On a class of elliptic operators with unbounded coefficients in convex domains

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Abstract

We study the realization A of the operator $\mathcal{A} = \frac{1}{2}\,\Delta - \langle DU, D\cdot\rangle$ in $L^2(\Omega,\mu)$, where Ω is a possibly unbounded convex open set in $\mathbb{R}^N$, U is a convex unbounded function such that $\lim_{x\to\partial\Omega,\, x\in\Omega} U(x) = +\infty$ and $\lim_{|x|\to+\infty,\, x\in\Omega} U(x) = +\infty$, $DU(x)$ is the element with minimal norm in the subdifferential of U at x, and $\mu(dx) = c\exp(-2U(x))dx$ is a probability measure, infinitesimally invariant for $\mathcal{A}$. We show that A, with domain $D(A) = \{u \in H^2(\Omega,\mu) : \langle DU, Du\rangle \in L^2(\Omega,\mu)\}$ is a dissipative self-adjoint operator in $L^2(\Omega,\mu)$. Note that the functions in the domain of A do not satisfy any particular boundary condition. Log-Sobolev and Poincaré inequalities allow then to study smoothing properties and asymptotic behavior of the semigroup generated by A.

1 Introduction

In this paper we give a contribution to the theory of second order elliptic operators with unbounded coefficients, that underwent a great developement in the last few years. See e.g. [1, 7, 5, 6, 8, 12, 13].

Here we consider the operator

$$\mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle \tag{1.1}$$

in a convex open set $\Omega \subset \mathbb{R}^N$, where U is a convex function such that

$$\lim_{x\to\partial\Omega,\, x\in\Omega} U(x) = +\infty, \qquad \lim_{|x|\to+\infty,\, x\in\Omega} U(x) = +\infty. \tag{1.2}$$

Since we do not impose any growth condition on U, the usual L^p and Sobolev spaces with respect to the Lebesgue measure are not the best setting for the operator $\mathcal{A}$. It is more convenient to introduce the measure

$$\mu(dx) = \left(\int_\Omega e^{-2U(x)}dx\right)^{-1} e^{-2U(x)}dx, \tag{1.3}$$

which is infinitesimally invariant for $\mathcal{A}$, i.e.

$$\int_\Omega \mathcal{A}u(x)\mu(dx) = 0, \;\; u \in C_0^\infty(\mathbb{R}^N),$$

and lets $\mathcal{A}$ be formally self-adjoint in $L^2(\Omega,\mu)$, as an easy computation shows. We prove in fact that the realization A of $\mathcal{A}$ in $L^2(\Omega,\mu)$, with domain

$$D(A) = \{u \in H^2(\Omega,\mu) : \mathcal{A}u \in L^2(\Omega,\mu)\} = \{u \in H^2(\Omega,\mu) : \langle DU, Du\rangle \in L^2(\Omega,\mu)\}$$

is a self-adjoint and dissipative operator, provided $C_0^\infty(\Omega)$ is dense in $H^1(\Omega,\mu)$. We recall that $H^1(\Omega,\mu)$ is naturally defined as the set of all $u \in H^1_{loc}(\Omega)$ such that u, $D_iu \in L^2(\Omega,\mu)$, for $i = 1,\dots,N$. While it is easy to see that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega,\mu)$, well-known counterexamples show that $C_0^\infty(\Omega)$ is not dense in $H^1(\Omega,\mu)$ in general.

Once we know that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega,\mu)$, it is not hard to show that for each $u \in D(A)$ and $\psi \in H^1(\Omega,\mu)$ we have

