ and $\widetilde{A}(t)=A(t)-\gamma I$ we have

$$\|B(t)R(\lambda,\widetilde{A}(t))\|\leqslant\eta \quad\text{and}\quad R(\lambda,\widetilde{A}(t)+B(t))=R(\lambda,\widetilde{A}(t))\,[I-B(t)R(\lambda,\widetilde{A}(t))]^{-1}$$

for $\lambda\in\Sigma_{0,\theta}$ and $t\in\mathbb{R}$. Thus (2.1) holds for $A(t)+B(t)$. Observe that

$$\begin{aligned}(I-R(\lambda,\widetilde{A}(t))B(t))^{-1}&=I+R(\lambda,\widetilde{A}(t))[I-B(t)R(\lambda,\widetilde{A}(t))]^{-1}B(t)\in L(D(A(t))),\\ R(\lambda,\widetilde{A}(t)+B(t))&=\big\{I+R(\lambda,\widetilde{A}(t))[I-B(t)R(\lambda,\widetilde{A}(t))]^{-1}B(t)\big\}R(\lambda,\widetilde{A}(t)).\end{aligned}$$

These equalities yield

$$\begin{aligned}&(\widetilde{A}(t)+B(t))R(\lambda,\widetilde{A}(t)+B(t))\,[(\widetilde{A}(t)+B(t))^{-1}-(\widetilde{A}(s)+B(s))^{-1}]\\ &\quad=(\widetilde{A}(t)+B(t))\big\{I+R(\lambda,\widetilde{A}(t))[I-B(t)R(\lambda,\widetilde{A}(t))]^{-1}B(t)\big\}\widetilde{A}(t)^{-1}\\ &\qquad\cdot\widetilde{A}(t)R(\lambda,\widetilde{A}(t))\,\big\{\widetilde{A}(t)^{-1}[(I+B(t)\widetilde{A}(t)^{-1})^{-1}-(I+B(s)\widetilde{A}(s)^{-1})^{-1}]\\ &\qquad\qquad+(\widetilde{A}(t)^{-1}-\widetilde{A}(s)^{-1})(I+B(s)\widetilde{A}(s)^{-1})^{-1}\big\}\end{aligned}$$

for $\lambda\in\Sigma_{0,\theta}$ and $t\neq s\in\mathbb{R}$. Since $\mathbb{R}\ni t\mapsto(I+B(t)\widetilde{A}(t)^{-1})^{-1}\in L(X)$ is uniformly bounded and Hölder continuous, we obtain (2.2) for $A(t)+B(t)$. The last assertion is a direct consequence of [**17**, Proposition 4.3] or [**25**, Corollary 6.8]; see also [**42**, Remark 4.5]. (Possibly one has to increase γ.) □

THEOREM 4.2. *Assume that* (2.1), (2.2), (2.9) *and* (4.1) *hold with* $a<((1+K)\times(1+R))^{-1}$ (*where* R *is given by* (2.10)) *and that* $\mathbb{R}\ni t\mapsto B(t)R(\omega,A(t))\in L(X)$ *is uniformly Hölder continuous. Suppose that* $\mathcal{L}$ *is a Fredholm operator. If* a *and* b *from* (4.1) *are small enough, then* $\widetilde{\mathcal{L}}$ *is a Fredholm operator with the same index as* $\mathcal{L}$.

Proof. Combining estimate (4.1) with Corollary 2.6, we obtain

$$\begin{aligned}\|B(\cdot)u(\cdot)\|_{L^p(\mathbb{R};X)}&\leqslant a\|A(\cdot)u(\cdot)\|_{L^p(\mathbb{R};X)}+b\|u\|_{L^p(\mathbb{R};X)}\\ &\leqslant aC_3\|\mathcal{L}u\|_{L^p(\mathbb{R};X)}+(aC_3\gamma+b)\|u\|_{L^p(\mathbb{R};X)}.\end{aligned}$$

Now Theorem IV.5.22 of [**24**] shows that there exists $\kappa>0$ such that if

$$b+aC_3\gamma+aC_3\kappa<\kappa,$$

then $\widetilde{\mathcal{L}}$ is a Fredholm operator with the same index. □

4.2. Relatively compact perturbations

Again we start with a perturbation result for our basic assumptions.

LEMMA 4.3. *Assume that $A(t)$, for $t \in \mathbb{R}$, are densely defined and satisfy (2.1), (2.2) and (2.7). Suppose that $B(t)$, for $t \in \mathbb{R}$, fulfill (4.1) and that $B(t)R(\omega, A(t)) \in L(X)$ are compact and uniformly Hölder continuous for $t \in \mathbb{R}$ and converge in $L(X)$ to operators $B_{\pm\infty}R(\omega, A_{\pm\infty})$ as $t \to \pm\infty$. Then the operators $A(t) + B(t)$ with domain $D(A(t))$, for $t \in \mathbb{R}$, satisfy (2.1) and (2.2), possibly with different constants. If also (2.9) holds, then $A(t) + B(t)$ satisfy (2.9), possibly with different constants.*

Proof. Replacing $A(t)$ by $A(t) - \omega I$, we may suppose that $\omega = 0$. Let $\eta \in (0, \frac{1}{2}]$ and set $\varepsilon = \eta(3K + 5)^{-1}$. Let $t \in \mathbb{R}$, and $x \in D(A(t))$ with $\|x\| + \|A(t)x\| \leqslant 1$. Then we have

$$\|R(\lambda, A(t))B(t)x\| \leqslant \frac{K \max\{a, b\}}{|\lambda|} \leqslant \varepsilon, \tag{4.2}$$

for $\lambda \in \Sigma_{\gamma,\theta}$ provided that γ is sufficiently large, say $\gamma \geqslant \gamma_1(\eta) > 0$. By assumption there exist $-\infty = t_1 < t_2 < \ldots < t_{n-1} < t_n = +\infty$ such that for each $t \in \mathbb{R}$ we find t_k with

$$\|B(t)A(t)^{-1} - B(t_k)A(t_k)^{-1}\| \leqslant \varepsilon.$$

Since the operators $B(t_i)A(t_i)^{-1}$, for $i = 1, \ldots, n$, are compact, there exist vectors $y_1, \ldots, y_m \in X$ such that for t and x as above there is an index $j \in \{1, \ldots, m\}$ with

$$\|B(t_k)A(t_k)^{-1}A(t)x - y_j\| \leqslant \varepsilon.$$

Further observe that $e^{sA(\tau)}y_j \to y_j$ as $s \to 0$ uniformly in $\tau \in \mathbb{R}$ due to our assumptions and the Trotter–Kato theorem; [**24**, Theorem IX.2.16]. Therefore

