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ON A CLASS OF PARABOLIC FREE BOUNDARY PROBLEMS

Alessandra Lunardi
Dipartimento di Matematica, Università di Parma
Via D'Azeglio 85/A, 43100 Parma, Italy
E-mail: lunardi@unipr.it, www: http://math.unipr.it/∼lunardi

Abstract. We describe several results and open problems about a class of parabolic free boundary equations, arising mainly as mathematical models in combustion theory.

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1 Introduction

My talk is about a class of parabolic free boundary problems arising as mathematical models in combustion theory. Their common feature is the fact that the velocity of the free (unknown) boundary is not given explicitly as a function of the other unknowns. The simplest significant example is the free boundary heat equation,

$$\begin{cases} u_t = \Delta u, & t > 0, \; x \in \Omega_t, \\ \\ u = 0, \;\; \dfrac{\partial u}{\partial \nu} = -1, & t > 0, \; x \in \partial\Omega_t, \end{cases} \tag{1}$$

where the unknowns are the open sets $\Omega_t \subset \mathbb{R}^N$ for $t > 0$, and the real valued function u, defined for $t > 0$ and $x \in \Omega_t$. The initial set Ω_0 and the initial function $u(0, \cdot) = u_0 : \Omega_0 \mapsto \mathbb{R}$ are given data.

These problems are well understood in space dimension $N = 1$, in which case a satisfactory theory is available since many years, see e.g. the review paper [22]. On the contrary, in the multidimensional case many of the basic questions have not been answered yet. For instance, wellposedness of the initial value problem for (1) is still an open problem. In particular, there are not uniqueness results for the classical solution to (1) for general initial data.

Many results for (1) can be extended to evolution problems of the type

$$\begin{cases} u_t(t,x) = \mathcal{L}u(t,x) + f(t,x,u(t,x),Du(t,x)), \;\; t \geq 0, \; x \in \Omega_t, \\ \\ u(t,x) = g_0(t,x), \;\; t \geq 0, \; x \in \partial\Omega_t, \\ \\ \dfrac{\partial u}{\partial \nu}(t,x) = g_1(t,x), \;\; t \geq 0, \; x \in \partial\Omega_t, \\ \\ u(0,x) = u_0(x), \;\; x \in \Omega_0, \end{cases} \tag{2}$$

where the boundary data g_0, g_1 satisfy the transversality condition

$$\frac{\partial g_0}{\partial \nu}(0,x) \neq g_1(0,x), \;\; x \in \partial\Omega_0. \tag{3}$$

Again, the unknowns are the open sets $\Omega_t \subset \mathbb{R}^N$ for $t > 0$, and the function u. The data are: the second order elliptic operator $\mathcal{L} = \sum_{i,j=1}^N a_{ij}(t,x)D_{ij}$, the functions $f : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, g_0, $g_1 : [0,T] \times \mathbb{R}^N \mapsto \mathbb{R}$, the (possibly unbounded) initial set $\Omega_0 \subset \mathbb{R}^N$, and the initial function $u_0 : \overline{\Omega}_0 \mapsto \mathbb{R}$.

Problem (1) is motivated by models in combustion theory ([8, 23]) for equidiffusional premixed flames, where $u = \lambda(T^* - T)$, T is the temperature, T^* is the temperature of the flame, and $\lambda > 0$ is a normalization factor. It can be seen as the high activation energy limit of the regularizing problems $u_t = \Delta u - \beta_\epsilon(u)$ in $[0,T] \times \mathbb{R}^N$, where $T > 0$ is fixed and $\beta_\epsilon(s) = \beta_1(s/\epsilon)/\epsilon$ is supported in a small interval $[0, \epsilon]$. In [9], Caffarelli and Vazquez used this regularization to prove existence of global weak solutions to (1) for C^2 initial data

(Ω_0, u_0), under suitable geometric assumptions on u_0. Such solutions may not be unique, and enjoy some regularity properties. The free boundary is locally Lipschitz continuous, and u is 1/2-Hölder continuous with respect to time, Lipschitz continuous with respect to the space variables. This regularity is in some sense optimal, as the example of self-similar solutions show.