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle \mu(dx).$$

This crucial integration formula implies that A is symmetric and dissipative. The next step is to prove that $\lambda I - A$ is onto for $\lambda > 0$, so that A is m-dissipative. This is done by approximation, solving first, for each $\lambda > 0$ and $f \in C_0^\infty(\Omega)$,

$$\lambda u_\alpha(x) - (\mathcal{A}_\alpha u_\alpha)(x) = f(x), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where A_α is defined as $\mathcal{A}$, with U replaced by its Moreau-Yosida approximation U_α. To be more precise, first we extend f and U to the whole $\mathbb{R}^N$ setting $f(x) = 0$ and $U(x) = +\infty$ for x outside Ω; since the extension of U is lower semicontinuous and convex the Moreau-Yosida approximations U_α are well defined and differentiable with Lipschitz continuous gradient in $\mathbb{R}^N$. Then (1.4) has a unique solution $u_\alpha \in H^2(\mathbb{R}^N,\mu_\alpha)$, with $\mu_\alpha(dx) = (\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx)^{-1}e^{-2U_\alpha(x)}dx$, and the norm of u_α in $H^2(\mathbb{R}^N,\mu_\alpha)$ is bounded by $C(\lambda)\|f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}$, where the constant $C(\lambda)$ is independent of α, due to the estimates for equations in the whole $\mathbb{R}^N$ already proved in [5]. Using the convergence properties of U_α and of DU_α to U and to DU respectively, we arrive at a solution $u \in H^2(\Omega,\mu)$ of

$$\lambda u(x) - (\mathcal{A}u)(x) = f(x), \quad x \in \Omega, \tag{1.5}$$

that belongs to $D(A)$, satisfies $\|u\|_{H^2(\Omega,\mu)} \le C(\lambda)\|f\|_{L^2(\Omega,\mu)}$ and is the unique solution to the resolvent equation because A is dissipative. If f is just in $L^2(\Omega,\mu)$, (1.5) is solved approaching f by a sequence of functions in $C_0^\infty(\Omega)$.

A lot of nice consequences follow: A generates an analytic contraction semigroup $T(t)$ in $L^2(\Omega,\mu)$, which is a Markov semigroup and may be extended in a standard way to a contraction semigroup in $L^p(\Omega,\mu)$ for each $p \ge 1$. The measure μ is invariant for $T(t)$, i.e.

$$\int_\Omega (T(t)f)(x)\mu(dx) = \int_\Omega f(x)\mu(dx), \quad f \in L^1(\Omega,\mu),$$

and moreover $T(t)f$ converges to the mean value $\overline{f} = \int_\Omega f(x)\mu(dx)$ of f as $t \to +\infty$, for each $f \in L^2(\Omega,\mu)$.

If, in addition, $U - \omega|x|^2/2$ is still convex for some $\omega > 0$, $T(t)$ enjoys further properties. 0 comes out to be a simple isolated eigenvalue in $\sigma(A)$, the rest of the spectrum is contained in $(-\infty,-\omega]$, and $T(t)f$ converges to $\overline{f}$ at an exponential rate as $t \to +\infty$. Moreover, $T(t)$ is a bounded operator (with norm not exceeding 1) from $L^p(\Omega,\mu)$ to $L^{q(t)}(\Omega,\mu)$, with $q(t) = 1 + (p-1)e^{2\omega t}$. This hypercontractivity property is the best we can expect in weighted Lebesgue spaces with general weight, and there is no hope that $T(t)$ maps, say, $L^2(\Omega,\mu)$ into $L^\infty(\Omega)$. Similarly, Sobolev embeddings are not available in general. The best we can prove is a logarithmic Sobolev inequality,

$$\int_\Omega f^2(x)\log(f^2(x))\mu(dx) \le \frac{1}{\omega}\int_\Omega |Df(x)|^2\mu(dx) + \overline{f^2}\log(\overline{f^2}), \quad f \in H^1(\Omega,\mu).$$

2 Preliminaries: operators in the whole $\mathbb{R}^N$

Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ be a convex C^1 function, satisfying

$$\lim_{|x|\to+\infty} U(x) = +\infty. \tag{2.1}$$

Then there are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geq a + b|x|$, for each $x \in \mathbb{R}^N$. It follows that the probability measure $\nu(dx) = e^{-2U(x)}dx / \int_{\mathbb{R}^N} e^{-2U(x)}dx$ is well defined.

The spaces $H^1(\mathbb{R}^N, \nu)$ and $H^2(\mathbb{R}^N, \nu)$, consist of the functions $u \in H^1_{loc}(\mathbb{R}^N)$ (respectively, $u \in H^2_{loc}(\mathbb{R}^N)$) such that u and its first (resp., first and second) order derivatives are in $L^2(\mathbb{R}^N, \nu)$.