$$\|y_j - y_{jr}\| \leqslant \varepsilon, \quad \text{where } y_{jr} := \frac{1}{r}\int_0^r e^{sA(t)}y_j\, ds,$$

for some $r \in (0, 1]$ not depending on t and j. Combining these facts, we deduce that

$$\begin{aligned}
\|A(t)R(\lambda, A(t))B(t)x\| &\leqslant \|A(t)R(\lambda, A(t))(B(t)A(t)^{-1} - B(t_k)A(t_k)^{-1})A(t)x\| \\
&\quad + \|A(t)R(\lambda, A(t))(B(t_k)A(t_k)^{-1}A(t)x - y_j)\| \\
&\quad + \|A(t)R(\lambda, A(t))(y_j - y_{j,r})\| \\
&\quad + \|R(\lambda, A(t))(e^{rA(t)}y_j - y_j)/r\| \\
&\leqslant 3(1 + K)\varepsilon + c_\eta\, |\lambda|^{-1}
\end{aligned}$$

for a constant c_η not depending on x, t, λ as above. Taking a sufficiently large $\gamma = \gamma(\eta) \geqslant \gamma_1(\eta)$, we thus obtain

$$\|A(t)R(\lambda, A(t))B(t)x\| \leqslant (3K + 4)\varepsilon \tag{4.3}$$

for all $t \in \mathbb{R}$ and $x \in D(A(t))$ with $\|x\| + \|A(t)x\| \leqslant 1$, and $\lambda \in \Sigma_{\gamma,\theta}$. We set

$\widetilde{A}(t) = A(t) - \gamma I$. Combining (4.2) and (4.3), we conclude that

$$\|R(\lambda, \widetilde{A}(t))B(t)\|_{L(D(A(t)))} \leqslant \eta \leqslant \tfrac{1}{2} \tag{4.4}$$

for $t \in \mathbb{R}$ and $\lambda \in \Sigma_{0,\theta}$, where $D(A(t))$ is endowed with the graph norm. In particular, there exist the resolvent operators

$$R(\lambda, \widetilde{A}(t) + B(t)) = \sum_{n=0}^{\infty} [R(\lambda, \widetilde{A}(t))B(t)]^n R(\lambda, \widetilde{A}(t)), \tag{4.5}$$

and (2.1) holds for $A(t) + B(t)$. We further have

$$C(t) := (I + B(t)\widetilde{A}(t)^{-1})^{-1} = I - B(t)[I + \widetilde{A}(t)^{-1}B(t)]^{-1}\widetilde{A}(t)^{-1}.$$

The operators $C(t) \in L(X)$ are uniformly bounded by (4.4), and due to

$$C(t) - C(s) = C(t)[B(s)\widetilde{A}(s)^{-1} - B(t)\widetilde{A}(t)^{-1}]C(s)$$

the map $t \mapsto C(t) \in L(X)$ is globally Hölder continuous. Now one can show (2.2) as in the proof of Lemma 4.1.

To establish the second assertion, take $\eta < (1 + R)^{-1}$ and increase γ if necessary. Based on (4.5), the second assertion can then be established as [**17**, Proposition 4.3] or [**25**, Corollary 6.8]. □

We first consider a path of operators of the type $A(t) = A + B(t)$, where A is a fixed $\mathcal{R}$-sectorial operator, $D(A(t)) = D(A)$ for $t \in \mathbb{R}$, and $B(t) : D(A) \mapsto X$ is compact for every $t \in \mathbb{R}$ satisfying the assumptions of Lemma 4.3. Moreover the operators $A + B_{+\infty}$ and $A + B_{-\infty}$ shall be hyperbolic. Then the operators $A(t)$ fulfill (2.1), (2.2) and (2.9) by Lemma 4.3, and thus they generate an evolution operator $G(t, s)$. We further introduce the stepwise constant path A_0 defined by

$$A_0(t) := \begin{cases} A + B_{-\infty} & \text{for } t < 0, \\ A + B_{+\infty} & \text{for } t \geqslant 0. \end{cases}$$

Clearly, A_0 generates the evolution operator

$$G_0(t, s) = \begin{cases} e^{(t-s)(A+B_{+\infty})} & \text{for } t \geqslant s \geqslant 0, \\ e^{t(A+B_{+\infty})}e^{-s(A+B_{-\infty})} & \text{for } t \geqslant 0 > s, \\ e^{(t-s)(A+B_{-\infty})} & \text{for } 0 > t \geqslant s. \end{cases}$$

Further, we have exponential dichotomies in $[0, +\infty)$ and in $(-\infty, 0]$ with the constant projections

$$P_{+\infty} = \frac{1}{2\pi i}\int_\gamma R(\lambda, A + B_{+\infty})\, d\lambda \quad \text{and} \quad P_{-\infty} = \frac{1}{2\pi i}\int_\gamma R(\lambda, A + B_{-\infty})\, d\lambda,$$

respectively. Here γ is any regular curve lying in $\{\mathrm{Re}\ \lambda > 0\}$, surrounding both $\sigma(A + B_{+\infty}) \cap \{\mathrm{Re}\ \lambda > 0\}$ and $\sigma(A + B_{-\infty}) \cap \{\mathrm{Re}\ \lambda > 0\}$ and having index 1 with respect to both sets. The stable and unstable manifolds W_0^s and W_0^u corresponding to A_0 at $t = 0$ are given by

$$W_0^s = (I - P_{+\infty})(X) \quad \text{and} \quad W_0^u = P_{-\infty}(X).$$

As in the general case, we define the operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 : D(\mathcal{L}_0) = L^p(\mathbb{R}; D(A)) \cap W^{1,p}(\mathbb{R}; X) \mapsto L^p(\mathbb{R}; X);$$
$$(\mathcal{L}_0 u)(t) = u'(t) - A_0(t)u(t) \quad \text{for } t \in \mathbb{R}.$$

The jump of A_0 at $t = 0$ affects only local regularity properties when crossing $t = 0$. All results of §§2 and 3 concerning the operators $\mathcal{L}_0^+$ and $\mathcal{L}_0^-$, with $T = 0$, still hold, as well as their consequences. In particular, if $P_{+\infty}(X)$ and $P_{-\infty}(X)$ are finite dimensional, Corollary 3.10 shows that $\mathcal{L}_0$ is a Fredholm operator with index equal to $\dim P_{-\infty}(X) - \dim P_{+\infty}(X)$. If one of the subspaces $P_{+\infty}(X)$ and $P_{-\infty}(X)$ is not finite dimensional, then $\mathcal{L}_0$ is a Fredholm operator as well; but the proof is less immediate.