Self-similar solutions are solutions of the type

$$u(t,x) = (T-t)^\alpha f(\eta)\,,\ \eta = \frac{|x|}{(T-t)^\beta},\ \ \Omega_t = \{|x| < r(T-t)^\beta\},\ \ 0 \le t < T \tag{4}$$

with $f : [0, +\infty) \mapsto \mathbb{R}$, $r > 0$. It is not hard to see that the couple (u, Ω_t) defined in 4 is a solution to (1) if and only $\alpha = \beta = 1/2$, and the function $g(x) = f(|x|)$ is an eigenfunction of an Ornstein-Uhlenbeck type operator in the ball $B(0, r)$,

$$\Delta g - \frac{1}{2}\langle x, Dg(x)\rangle + \frac{1}{2} g = 0,$$

with two boundary conditions, $g = 0$, $\partial g/\partial\nu = -1$. In other words, f has to verify

$$f''(\eta) + \frac{N-1}{\eta} f'(\eta) + \frac{1}{2} f(\eta) = \frac{1}{2}\eta f'(\eta) \quad \text{for } 0 < \eta \le r\,; f'(0) = f(r) = 0\ ,\ f'(r) = -1\,. \tag{5}$$

In [9] it was proved that there exist a unique $r > 0$ and a unique C^2 function $f : [0, r] \mapsto \mathbb{R}$ satisfying (5) and such that $f(\eta) > 0$ for $0 \le \eta < r$. Moreover f is analytic. But, in spite of the regularity of f, at the extintion time T the solution u is not better than 1/2-Hölder continuous in time and Lipschitz continuous in space.

I will describe a completely different approach, developed in collaboration with C.-M. Brauner, J. Hulshof, O. Baconneau, that leads to existence, uniqueness, and smoothness for $t > 0$ of local regular solutions (with u bounded, in the case of unbounded domains) to problem (1), and more generally to problem (2).

It consists in transforming problem (1) into a fully nonlinear parabolic problem in the fixed domain Ω_0 for an auxiliary unknown w, for which the usual techniques of fully nonlinear parabolic problems in Hölder spaces (see e.g. [19, Ch. 8]) give a local existence and uniqueness result. Coming back to (1), for Ω_0 with $C^{3+\alpha+k}$ boundary and $u_0 \in C^{3+\alpha+k}(\overline{\Omega}_0)$, $0 < \alpha <$, $k = 0, 1$, we obtain a local solution with $\partial\Omega_t \in C^{2+\alpha+k}$ and $u(t, \cdot)$ in $C^{2+\alpha+k}(\overline{\Omega}_t)$.

The coordinate transformation that fixes the boundary is rather natural. Set $\Gamma = \partial\Omega_0$, and for $a > 0$ define the map

$$X : \Gamma \times [-a, a] \to \mathbb{R}^N, \quad X(\xi', r) = \xi' + r\nu(\xi'). \tag{6}$$

If a is sufficiently small, then (6) is a diffeomorphism to a compact neighborhood $\mathcal{R}$ of Γ. In $\mathcal{R}$ every ξ can be written in a unique way as $\xi = X(\xi', r)$ with $\xi' \in \Gamma$ and $r \in [-a, a]$. So, $\xi' = \xi'(\xi)$ is the nearest point to ξ in Γ, and $r = r(\xi)$ is the signed distance from ξ to Γ.

We look for Ω_t close to Ω in some time interval I, in the sense that its boundary is sought as

$$\partial\Omega_t = \{x = \xi' + s(t, \xi')\nu(\xi'),\ \xi' \in \Gamma\},$$

where $s : \Gamma \times I \to [-a, a]$ is one of the unknowns of the problem.