We recall some results proved in [5] on the realization A of $\mathcal{A}$ in $L^2(\mathbb{R}^N, \nu)$. It is defined by

$$\begin{cases} D(A) &= \{u \in H^2(\mathbb{R}^N, \nu) :\ \mathcal{A}u \in L^2(\mathbb{R}^N, \nu)\} \\ \\ &= \{u \in H^2(\mathbb{R}^N, \nu) :\ \langle DU, Du\rangle \in L^2(\mathbb{R}^N, \nu)\}, \\ \\ (Au)(x) = \mathcal{A}u(x), \ \ x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

Theorem 2.1 *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ be a convex function satisfying assumption* (2.1)*. Then the resolvent set of A contains* $(0, +\infty)$ *and*

$$\begin{cases} (i) \quad \|R(\lambda, A)f\|_{L^2(\mathbb{R}^N,\nu)} \leq \dfrac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \\ \\ (ii) \quad \|\,|DR(\lambda, A)f|\,\|_{L^2(\mathbb{R}^N,\nu)} \leq \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \\ \\ (iii) \quad \|\,|D^2R(\lambda, A)f|\,\|_{L^2(\mathbb{R}^N,\nu)} \leq 4\|f\|_{L^2(\mathbb{R}^N,\nu)}. \end{cases} \tag{2.3}$$

Theorem 2.2 *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ satisfy* (2.1)*, and be such that $x \mapsto U(x) - \omega|x|^2/2$ is convex, for some $\omega > 0$. Then, setting $\overline{u} = \int_{\mathbb{R}^N} u(x)\nu(dx)$, we have*

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}|^2\nu(dx) \leq \frac{1}{2\omega}\int_{\mathbb{R}^N} |Du(x)|^2\nu(dx),$$

$$\int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\nu(dx) \leq \frac{1}{\omega}\int_{\mathbb{R}^N} |Du(x)|^2\nu(dx) + \overline{u^2}\log(\overline{u^2}),$$

for each $u \in H^1(\mathbb{R}^N, \nu)$ (we adopt the convention $0\log 0 = 0$).

3 The operator A

Let $U : \Omega \mapsto \mathbb{R}$ be a convex function satisfying assumption (1.2), and let us extend it to the whole $\mathbb{R}^N$ setting

$$U(x) = +\infty, \ \ x \notin \Omega. \tag{3.1}$$

The extension, that we shall still call U, is lower semicontinuous and convex. For each $x \in \mathbb{R}^N$, the subdifferential $\partial U(x)$ of U at x is the set $\{y \in \mathbb{R}^N :\ U(\xi) \geq U(x) + \langle y, \xi - x\rangle,\ \forall\, \xi \in \mathbb{R}^N\}$. At each $x \in \Omega$, since U is real valued and continuous, $\partial U(x)$ is not empty and it has a unique element with minimal norm, that we denote by $DU(x)$. Of course if U is differentiable at x, $DU(x)$ is the usual gradient. At each $x \notin \Omega$, $\partial U(x)$ is empty and $DU(x)$ is not defined.

Lemma 3.1 *There are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geq a + b|x|$ for each $x \in \Omega$.*

Proof — The statement is obvious if Ω is bounded. If Ω is unbounded, we may assume without loss of generality that $0 \in \Omega$. Assume by contradiction that there is a sequence x_n with $|x_n| \to +\infty$ such that $\lim_{n\to\infty} U(x_n)/|x_n| = 0$. Let R be so large that $\min\{U(x) - U(0) :\ x \in \Omega,\ |x| = R\} > 0$. Since U is convex, for n large enough we have

$$U\left(\frac{R}{|x_n|}x_n\right) \leq \frac{R}{|x_n|}U(x_n) + \left(1 - \frac{R}{|x_n|}\right)U(0)$$

so that

$$\limsup_{n\to\infty} U\left(\frac{R}{|x_n|}x_n\right) - U(0) \leq \lim_{n\to\infty} \frac{R}{|x_n|}U(x_n) - \frac{R}{|x_n|}U(0) = 0,$$

a contradiction. □

We set as usual $e^{-\infty} = 0$. The function

$$x \mapsto e^{-2U(x)}, \;\; x \in \mathbb{R}^N,$$

is continuous, it is positive in Ω, and it vanishes outside Ω. Lemma 3.1 implies that it is in $L^1(\Omega)$. Therefore, the probability measure (1.3) is well defined, and it has Ω as support.