Proposition 4.4. *Assume that A satisfies* (2.1) *and* (2.9), *that the mappings $B_{\pm\infty} : D(A) \to X$ are compact, and that $\sigma(A + B_{\pm\infty}) \cap i\mathbb{R} = \emptyset$. Then $\mathcal{L}_0$ is a Fredholm operator with index*

$$\begin{aligned}\operatorname{ind} \mathcal{L}_0 &= \operatorname{ind}((I - P_{+\infty})(X), P_{-\infty}(X)) \\ &= \dim((I - P_{+\infty})(X) \cap P_{-\infty}(X)) - \dim([(I - P_{+\infty})(X)]^\perp \cap [P_{-\infty}(X)]^\perp).\end{aligned}$$

Proof. Statement (v) of Theorem 3.8, applied to A_0, implies that $\mathcal{L}_0$ is a Fredholm operator if and only if $((I - P_{+\infty})(X), P_{-\infty}(X))$ is a Fredholm couple, and then

$$\begin{aligned}\operatorname{ind} \mathcal{L}_0 &= \operatorname{ind}((I - P_{+\infty})(X), P_{-\infty}(X)), \\ \dim \operatorname{Ker} \mathcal{L}_0 &= \dim((I - P_{+\infty})(X) \cap P_{-\infty}(X)).\end{aligned}$$

To prove that the couple is Fredholm, we observe that

$$\begin{aligned}P_{+\infty} - P_{-\infty} &= \frac{1}{2\pi i}\int_\gamma (R(\lambda, A + B_{+\infty}) - R(\lambda, A + B_{-\infty}))\, d\lambda \\ &= \frac{1}{2\pi i}\int_\gamma R(\lambda, A + B_{+\infty})(B_{+\infty} - B_{-\infty})R(\lambda, A + B_{-\infty})\, d\lambda\end{aligned}$$

is a compact operator, because $B_{+\infty} - B_{-\infty} : D(A) \to X$ is compact.

Thus the range of $I - P_{+\infty} + P_{-\infty}$ is closed and has a finite-dimensional complement. As a result, the larger set $(I - P_{+\infty})(X) + P_{-\infty}(X)$ is closed and has a finite codimension, too. In addition, the space $P_{-\infty}(X) \cap (I - P_{+\infty})(X)$ is finite dimensional since it is a subspace of the kernel of $I - (I - P_{+\infty})P_{-\infty}$, and $(I - P_{+\infty})P_{-\infty} = (P_{-\infty} - P_{+\infty})P_{-\infty}$ is compact.

As a consequence, $\mathcal{L}_0$ is a Fredholm operator, and it remains to show that

$$\operatorname{codim} \operatorname{Range} \mathcal{L}_0 = \dim([(I - P_{+\infty})(X)]^\perp \cap [P_{-\infty}(X)]^\perp). \tag{4.6}$$

To this purpose we recall that $\operatorname{codim} \operatorname{Range} \mathcal{L}_0 = \dim \operatorname{Ker} \mathcal{L}_0^*$ by [**24**, Theorem IV.5.13]. Due to (3.8), a function $v : \mathbb{R} \to X^*$ belongs to $\operatorname{Ker} \mathcal{L}_0^*$ if and only if $v \in L^q(\mathbb{R}, X^*)$ and $v(s) = G_0(t, s)^* v(t)$ for all $t \geqslant s$. Observe that $G_0(t, s)^*$ has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ by duality. Hence,

$$\begin{aligned}\|v(s)\| &= \|e^{-s(A+B_{-\infty})^*}v(0)\| \\ &\geqslant \|e^{-s(A+B_{-\infty})^*}P^*_{-\infty}v(0)\| - \|e^{-s(A+B_{-\infty})^*}(I-P^*_{-\infty})v(0)\| \\ &\geqslant N^{-1}e^{-\beta s}\|P^*_{-\infty}v(0)\| - Ne^{\beta s}\|(I-P^*_{-\infty})v(0)\| \quad \text{for } s\leqslant 0,\end{aligned}$$

and

$$\|(I-P^*_{+\infty})v(0)\| = \|e^{t(A+B_{+\infty})^*}(I-P^*_{+\infty})v(t)\| \leqslant Ne^{-\beta t}\|v(t)\| \quad \text{for } t\geqslant 0,$$

for $v\in \operatorname{Ker}\mathcal{L}_0^*$. Since $v\in L^q(\mathbb{R},X^*)$, we obtain $v(0)\in P^*_{+\infty}(X^*)\cap(I-P^*_{-\infty})(X^*)$. Then it is easy to see that the mapping

$$\Phi : \operatorname{Ker}\mathcal{L}_0^* \to P^*_{+\infty}(X^*)\cap(I-P^*_{-\infty})(X^*); \quad v \mapsto v(0),$$

is an isomorphism. Observe that

$$P^*_{+\infty}(X^*) = [(I-P_{+\infty})(X)]^\perp \quad \text{and} \quad (I-P^*_{-\infty})(X^*) = [P_{-\infty}(X)]^\perp.$$

Thus we have shown (4.6). □

Observe that if X is a Hilbert space and $P_{\pm\infty}$ are self-adjoint, then the last argument of the above proof shows that

$$\operatorname{ind}\mathcal{L}_0 = \dim((I-P_{+\infty})(X)\cap P_{-\infty}(X)) - \dim(P_{+\infty}(X)\cap(I-P_{-\infty})(X)).$$

As a second step, we write $A(t)=A_0(t)+\widetilde{B}(t)$, with

$$\widetilde{B}(t) = \begin{cases} B(t)-B_{-\infty} & \text{for } t<0, \\ B(t)-B_{+\infty} & \text{for } t>0. \end{cases}$$

Thus the perturbation $\widetilde{B}(t)$ is not only compact for each t, but it tends to 0 as $t\to\pm\infty$. Unfortunately, this is not enough to guarantee that the induced perturbation $D(\mathcal{L}_0) \mapsto L^p(\mathbb{R},X)$, $u \mapsto \widetilde{B}(\cdot)u(\cdot)$, is relatively compact, and hence we cannot directly deduce that $\mathcal{L}$ is a Fredholm operator because it is a compact perturbation of a Fredholm operator. Note that $u \mapsto \widetilde{B}(\cdot)u(\cdot)$ is relatively compact if the embedding $D\hookrightarrow X$ is compact, but not in general.

However, we can circumvent this difficulty by working in $\ell^p(\mathbb{Z},X)$ instead of in $L^p(\mathbb{R},X)$, thanks to the following theorem taken from [**26**, Theorem 1.4].