Then we extend the vector field

$$\Phi(t,\xi) = s(t,\xi)\nu(\xi), \;\; \xi \in \Gamma, \tag{7}$$

to the whole of $\mathbb{R}^N$ in a standard way, by setting

$$\Phi(t,\xi) = \begin{cases} \alpha(r)s(t,\xi')\nu(\xi') & \text{if } \xi \in \mathcal{R}, \\ \\ 0 \quad \text{otherwise} \;, \end{cases} \tag{8}$$

where $r = r(\xi)$, $\xi' = \xi'(\xi)$, and $\alpha : \mathbb{R} \mapsto [0,1]$ is a smooth mollifier which is equal to 1 near 0 and has compact support in $(-a, a)$.

The extension Φ is used now to transform (1) to a problem on the fixed domain Ω_0. We define the coordinate transformation

$$x = \xi + \Phi(t,\xi), \tag{9}$$

which differs from the identity only in a small neighborhood of Γ, and maps Ω_0 onto Ω_t.

After the change of coordinates, we get a Cauchy problem for the couple $(s, \widetilde{u})$ where $\widetilde{u}(t,\xi) = u(t,x)$ is again the function u in the new coordinates,

$$\begin{cases} \widetilde{u}_t - \langle D\widetilde{u}, (I + {}^tD\Phi)^{-1}\Phi_t \rangle = \mathcal{A}(s)\widetilde{u}, & t > 0, \; x \in \overline{\Omega}_0, \\ \\ \widetilde{u} = 0, \;\; \mathcal{B}(s)\widetilde{u} = -1, & t > 0, \; x \in \Gamma, \\ \\ s(0,\cdot) = 0, \;\; \widetilde{u}(0,\cdot) = u_0, & x \in \Gamma, \\ \\ \widetilde{u}(0,\cdot) = u_0, & x \in \overline{\Omega}_0. \end{cases} \tag{10}$$

Here $\mathcal{A}(s)$ is the Laplacian in the new coordinates, and $\mathcal{B}(s)$ is the normal derivative in the new coordinates.

System (10) still has to be decoupled. To this aim, we introduce a new unknown w by splitting $\widetilde{u}$ as

$$\widetilde{u}(t,\xi) = u_0(\xi) + \langle Du_0(\xi), \Phi(t,\xi) \rangle + w(t,\xi). \tag{11}$$

At $t = 0$ we have $\widetilde{u}(0,\xi) = u_0(\xi)$, $\Phi(0,\cdot) \equiv 0$, so that

$$w(0,\xi) = 0, \;\; \xi \in \overline{\Omega}_0. \tag{12}$$

(11) allows to get s in terms of w thanks to the boundary condition $u = 0$ at $\partial\Omega_t$, which gives

$$s(t,\xi) = w(t,\xi), \;\; t \geq 0, \; \xi \in \Gamma, \tag{13}$$

so that

$$\Phi(t,\xi) = w(t,\xi')\widetilde{\nu}(\xi), \;\; t \geq 0, \; \xi \in \overline{\Omega}_0, \tag{14}$$

where $\widetilde{\nu}(\xi)$ is the extension of the normal vector field in formula (8): $\widetilde{\nu}(\xi) = \alpha(r)\nu(\xi')$ if $\xi \in \mathcal{R}$, $\widetilde{\nu}(\xi) = 0$ otherwise. Replacing (14) in (10) we get

$$w_t = \mathcal{F}_1(\xi, w, Dw, D^2w) + \mathcal{F}_2(\xi, w, Dw)s_t, \;\; t \geq 0, \; \xi \in \overline{\Omega}_0, \tag{15}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ are obtained respectively from $\mathcal{A}(s)\widetilde{u} = \mathcal{A}(s)(u_0 + \langle Du_0, \Phi\rangle + w)$ and from $\langle D\widetilde{u}, (I + {}^tD\Phi)^{-1}\Phi_t\rangle - \langle Du_0, \Phi_t\rangle$, replacing $\Phi = w(t, \xi')\widetilde{\nu}(\xi)$.