Lemma 3.2 *$C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$.*

Proof — It is well known that every function $u \in L^2(\Omega)$ with compact support may be approximated in $L^2(\Omega)$ by a sequence of C_0^∞ functions obtained by convolution with smooth mollifiers. Since u has compact support, such a sequence approximates u also in $L^2(\Omega, \mu)$.

Therefore it is sufficient to show that every $u \in L^2(\Omega, \mu)$ may be approximated by a sequence of L^2 functions with compact support, contained in Ω. In this case also the functions obtained by convolution with smooth mollifiers have support in Ω.

Let $\theta_n : \mathbb{R} \mapsto \mathbb{R}$ be a sequence of smooth functions such that $0 \leq \theta_n(y) \leq 1$ for each y, $\theta_n \equiv 1$ for $y \leq n$, $\theta_n \equiv 0$ for $y \geq 2n$. We set

$$u_n(x) = u(x)\theta_n(U(x)),\ x \in \Omega, \quad u_n(x) = 0,\ x \notin \Omega.$$

Then u_n has compact support in Ω, and $u_n \to u$ in $L^2(\mathbb{R}^N, \mu)$. Indeed,

$$\int_{\mathbb{R}^N} |u_n - u|^2 \mu(dx) \leq \int_{\{x\in\Omega:\, U(x)\geq n\}} |u|^2 \mu(dx)$$

which goes to 0 as $n \to \infty$. □

We remark that in general $C_0^\infty(\Omega)$ is not dense in $H^1(\mathbb{R}^N, \mu)$. See next example 4.1.

We introduce now the main tool in our study, i.e. the *Moreau-Yosida approximations* of U,

$$U_\alpha(x) = \inf\left\{U(y) + \frac{1}{2\alpha}|x-y|^2 :\ y \in \mathbb{R}^N\right\},\ \ x \in \mathbb{R}^N,\ \alpha > 0,$$

that are real valued on the whole $\mathbb{R}^N$ and enjoy good regularity properties: they are convex, differentiable, and for each $x \in \mathbb{R}^N$ we have (see e.g. [2, prop. 2.6, prop. 2.11])

$$U_\alpha(x) \leq U(x), |DU_\alpha(x)| \leq |DU(x)|,$$

$$\lim_{\alpha\to 0} U_\alpha(x) = U(x),\ \ x \in \mathbb{R}^N,$$

$$\lim_{\alpha\to 0} DU_\alpha(x) = DU(x),\ x \in \Omega; \quad \lim_{\alpha\to 0} |DU_\alpha(x)| = +\infty,\ x \notin \Omega.$$

Moreover DU_α is Lipschitz continuous for each α, with Lipschitz constant $1/\alpha$.

Let us define now the realization A of $\mathcal{A}$ in $L^2(\Omega,\mu)$ by

$$\begin{cases} D(A)=\{u\in H^2(\Omega,\mu):\ \langle DU,Du\rangle\in L^2(\Omega,\mu)\}, \\ \\ (Au)(x)=\mathcal{A}u(x),\ \ x\in\Omega. \end{cases} \tag{3.2}$$

We shall show that A is a self-adjoint dissipative operator, provided $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N,\mu)$. The fact that A is symmetric is a consequence of the next lemma.

Lemma 3.3 *If $C_0^\infty(\Omega)$ is dense in $H^1(\mathbb{R}^N,\mu)$, then for each $u\in D(A)$, $\psi\in H^1(\mathbb{R}^N,\mu)$ we have*

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\nu(dx)=-\frac{1}{2}\int_\Omega \langle Du(x),D\psi(x)\rangle\mu(dx). \tag{3.3}$$

Proof — Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N,\mu)$ it is sufficient to show that (3.3) hold for each $\psi\in C_0^\infty(\mathbb{R}^N)$.