THEOREM 4.5. *Let $U(t,s)$ be an exponentially bounded, strongly continuous evolution operator in a reflexive Banach space X, and let $\mathcal{G}: D(\mathcal{G}) \mapsto L^p(\mathbb{R},X)$ be the generator of the corresponding evolution semigroup in $L^p(\mathbb{R},X)$ given by $(T(t)f)(s)=U(s,s-t)f(s-t)$. Define the operator $\mathcal{D}\in L(\ell^p(\mathbb{Z},X))$, with $1\leqslant p<\infty$, by*

$$\mathcal{D}x = (x_n - U(n,n-1)x_{n-1})_{n\in\mathbb{Z}}, \quad x=(x_n)_{n\in\mathbb{Z}}\in\ell^p(\mathbb{Z},X).$$

Then $\mathcal{G}$ is a Fredholm operator if and only if $\mathcal{D}$ is a Fredholm operator, in which case they have the same index.

As observed at the end of § 3, under our assumptions $-\mathcal{L}$ generates the evolution semigroup associated to the evolution operator $G(t,s)$. Using similar arguments one also sees that $-\mathcal{L}_0$ is the generator of the evolution semigroup corresponding $G_0(t,s)$. By Proposition 4.4 and Theorem 4.5, the operator

$$\mathcal{D}_0 : \ell^p(\mathbb{Z}, X) \mapsto \ell^p(\mathbb{Z}, X)$$

defined by

$$\mathcal{D}_0 x = (x_n - G_0(n, n-1)x_{n-1})_{n\in\mathbb{Z}}, \quad x = (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}, X),$$

is a Fredholm operator with index equal to $\operatorname{ind}\mathcal{L}_0$. We will show that the perturbation

$$x \mapsto Sx := (G(n, n-1) - G_0(n, n-1))x_{n-1})_{n\in\mathbb{Z}}, \quad x = (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}, X), \quad (4.7)$$

is compact in $\ell^p(\mathbb{Z}, X)$. Then the operator $\mathcal{D}$ defined by

$$\mathcal{D}x = (x_n - G(n, n-1)x_{n-1})_{n\in\mathbb{Z}}, \quad x = (x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}, X),$$

is a Fredholm operator in $\ell^p(\mathbb{Z}, X)$ with index $\operatorname{ind}\mathcal{L}_0$ by Theorem IV.5.26 of [**24**]. Using Theorem 4.5 again, we conclude that $\mathcal{L}$ is also a Fredholm operator, with the same index as $\mathcal{L}_0$. To prove that the perturbation S is compact, we need the following two results. Here we assume that A satisfies (2.1) and is densely defined, that $B(t) : D(A) \mapsto X$ is compact, uniformly bounded and globally Hölder continuous in $L(D(A), X)$ for $t \in \mathbb{R}$, and that $B(t)$ converge to $B_{\pm\infty}$ in $L(D(A), X)$ as $t \to \pm\infty$. The next lemma is a special case of results in [**10**, **20**, **37**, **38**]; see for example, [**38**, Proposition 2.6].

Lemma 4.6. *We have* $\lim_{|n|\to\infty} \|G(n, n-1) - G_0(n, n-1)\|_{L(X)} = 0$.

Proposition 4.7. *The operator* $S : \ell^p(\mathbb{Z}, X) \mapsto \ell^p(\mathbb{Z}, X)$ *is compact.*

Proof. To prove that the range of the unit ball $B(0,1) \subset \ell^p(\mathbb{Z}, X)$ is totally bounded, it is enough to show that for each $\varepsilon > 0$ the following statements hold:

(a) there exists $N \in \mathbb{N}$ such that for each $x \in B(0,1)$ we have

$$\sum_{|n|\geqslant N} \|(G(n, n-1) - G_0(n, n-1))x_{n-1}\|^p \leqslant \varepsilon,$$

(b) for each $n \in \mathbb{Z}$ there is a compact set $K \subset X$ such that

$$\{(G(n, n-1) - G_0(n, n-1))x_{n-1} : x \in B(0,1)\}$$

is contained in $K + B_X(0, \varepsilon)$.

Point (a) is an obvious consequence of Lemma 4.6. Concerning point (b), we write

$$\begin{aligned} G(n, n-1) - G_0(n, n-1) = {} & G(n, n-1+h)(G(n-1+h, n-1) \\ & - G_0(n-1+h, n-1)) \\ & + \int_{n-1+h}^{n-1} G(n,s)\widetilde{B}(s)G_0(s, n-1)\, ds \end{aligned}$$

for each $h \in (0,1)$. This identity follows from the variation-of-constants formula in the interval $[n-1+h, n]$ and

$$\frac{d}{dt}[G(t, n-1) - G_0(t, n-1)] - A(t)(G(t, n-1) - G_0(t, n-1)) = \widetilde{B}(t)G_0(t, n-1)$$

for $n-1 < t \leqslant n$. If $n-1 \geqslant 0$, then

$$G(n-1+h, n-1) - G_0(n-1+h, n-1) = e^{hA(n-1)} - e^{h(A+B_{+\infty})} + \int_{n-1}^{n-1+h} Z(s, n-1)\, ds,$$

where for all $r > \sigma \in \mathbb{R}$, $Z(r, \sigma)$ is the operator in the representation formula (2.4). Therefore,

$$\begin{aligned} G(n, n-1) - G_0(n, n-1) &= G(n, n-1+h)(e^{hA(n-1)} - e^{h(A+B_{+\infty})}) \\ &+ \int_{n-1}^{n-1+h} G(n, n-1+h) Z(s, n-1)\, ds \\ &+ \int_{n-1+h}^{n-1} G(n, s)\widetilde{B}(s) G_0(s, n-1)\, ds. \end{aligned}$$

Further, the operators

$$\begin{aligned} &G(n, n-1+h)(e^{hA(n-1)} - e^{h(A+B_{+\infty})}) \\ &= \frac{1}{2\pi i}\int_\gamma e^{\lambda h} G(n, n-1+h) R(\lambda, A(n-1))(B(n-1) - B_{+\infty}) R(\lambda, A+B_{+\infty}) d\lambda, \end{aligned}$$

and

$$\int_{n-1+h}^{n-1} G(n, s)\widetilde{B}(s) G_0(s, n-1)\, ds$$

are compact, and

$$\begin{aligned} \left\| G(n, n-1+h)\int_{n-1}^{n-1+h} Z(s, n-1)\, ds \right\|_{L(X)} &\leqslant M_0 c \int_{n-1}^{n-1+h} (s-n+1)^{\delta-1}\, ds \\ &= cM_0 h^\delta/\delta. \end{aligned}$$

Fix $h \in (0,1)$ such that $cM_0 h^\delta/\delta \leqslant \varepsilon$. Then $G(n, n-1) - G_0(n, n-1)$ is the sum of a compact operator plus an operator with norm less than ε, and (b) follows for $n \geqslant 1$. If $n \leqslant 0$, we use the same argument, replacing $B_{+\infty}$ by $B_{-\infty}$. □

So we have shown the following theorem.

THEOREM 4.8. *Let A be a fixed operator satisfying* (2.1) *and* (2.9). *Assume that $B(t) : D(A) \to X$ is compact and uniformly bounded and Hölder continuous in $L(D(A), X)$ for $t \in \mathbb{R}$ and converges to $B_{\pm\infty}$ in $L(D(A), X)$ as $t \to \pm\infty$. Suppose that $\sigma(A + B_{\pm\infty}) \cap i\mathbb{R} = \emptyset$. Then $\mathcal{L}$ is a Fredholm operator with index*

$$\operatorname{ind} \mathcal{L} = \dim((I - P_{+\infty})(X) \cap P_{-\infty}(X)) - \dim([(I - P_{+\infty})(X)]^\perp \cap [P_{-\infty}(X)]^\perp).$$

Using the same arguments, one can establish a second result on compact perturbations.

THEOREM 4.9. *Let $A(t)$, with $t \in \mathbb{R}$, satisfy* (2.1), (2.2), (2.7), (2.8) *and* (2.9), *and let $\mathcal{L}$ be Fredholm. Assume that $B(t) : D(A(t)) \to X$ is compact and that $t \mapsto B(t)R(\omega, A(t)) \in L(X)$ is uniformly Hölder continuous and converges to* 0 *in*

$L(X)$ as $t \to \pm\infty$. Then $\widetilde{\mathcal{L}}$ is a Fredholm operator with index

$$\operatorname{ind}\widetilde{\mathcal{L}} = \operatorname{ind}\mathcal{L}.$$

5. *Examples*

5.1. *Parabolic systems on bounded domains*

Due to recent advances in [**17**] we could treat very general parabolic boundary value systems of order $2m$. For the sake of simplicity, we concentrate on second order systems with Robin type boundary conditions and we require more regularity assumptions than necessary. Let $N \in \mathbb{N}$, $q \in (1, +\infty)$, and Ω be an open bounded subset of $\mathbb{R}^d$ with boundary Γ of class C^2. We study the differential operators

$$\mathcal{A}(t) = \mathcal{A}(t,x,D) = -\sum_{k,l=1}^{d} a_{kl}(t,x)\partial_k\partial_l + \sum_{k=1}^{d} a_k(t,x)\partial_k + a_0(t,x),$$

for $t \in \mathbb{R}$ and $x \in \overline{\Omega}$, and the boundary operators

$$\mathcal{B}(t) = \mathcal{B}(t,x,D) = \sum_{k=1}^{d} b_k(t,x)\gamma\partial_k + b_0(t,x)\gamma$$

for $t \in \mathbb{R}$ and $x \in \Gamma$. The derivatives are understood in the distributional sense and γ is the trace operator. The coefficients are complex $(N \times N)$-matrices satisfying

$$a_{kl}, a_j \in C_b^{\alpha_1}(\mathbb{R}; C(\overline{\Omega}, \mathbb{C}^{N\times N})), \qquad b_j \in C_b^{\alpha_2}(\mathbb{R}; C^1(\Gamma; \mathbb{C}^{N\times N}))$$

for $k,l = 1,\ldots,d$, $j = 0,\ldots,d$ and constants $\alpha_1 \in (0,1)$ and $\alpha_2 \in (\frac{1}{2}, 1)$, where C_b^{α} denotes the space of uniformly bounded and globally HÖlder continuous functions. We further suppose that

$$a_\alpha(t,\cdot) \to a_\alpha(\pm\infty,\cdot) \quad \text{in } C(\overline{\Omega}; \mathbb{C}^{N\times N}), \qquad b_j(t,\cdot) \to b_j(\pm\infty,\cdot) \quad \text{in } C^1(\Gamma; \mathbb{C}^{N\times N})$$

as $t \to \pm\infty$, for $\alpha = (k,l)$ or $\alpha = j$, and $k,l = 1,\ldots,d$, $j = 0,\ldots,d$. The principal symbols of $\mathcal{A}(t,x,D)$ and $\mathcal{B}(t,x,D)$ are defined by

$$a_\#(t,x,\xi) = \sum_{k,l=1}^{d} a_{kl}(t,x)\xi_k\xi_l \quad \text{and} \quad b_\#(t,x,\xi) = \sum_{k=1}^{d} b_k(t,x)\xi_k$$

for $\xi \in \mathbb{R}^d$, $t \in [-\infty, +\infty]$, and $x \in \overline{\Omega}$, respectively, $x \in \Gamma$.

We further suppose that $(\mathcal{A}(t), \mathcal{B}(t))$ are normally elliptic; cf. [**5**, **17**, **18**] and the references therein. This means that

$$\sigma(a_\#(t,x,\xi)) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$$

for $t \in [-\infty, +\infty]$, $x \in \overline{\Omega}$, and $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, and the Lopatinskii–Shapiro condition (see for example [**5**]) holds: for all $t \in [-\infty, +\infty]$, $x \in \Gamma$, tangent vectors ξ of Γ at $x \in \Gamma$, and $\operatorname{Re}\lambda \geqslant 0$ with $(\xi,\lambda) \neq (0,0)$, $v = 0$ is the unique solution in $C_0(\mathbb{R}_+, \mathbb{C}^N)$ of the ODE

$$\begin{aligned} \lambda v(\tau) + a_\#(t,x,\xi + \nu(x)i\partial_\tau)v(\tau) &= 0 \quad \text{for } \tau \geqslant 0,\\ b_\#(t,x,\xi + \nu(x)i\partial_\tau)v(0) &= 0, \end{aligned}$$

where $\nu(x)$ is the outer normal vector at $x \in \Gamma$. The elliptic boundary value problem $(\mathcal{A}(t), \mathcal{B}(t))$, with $t \in \mathbb{R}$, is normally elliptic if for instance, denoting by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{C}^N$, we have $\mathrm{Re}\langle a_{\#}(t, x, \xi)\eta, \eta \rangle > 0$ for $\eta \in \mathbb{C}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, and $b_j(t, x) = \beta_j(t, x) b(t, x)$ for $j = 1, \ldots, d$, invertible matrices $b(t, \cdot) \in C^1(\Gamma, \mathbb{C}^{N\times N})$, and an outward pointing, nowhere vanishing vector field $\beta(t, \cdot) \in C^1(\Gamma, \mathbb{R}^d)$; see [**5**, Theorem 4.2].