If $\psi\in C_0^\infty(\Omega)$, then the function $\psi\exp(-2U)$ is continuously differentiable and it has compact support in Ω. Integrating by parts $(\Delta u)(x)\psi(x)\exp(-2U(x))$ we get

$$\frac{1}{2}\int_\Omega (\Delta u)(x)\psi(x)e^{-2U(x)}dx=-\frac{1}{2}\int_\Omega \langle Du(x),D(\psi(x)e^{-2U(x)})\rangle dx$$

$$=-\frac{1}{2}\int_\Omega \langle Du(x),D\psi(x)\rangle e^{-2U(x)}dx+\frac{1}{2}\int_\Omega \langle Du(x),2DU(x)\rangle\psi(x)e^{-2U(x)}dx$$

so that (3.3) holds. □

Taking $\psi=u$ in (3.3) shows that A is symmetric.

Once we have the integration formula (3.3) and the powerful tool of the Moreau-Yosida approximations at our disposal, the proof of the dissipativity of A is similar to the proof of theorem 2.4 of [5]. However we write down all the details for the reader's convenience.

Theorem 3.4 *Let $U:\Omega\mapsto\mathbb{R}$ be a convex function satisfying assumption (1.2), and be such that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega,\mu)$. Then the resolvent set of A contains $(0,+\infty)$ and*

$$\begin{cases} (i) \quad \|R(\lambda,A)f\|_{L^2(\Omega,\mu)}\le \frac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \\ \\ (ii) \quad \|\,|DR(\lambda,A)f|\,\|_{L^2(\Omega,\mu)}\le \frac{2}{\sqrt{\lambda}}\|f\|_{L^2(\Omega,\mu)}, \\ \\ (iii) \quad \|\,|D^2R(\lambda,A)f|\,\|_{L^2(\Omega,\mu)}\le 4\|f\|_{L^2(\Omega,\mu)}. \end{cases} \tag{3.4}$$

Moreover the resolvent $R(\lambda,A)$ is positivity preserving, and $R(\lambda,A)\mathbb{1}=\mathbb{1}/\lambda$.

Proof — For $\lambda>0$ and $f\in L^2(\Omega,\mu)$ consider the resolvent equation

$$\lambda u-Au=f. \tag{3.5}$$

It has at most a solution, because if $u\in D(A)$ satisfies $\lambda u=Au$ then by (3.3) we have

$$\int_\Omega \lambda(u(x))^2\mu(dx)=\int_\Omega (Au)(x)u(x)\mu(dx)=-\frac{1}{2}\int_\Omega |Du(x)|^2\mu(dx)\le 0,$$

so that $u=0$.

To find a solution to (3.5), we approximate U by the Moreau–Yosida approximations U_α defined above, we consider the measures $\nu_\alpha(dx)=e^{-2U_\alpha(x)}dx/\int_{\mathbb{R}^N}e^{-2U_\alpha(x)}dx$ in $\mathbb{R}^N$ and the operators $\mathcal{A}_\alpha$ defined by $\mathcal{A}_\alpha u=\Delta u/2-\langle DU_\alpha,Du\rangle$.

Since the functions U_α are convex and satisfy (2.1), the results of theorem 2.1 hold for the operators $A_\alpha\colon D(A_\alpha) = H^2(\mathbb{R}^N,\nu_\alpha) \mapsto L^2(\mathbb{R}^N,\nu_\alpha)$. In particular, for each $f \in C_0^\infty(\mathbb{R}^N)$ with support contained in Ω, the equation

$$\lambda u_\alpha - A_\alpha u_\alpha = f, \tag{3.6}$$

has a unique solution $u_\alpha \in D(A_\alpha)$. Moreover, each u_α is bounded with bounded and Hölder continuous second order derivatives, thanks to the Schauder estimates and the maximum principle that hold for operators with Lipschitz continuous coefficients, see [10].

Estimates (2.3) imply that

$$\begin{cases} \|u_\alpha\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \leq \dfrac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}, \\[2ex] \|\,|Du_\alpha|\,\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \leq \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}, \\[2ex] \|\,|D^2u_\alpha|\,\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \leq 4\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}, \end{cases} \tag{3.7}$$

so that

$$\|u_\alpha\|_{H^2(\mathbb{R}^N,\nu_\alpha)} \leq C\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}$$

with $C = C(\lambda)$ independent of α. Since $U_\alpha(x)$ goes to $U(x)$ monotonically as $\alpha \to 0$, then $\exp(-2U_\alpha(x))$ goes to $\exp(-2U(x))$ monotonically, and $(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx)^{-1}$ goes to $(\int_{\mathbb{R}^N} e^{-2U(x)}dx)^{-1}$, $\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}$ goes to $\|f\|_{L^2(\mathbb{R}^N,\mu)}$ as $\alpha \to 0$. It follows that the norm $\|u_\alpha\|_{H^2(\mathbb{R}^N,\nu_\alpha)}$ is bounded by a constant independent of α, and consequently also the norm $\|u_\alpha\|_{H^2(\mathbb{R}^N,\mu)}$ is bounded by a constant independent of α. Therefore there is a sequence u_{α_n} that converges weakly in $H^2(\mathbb{R}^N,\mu)$ to a function $u \in H^2(\mathbb{R}^N,\mu)$, and converges to u in $H^1(K)$ for each compact subset $K \subset \Omega$. This implies easily that u solves (3.5). Indeed, let $\phi \in C_0^\infty(\Omega)$. For each $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^N} (\lambda u_{\alpha_n} - \frac{1}{2}\Delta u_{\alpha_n} + \langle DU_{\alpha_n}, Du_{\alpha_n}\rangle - f)\phi\, e^{-2U}dx = 0.$$

Letting $n \to \infty$, we get immediately that $\int_{\mathbb{R}^N}(\lambda u_{\alpha_n} - \frac{1}{2}\Delta u_{\alpha_n})\phi e^{-2U(x)}dx$ goes to $\int_{\mathbb{R}^N}(\lambda u - \frac{1}{2}\Delta u)\phi e^{-2U(x)}dx$. Moreover $\int_{\mathbb{R}^N}\langle DU_{\alpha_n}, Du_{\alpha_n}\rangle\phi e^{-2U(x)}dx$ goes to $\int_{\mathbb{R}^N}\langle DU, Du\rangle\phi e^{-2U(x)}dx$ because DU_{α_n} goes to DU in $L^2(\operatorname{supp}\phi)$. Therefore letting $n \to \infty$ we get

$$\int_{\mathbb{R}^N} (\lambda u - \mathcal{A}u - f)\phi\, e^{-2U}dx = 0$$

for each $\phi \in C_0^\infty(\mathbb{R}^N)$, and hence $\lambda u - \mathcal{A}u = f$ almost everywhere in Ω. So, $u_{|\Omega} \in D(A)$ is the solution of the resolvent equation, and letting $\alpha \to 0$ in (3.7) we get

$$\begin{cases} \|u\|_{L^2(\Omega,\mu)} \leq \dfrac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \quad \||Du|\|_{L^2(\Omega,\mu)} \leq \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\Omega,\nu)}, \\[2ex] \||D^2u|\|_{L^2(\Omega,\mu)} \leq 4\|f\|_{L^2(\Omega,\mu)}. \end{cases} \tag{3.8}$$

Let now $f \in L^2(\Omega,\mu)$ and let f_n be a sequence of $C_0^\infty(\Omega)$ functions going to f in $L^2(\Omega,\mu)$ as $n \to \infty$. Thanks to estimates (3.8), the solutions u_n of

$$\lambda u_n - Au_n = f_n$$

are a Cauchy sequence in $H^2(\Omega,\mu)$, and converge to a solution $u \in H^2(\Omega,\mu)$ of (3.5). Due again to estimates (3.8), u satisfies (3.4).