On $X = L^q(\Omega; \mathbb{C}^N)$ we now define

$$A(t)u = -\mathcal{A}(t, \cdot, D)u, \quad u \in D(A(t)) = \{u \in W^{2,q}(\Omega; \mathbb{C}^N) : \mathcal{B}(t, \cdot, D)u = 0 \text{ on } \Gamma\}$$

for $t \in [-\infty, +\infty]$. The Agmon–Douglis–Nirenberg estimates in the version of Theorem 2.3 of [**5**] (where the normal ellipticity assumption is used) imply condition (2.1) for $t \in [-\infty, +\infty]$. See also [**17**, Theorem 8.2].

Moreover, the graph norms of the operators $A(t)$, with $t \in [-\infty, +\infty]$, and the norm of $W^{2,q}(\Omega; \mathbb{C}^N)$ are uniformly equivalent. Theorem 8.2 of [**17**] further implies that all operators $A(t)$ satisfy (2.9) for fixed $t \in [-\infty, +\infty]$. Let us check that the corresponding $\mathcal{R}$-bounds are uniformly bounded in t. Take $t, s \in [-\infty, +\infty]$ and $f \in L^q([0, 1], X)$. Set

$$u(\tau, x) = \int_0^\tau (e^{(\tau-\sigma)A(s)} f(\sigma))(x)\, d\sigma \quad \text{for } 0 \leqslant \tau \leqslant 1.$$

(In fact, u depends also on s but we drop the dependence on s for notational simplicity.) We then have, for $\tau \in [0, 1]$,

$$\begin{aligned} \partial_\tau u(\tau, x) + \mathcal{A}(t, x, D)u(\tau, x) & \\ &= f(\tau, x) + (\mathcal{A}(t, x, D) - \mathcal{A}(s, x, D))u(\tau, x) \quad \text{for } x \in \Omega \text{ a.e.}, \\ \mathcal{B}(t, x, D)u(\tau, x) &= (\mathcal{B}(t, x, D) - \mathcal{B}(s, x, D))u(\tau, x) \quad \text{for } x \in \Gamma \text{ a.e.}, \\ u(0, x) &= 0 \quad \text{for } x \in \Omega. \end{aligned}$$

Given $\varepsilon > 0$, we find a neighborhood $U(t, \varepsilon)$ of $t \in [-\infty, +\infty]$ such that

$$\|a_\alpha(t, \cdot) - a_\alpha(s, \cdot)\|_{L^\infty(\Omega; \mathbb{C}^{N\times N})} \leqslant \varepsilon \quad \text{and} \quad \|b_j(t, \cdot) - b_j(s, \cdot)\|_{C^1(\Gamma; \mathbb{C}^{N\times N})} \leqslant \varepsilon$$

for $s \in U(t, \varepsilon)$, for $\alpha = (k, l)$ or $\alpha = j$, and $k, l = 1, \ldots, d$, $j = 0, \ldots, d$. Theorem 2.1 of [**18**] combined with the extension results in [**18**, § 3] then implies that

$$\begin{aligned} &\|u\|_{W^{1,q}([0,1];X)} + \|u\|_{L^q([0,1];W^{2,q}(\Omega;\mathbb{C}^N))} \\ &\qquad \leqslant c_t \big(\|f\|_{L^q([0,1];X)} + \varepsilon \|u\|_{W^{1,q}([0,1];X)} + \varepsilon \|u\|_{L^q([0,1];W^{2,q}(\Omega;\mathbb{C}^N))} \big). \end{aligned}$$

Choosing $\varepsilon = (2c_t)^{-1}$, and taking into account the fact that the norm of $W^{2,q}(\Omega; \mathbb{C}^N)$ is equivalent to the graph norm of $A(s)$, with equivalence constants independent of s, we deduce that

$$\|u\|_{W^{1,q}([0,1];X)} + \|A(s)u\|_{L^q([0,1];X)} \leqslant c'_t \, \|f\|_{L^q([0,1];X)} \quad \text{for } s \in U(t, (2c_t)^{-1}).$$

The compactness of $[-\infty, +\infty]$ thus yields

$$\|u\|_{W^{1,q}([0,1];X)} + \|A(s)u\|_{L^q([0,1];X)} \leqslant c' \, \|f\|_{L^q([0,1];X)}$$

for $s \in \mathbb{R}$. This uniform estimate shows (2.9) due to (the proofs of) Theorem 4.2 and Remark 2.3 of [**42**]; see also [**17**, Proposition 3.17].

In order to check (2.2) we proceed as in [**2**]; see also [**3**]. We extend the coefficients b_j to Ω preserving their norms. For $f \in X$, $t, s \in \mathbb{R}$, and $|\arg \lambda| \leqslant \theta$, we set

$$v = -R(\omega, A(s))f \quad \text{and} \quad u = R(\lambda+\omega, A(t))(\lambda+\omega - A(s))v,$$

where ω is the constant in (2.1). Then

$$u - v = (A(t)-\omega)R(\lambda+\omega, A(t))(R(\omega, A(t)) - R(\omega, A(s)))f$$

and

$$\begin{aligned} (\lambda+\omega)u + \mathcal{A}(t,\cdot,D)u &= \lambda v - f, \qquad (\omega + \mathcal{A}(s,\cdot,D))v = -f \quad \text{on } \Omega,\\ \mathcal{B}(t,\cdot,D)u &= 0, \qquad\qquad\qquad\qquad \mathcal{B}(s,\cdot,D)v = 0 \quad \text{on } \Gamma. \end{aligned}$$

This shows that

$$\begin{aligned} (\lambda+\omega)(u-v) + \mathcal{A}(t,\cdot,D)(u-v) &= (\mathcal{A}(s,\cdot,D) - \mathcal{A}(t,\cdot,D))v \quad \text{on } \Omega,\\ \mathcal{B}(t,\cdot,D)(u-v) &= (\mathcal{B}(s,\cdot,D) - \mathcal{B}(t,\cdot,D))v \quad \text{on } \partial\Omega. \end{aligned}$$

The Agmon–Douglis–Nirenberg estimate from Theorem 2.3 of [**5**] and our assumptions now imply that

$$\begin{aligned} \|u-v\|_X &\leqslant \frac{c}{|\lambda+\omega|}\big(\|(\mathcal{A}(s,\cdot,D) - \mathcal{A}(t,\cdot,D))v\|_{L^q(\Omega;\mathbb{C}^N)}\\ &\qquad + |\lambda+\omega|^{1/2}\|(\mathcal{B}(s,\cdot,D) - \mathcal{B}(t,\cdot,D))v\|_{W^{1,q}(\Omega;\mathbb{C}^N)}\big)\\ &\leqslant c'\left(|\lambda+\omega|^{-1}|t-s|^{\alpha_1} + |\lambda+\omega|^{-1/2}|t-s|^{\alpha_2}\right)\|f\|_X. \end{aligned}$$

In the same way one derives (2.7). Observe that the operators $A(\pm\infty)$ have compact resolvents. Hence the spectra of these operators consist of eigenvalues only and do not depend on q (see for example [**16**, Theorem 1.6.3]). If we can check (2.8), then Corollary 3.11 shows that $\mathcal{L}$ is a Fredholm operator with index equal to $\dim P_{-\infty}(X) - \dim P_{+\infty}(X)$.