If in addition $f(x) \geq 0$ a.e. in Ω, we may take $f_n(x) \geq 0$ in Ω, see the proof of lemma 3.2. Each u_α, solution to (3.6) with f replaced by f_n, has nonnegative values thanks to the maximum principle for elliptic operators with Lipschitz continuous coefficients proved in [10]. Our limiting procedure gives $R(\lambda, A)f_n(x) \geq 0$ for each x, and $R(\lambda, A)f(x) \geq 0$ for each x. So, $R(\lambda, A)$ is a positivity preserving operator. □

4 Examples and consequences

Example 4.1 Let Ω be the unit open ball in $\mathbb{R}^N$, and let $U(x) = -\frac{\alpha}{2}\log(1-|x|)$ for $x \in \Omega$, with $\alpha > 0$. Then

$$\exp(-2U(x)) = (1-|x|)^\alpha, \;\; DU(x) = \frac{\alpha x}{2|x|(1-|x|)}, \;\; 0 < |x| < 1,$$

and it is known that $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$ iff $\alpha \geq 1$. See e.g. [14, thm. 3.6.1]. In this case the result of theorem 3.4 holds, and A is a self-adjoint dissipative operator in $L^2(\Omega, \mu)$. □

Under the assumptions of theorem 3.4, A is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$.

Since the resolvent $R(\lambda, A)$ is positivity preserving for $\lambda > 0$, also $T(t)$ is positivity preserving. Since $R(\lambda, A)\mathbb{1} = \mathbb{1}/\lambda$, then $T(t)\mathbb{1} = \mathbb{1}$ for each $t > 0$. Therefore, $T(t)$ is a Markov semigroup and it may be extended in a standard way to a contraction semigroup (that we shall still call $T(t)$) in $L^p(\Omega, \mu)$, $1 \leq p \leq \infty$. $T(t)$ is strongly continuous in $L^p(\Omega, \mu)$ for $1 \leq p < \infty$, and it is analytic for $1 < p < \infty$. See e.g [4, ch. 1]. The infinitesimal generator of $T(t)$ in $L^p(\Omega, \mu)$ is denoted by A_p. The characterization of the domain of A_p in $L^p(\Omega, \mu)$ is an interesting open problem.

An important optimal regularity result for evolution equations follows, see [9].

Corollary 4.2 *Let $1 < p < \infty$, $T > 0$. For each $f \in L^p((0,T); L^p(\Omega, \mu))$ (i.e. $(t,x) \mapsto f(t)(x) \in L^p((0,T) \times \Omega; dt \times \mu)$) the problem*

$$\begin{cases} u'(t) = A_p u(t) + f(t), \;\; 0 < t < T, \\ \\ u(0) = 0, \end{cases}$$

has a unique solution $u \in L^p((0,T); D(A_p)) \cap W^{1,p}((0,T); L^p(\Omega, \mu))$.

From lemma 3.3 we get, taking $\psi \equiv 1$,

$$\int_\Omega Au \, \mu(dx) = 0, \;\; u \in D(A),$$

and hence,

$$\int_\Omega T(t)f \, \mu(dx) = \int_\Omega f \, \mu(dx), \;\; t > 0,$$

for each $f \in L^2(\Omega, \mu)$. Since $L^2(\Omega, \mu)$ is dense in $L^1(\Omega, \mu)$, the above equality holds for each $f \in L^1(\Omega, \mu)$. In other words, μ is an invariant measure for the semigroup $T(t)$.

From lemma 3.3 we get also

$$u \in D(A), \;\; Au = 0 \Longrightarrow Du = 0,$$

and hence the kernel of A consists of the constant functions. Let us prove now that

$$\lim_{t \to +\infty} T(t)f = \int_\Omega f(y)\mu(dy) \quad \text{in } L^2(\Omega, \mu), \tag{4.1}$$

for all $f \in L^2(\Omega, \mu)$.

Indeed, since the function $t \to \varphi(t) = \int_\Omega (T(t)f)^2 \mu(dx)$ is nonincreasing and bounded, there exists the limit $\lim_{t\to+\infty} \varphi(t) = \lim_{t\to+\infty} \langle T(2t)f, f\rangle_{L^2(\Omega,\mu)}$. By a standard arguments it follows that there exists a symmetric nonnegative operator $Q \in \mathcal{L}(L^2(\Omega, \mu))$ such that

$$\lim_{t\to+\infty} T(t)f = Qf, \quad f \in L^2(H, \mu).$$

On the other hand, using the Mean Ergodic Theorem in Hilbert space (see e.g. [11, p. 24]) we get easily

$$\lim_{t\to+\infty} T(t)f = P\Big(\int_0^1 T(s)f ds\Big),$$

where P is the orthogonal projection on the kernel of A. Since the kernel of A consists of the constant functions, (4.1) follows.

From now on we make a strict convexity assumption on U:

$$\exists\, \omega > 0 \text{ such that } x \mapsto U(x) - \omega|x|^2/2 \text{ is convex.} \tag{4.2}$$

This will allow us to prove further properties for $T(t)$, through Poincaré and Log–Sobolev inequalities.

If (Λ, m) is any measure space and $u \in L^1(\Lambda, m)$ we set

$$\overline{u}_m = \int_\Lambda u(x)m(dx). \tag{4.3}$$

Proposition 4.3 *Let the assumptions of theorem 3.4 and (4.2) hold. Then*

$$\int_\Omega |u(x) - \overline{u}_\mu|^2 \mu(dx) \le \frac{1}{2\omega} \int_\Omega |Du(x)|^2 d\mu(dx), \quad u \in H^1(\Omega, \mu), \tag{4.4}$$

and

$$\int_\Omega u^2(x)\log(u^2(x))\mu(dx) \le \frac{1}{\omega} \int_\Omega |Du(x)|^2 \mu(dx) + \overline{u^2}_\mu \log(\overline{u^2}_\mu), \ \ u \in H^1(\Omega, \mu). \tag{4.5}$$

Proof — Let $u \in C_0^\infty(\mathbb{R}^N)$ have support in Ω. Let U_α be the Moreau-Yosida approximations of U, and set as usual $\nu_\alpha(dx) = (\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx)^{-1} e^{-2U_\alpha(x)}dx$. Since $x \mapsto U_\alpha(x) - \omega(1-\alpha)|x|^2$ is convex in the whole $\mathbb{R}^N$, by theorem 2.2 we have, for $\alpha \in (0, 1)$,

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}_\alpha|^2 \nu_\alpha(dx) \le \frac{1}{2\omega(1-\alpha)} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx), \tag{4.6}$$

(where $\overline{u}_\alpha$ stands for $\overline{u}_{\nu_\alpha}$) and

$$\int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\nu_\alpha(dx) \le \frac{1}{\omega(1-\alpha)} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx) + \overline{u^2}_\alpha \log(\overline{u^2}_\alpha). \tag{4.7}$$

Since

$$\lim_{\alpha\to 0} U_\alpha(x) = \begin{cases} U(x) & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega, \end{cases}$$

then $\overline{u}_\alpha$ goes to $\overline{u}_\mu = \int_\Omega u(x)\mu(dx)$, $\overline{u^2}_\alpha$ goes to $\overline{u^2}_\mu$ as α goes to 0, and letting α go to 0 in (4.6), (4.7) we obtain that u satisfies (4.4) and (4.5). Since $C_0^\infty(\Omega)$ is dense in $H^1(\Omega, \mu)$, the statement follows. □

Proposition 4.3 yields other properties of $T(t)$, listed in the next corollary. The proof is identical to the proof of [5, cor. 4.3], and we omit it.

Corollary 4.4 *Let the assumptions of theorem 3.4 and (4.2) hold. Then* 0 *is a simple isolated eigenvalue of* A*. The rest of the spectrum,* $\sigma(A)\setminus\{0\}$ *is contained in* $(-\infty,-\omega]$*, and*

$$\|T(t)u-\overline{u}_\mu\|_{L^2(\Omega,\mu)}\leq e^{-\omega t}\|u-\overline{u}_\mu\|_{L^2(\Omega,\mu)},\ \ u\in L^2(\Omega,\mu),\ t>0. \tag{4.8}$$

Moreover we have

$$\|T(t)\varphi\|_{L^{q(t)}(\Omega,\mu)}\leq\|\varphi\|_{L^p(\Omega,\mu)},\quad p\geq 2,\ \varphi\in L^p(\Omega,\mu), \tag{4.9}$$

where

$$q(t)=1+(p-1)e^{2\omega t},\quad t>0. \tag{4.10}$$

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