We give a rather simple example to illustrate the spectral condition (2.8). The example could occur if a reaction diffusion system with two species, diagonal diffusion, and conormal boundary conditions is linearized along a heteroclinic orbit. We consider the differential operator in divergence form

$$-\mathcal{A}(t,x,D) = \begin{pmatrix} \operatorname{div} a(t,x)\nabla + a_0(t,x) & b(t,x)\\ c(t,x) & \operatorname{div} d(t,x)\nabla + d_0(t,x) \end{pmatrix}$$

for $t \in [-\infty, +\infty]$ and $x \in \overline{\Omega}$, and the boundary operator

$$\mathcal{B}(t,x,D) = \begin{pmatrix} a(t,x)\nu(x)\cdot\nabla & 0\\ 0 & d(t,x)\nu(x)\cdot\nabla \end{pmatrix}$$

for $t \in [-\infty, +\infty]$ and $x \in \Gamma = \partial\Omega$. Here Ω with outer normal ν is given as above. We assume that a, a_0, b, c, d and d_0 are real-valued, $a(t,x), d(t,x) \geqslant \delta > 0$,

$$\begin{aligned} a, d &\in C_b^{\alpha+1/2}(\mathbb{R}; C^1(\overline{\Omega})) \cap C([-\infty,+\infty]; C^1(\overline{\Omega})),\\ a_0, b, c, d_0 &\in C_b^{\alpha}(\mathbb{R}; C(\overline{\Omega})) \cap C([-\infty,+\infty]; C(\overline{\Omega})) \end{aligned}$$

for some $\alpha > 0$, and that the coefficients at $t = \pm\infty$ are equal to constants. Then it is not hard to check that we are in the situation discussed above. We thus have to study the spectra of the operators

$$A(\pm\infty) = \begin{pmatrix} a(\pm\infty)\Delta + a_0(\pm\infty) & b(\pm\infty)\\ c(\pm\infty) & d(\pm\infty)\Delta + d_0(\pm\infty) \end{pmatrix}$$

on $L^q(\Omega)^2$ with domains

$$D(A(\pm\infty)) = \{(u,v) \in W^{2,q}(\Omega)^2 : \partial_\nu u = \partial_\nu v = 0 \text{ on } \Gamma\}.$$

Here ∂_ν is the derivative in the direction of the outer normal direction at $x \in \Gamma$. It is straightforward to check that $\lambda \in \mathbb{C}$ is an eigenvalue of $A(\pm\infty)$ if and only if there is an $n \in \mathbb{N}_0$ such that λ is an eigenvalue of the matrix

$$M_n^{\pm} = \begin{pmatrix} a(\pm\infty)\mu_n + a_0(\pm\infty) & b(\pm\infty) \\ c(\pm\infty) & d(\pm\infty)\mu_n + d_0(\pm\infty) \end{pmatrix}$$

where $\mu_n \leqslant 0$ are the distinct eigenvalues of the Neumann Laplacian on Ω. Thus we have to ensure that none of the matrices $M_n^{\pm}$, for $n \in \mathbb{N}_0$, has an eigenvalue on $i\mathbb{R}$. One obtains a purely imaginary eigenvalue of $A(\pm\infty)$ if and only if either $\det(M_n^{\pm}) = 0$ for some $n \in \mathbb{N}_0$, or $\operatorname{tr}(M_n^{\pm}) = 0$ and $\det(M_n^{\pm}) > 0$ for some $n \in \mathbb{N}_0$. Therefore (2.8) holds if, for example,

$$\mu_n \neq -(a_0(\pm\infty) + d_0(\pm\infty))/(a(\pm\infty) + d(\pm\infty)) \quad \text{for } n \in \mathbb{N}_0,$$
$$a(\pm\infty)d_0(\pm\infty) + a_0(\pm\infty)d(\pm\infty) < 0 \quad \text{and} \quad a_0(\pm\infty)d_0(\pm\infty) > b(\pm\infty)c(\pm\infty),$$

since then $\operatorname{tr} M_n^{\pm} \neq 0$ and $\det(M^{\pm}(\mu)) > 0$ for all $\mu \leqslant 0$. Taking $n = 0$, we see that the unstable subspaces of $A(\pm\infty)$ are non-trivial if $a_0(\pm\infty) + d_0(\pm\infty) > 0$, too.

5.2. *Generalized Ornstein–Uhlenbeck operators*

Let $\Phi : \mathbb{R}^d \mapsto \mathbb{R}$ be a convex function such that $\lim_{|x|\to+\infty} \Phi(x) = +\infty$, so that $\int_{\mathbb{R}^d} e^{-\Phi(x)} dx < +\infty$ and the probability measure

$$\mu(dx) = \left(\int_{\mathbb{R}^d} e^{-\Phi(x)}\, dx \right)^{-1} e^{-\Phi(x)}\, dx \tag{5.1}$$

is well defined. We choose $X = L^q(\mathbb{R}^d, \mu)$, with $1 < q < +\infty$. In the paper [**15**] it was shown that the operator

$$A_2 : D(A_2) = \{u \in W^{2,2}(\mathbb{R}^d, \mu) : \langle D\Phi, Du\rangle \in L^2(\mathbb{R}^d, \mu)\}, \quad A_2 u = \Delta u - \langle D\Phi, Du\rangle,$$

(where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^d$) is the infinitesimal generator of a symmetric Markov semigroup $T_2(t)$ in $L^2(\mathbb{R}^d, \mu)$. Moreover, $D(A_2)$ is compactly embedded in $L^2(\mathbb{R}^d, \mu)$ provided that

$$\exists\, \omega > 0 \text{ such that } x \mapsto \Phi(x) - \tfrac{1}{2}\omega|x|^2 \text{ is convex,} \tag{5.2}$$

which we shall assume throughout.

For $1 < q < 2$ (respectively, $2 < q < \infty$) we denote by $T_q(t)$ the standard extension (respectively, restriction) of $T_2(t)$ to X and by A_q its generator; cf. [**16**]. The question whether the domain of A_q is contained in $W^{2,q}(\mathbb{R}^d, \mu)$ if $q \neq 2$ is still open; sufficient conditions in order that $D(A_q) = W^{2,q}(\mathbb{R}^d, \mu)$ are derived in [**28**]. In any case, $D(A_q)$ is compactly embedded in X, the spectrum of A_q is independent of q (see for example [**16**, Theorem 1.6.3]), and it consists of a sequence of negative eigenvalues $-\lambda_1 > -\lambda_2 > \ldots$ having finite multiplicities $k_1, k_2, \ldots$ plus the simple eigenvalue $\lambda_0 = 0$.

We consider the path of sectorial operators

$$A(t) : D(A_q) \to X, \quad (A(t)u)(x) = (A_q u)(x) - \varphi(t,x)u(x) \quad \text{for } t \in \mathbb{R}$$

in $L^q(\mathbb{R}^d,\mu)$, where φ is a real-valued L^∞ function such that $\|\varphi(t,\cdot)-\varphi(s,\cdot)\|_\infty \leqslant C|t-s|^\alpha$ and

$$\exists \lim_{t\to\pm\infty} \varphi(t,\cdot\,) =: \varphi_\pm \quad \text{in } L^\infty(\mathbb{R}^d).$$

So, the domain of $A(t)$ is constant and (2.1) and (2.2) are satisfied. The limiting operators $A(-\infty)$ and $A(+\infty)$ are hyperbolic under standard assumptions on the ranges of $\varphi_\pm$.

LEMMA 5.1. *Let $a_\pm$ and $b_\pm$ be such that $a_- \leqslant \varphi_-(x) \leqslant b_-$ and $a_+ \leqslant \varphi_+(x) \leqslant b_+$ for all $x \in \mathbb{R}^d$. Assume that there exist $n, m \in \mathbb{N}\cup\{0\}$ such that $\lambda_{m+1} < a_- \leqslant b_- < \lambda_m$ and $\lambda_{n+1} < a_+ \leqslant b_+ < \lambda_n$. Then $A(-\infty)$ and $A(+\infty)$ are hyperbolic operators. The unstable spaces $P_{-\infty}(X)$ and $P_{+\infty}(X)$ have positive finite dimensions equal to $1+k_1+\ldots+k_m$ and $1+k_1+\ldots+k_n$, respectively, where k_j is the multiplicity of the eigenvalue $-\lambda_j$ of A_2.*

Proof. First, let $q=2$. Since A_2 is self-adjoint, the operators $A(-\infty)$ and $A(+\infty)$ are self-adjoint too in $L^2(\mathbb{R}^d,\mu)$. The domain $D(A_2)$ is compactly embedded in $L^2(\mathbb{R}^d,\mu)$. Therefore the spectra of $A(-\infty)$ and of $A(+\infty)$ consist of sequences of real eigenvalues, each eigenvalue has finite geometric multiplicity, and $A(-\infty)$ and $A(+\infty)$ are hyperbolic if and only if 0 is not an eigenvalue. In view of our assumptions, the minmax principle (see for example, [**34**, Theorem XIII.2]) implies that 0 is not an eigenvalue of $A(-\infty)$ and $A(+\infty)$. Moreover, the number of strictly positive eigenvalues of $A(-\infty)$ and $A(+\infty)$ (counting multiplicities) is equal to $1+k_1+\ldots+k_m$ and $1+k_1+\ldots+k_n$, respectively.

In the case that $q\neq 2$, Theorems 1.6.1 and 1.6.3 and Corollary 1.6.2 of [**16**] show that the spectra of $A(-\infty)$ and $A(+\infty)$ and the multiplicities of the eigenvalues do not depend on q. Thus the lemma is proved. □

We do not need the general theory of optimal L^p regularity for the proof of Theorem 2.2. An easy proof by perturbation is given below.

LEMMA 5.2. *Let $a<b\in\mathbb{R}$ and $1<p<+\infty$, let $\varphi\in L^\infty((a,b)\times\mathbb{R})$ and let $f\in L^p((a,b);X)$. Then the problem*

$$\begin{cases} u'(t) = A_p u(t) - \varphi(t,\cdot)u(t) + f(t) & \text{for } a<t<b, \\ u(a)=0, \end{cases} \tag{5.3}$$

has a unique solution u, which belongs to $W^{1,p}((a,b);X)$. For almost all $t\in(a,b)$, $u(t)$ belongs to $D(A_q)$, and there is $C_{p,b-a}$, independent of f, such that

$$\|u\|_{W^{1,p}((a,b);X)} + \|A(\,\cdot\,)u(\,\cdot\,)\|_{L^p((a,b);X)} \leqslant C_{p,b-a}\|f\|_{L^p((a,b);X)}. \tag{5.4}$$

Proof. Since A_p is the generator of a Markov semigroup, the result is true when $\varphi\equiv 0$, thanks to [**14**]. In the general case, the solution to (5.3) must satisfy

$$u(t) = \int_a^t T_p(t-s)(f(s)-\varphi(s,\cdot)u(s))\,ds \quad \text{for } a<t<b.$$

The operator Λ defined by $(\Lambda u)(t) = \int_a^t T_p(t-s)(f(s)-\varphi(s,\cdot)u(s))\,ds$ is easily seen to be a contraction in $L^p((a,b);X)$ with respect to the norm

$$|u|_{L^p((a,b);X)} := \left(\int_a^b (e^{-\omega s}\|u(s)\|_X)^p\,ds\right)^{1/p},$$

provided ω is large enough. Therefore it has a unique fixed point u. Since

$$s \mapsto \widetilde{f}(s) := f(s) + \varphi(s,\cdot)u(s) \in L^p((a,b);X),$$

the statement follows by applying the result of [**14**] to $u = \int_a^t T_p(t-s)\widetilde{f}(s)\,ds$. □

If the assumptions of Lemma 5.1 hold, our operator $\mathcal{L}$ has the domain

$$D(\mathcal{L}) = W^{1,p}(\mathbb{R};X) \cap L^p(\mathbb{R};D(A_q)).$$

Applying Corollary 3.11 gives the fact that $\mathcal{L}$ is a Fredholm operator with index equal to $\dim P_{-\infty}(X) - \dim P_{+\infty}(X)$.

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Davide Di Giorgio
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa
Italy

digiorgi@mail.dm.unipi.it

Alessandra Lunardi
Dipartimento di Matematica
Università di Parma
Via D'Azeglio 85/A
43100 Parma
Italy

lunardi@unipr.it
http://math.unipr.it/∼lunardi

Roland Schnaubelt
FB Mathematik und Informatik
Martin–Luther–Universität Halle–Wittenberg
06099 Halle
Germany

schnaubelt@mathematik.uni-halle.de