Equation (15) still contains s_t, that we eliminate using again the identity $s = w$ at the boundary which gives $s_t = w_t$. Replacing in (15) for $\xi \in \Gamma$ we get

$$s_t(1 - \mathcal{F}_2(\xi, w, Dw)) = \mathcal{F}_1(\xi, w, Dw, D^2w), \ \ t \geq 0, \ \xi \in \Gamma.$$

At $t = 0$ we have $w \equiv 0$, and $\mathcal{F}_2$ vanishes at $(\xi, 0, 0)$, so that, at least for t small, $\mathcal{F}_2(\cdot, w(t, \cdot), Dw(t, \cdot))$ is different from 1 and we get s_t in terms of w,

$$s_t = \mathcal{F}_3(\xi, w, Dw, D^2w) = \frac{\mathcal{F}_1(\xi, w, Dw, D^2w)}{1 - \mathcal{F}_2(\xi, w, Dw)}, \ \ t \geq 0, \ \xi \in \Gamma,$$

which, replaced in (15), gives the final equation for w,

$$w_t = \mathcal{F}(w)(\xi), \ \ t \geq 0, \ \xi \in \overline{\Omega}_0,$$

where

$$\mathcal{F}(w)(\xi) = \mathcal{F}_1(\xi, w, Dw, D^2w) + \mathcal{F}_2(\xi, w, Dw)\mathcal{F}_3(\xi, w, Dw, D^2w).$$

Note that $\mathcal{F}(0)(\xi) = \Delta u_0(\xi)$, and $\mathcal{F}(w)(\xi) = \Delta w + \Delta u_0(\xi)$ if ξ is far from the boundary Γ. Even near the boundary, the equation for w is parabolic for w small, in the sense that the linear part of $\mathcal{F}$ side near $w \equiv 0$ is still Δw plus a first order differential operator applied to w. But the nonlinear part of $\mathcal{F}$ involves second order space derivatives of w in a nonlocal way, through their traces at the boundary. Moreover, $\mathcal{F}$ depends on u_0 through its derivatives up the third order, due to the splitting (11) that contains the first order derivatives of u_0.

The boundary condition for w comes from the boundary condition $\partial u/\partial n = -1$ in (1). We get

$$\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1}D(u_0 + \langle Du_0, \Phi\rangle + w\rangle + |(I + {}^tD\Phi)^{-1}\nu| = 0, \tag{16}$$

which gives

$$\mathcal{G}(w)(\xi) = 0, \ \ t \geq 0, \ \xi \in \Gamma,$$

when we replace $\Phi = w(t, \xi')\widetilde{\nu}(\xi)$ in (16). The function $\mathcal{G}$ is smooth, it vanishes at $w \equiv 0$, and its linear part near $w \equiv 0$ is

$$\mathcal{B}w := \frac{\partial w}{\partial \nu} + \frac{\partial^2 u_0}{\partial \nu^2}w.$$

Therefore, the final problem for the only unknown w is rewritten as

$$\begin{cases} w_t = \mathcal{F}(w), \ \ t \geq 0, \ \xi \in \overline{\Omega}_0, \\ \\ \mathcal{G}(w) = 0, \ \ t \geq 0, \ \xi \in \Gamma, \\ \\ w(0, \cdot) = 0, \ \ \xi \in \overline{\Omega}_0. \end{cases} \tag{17}$$

It may be seen as a fully nonlinear evolution equation, with fully nonlinear boundary condition, which is parabolic near $w \equiv 0$. The usual techniques of parabolic problems may be used to find a local solution of (17). Precisely, there is $R_0 > 0$ such that for every $R \geq R_0$ and for every sufficiently small $T > 0$ problem (17) has a unique solution in the ball $B(0, R) \subset C^{1+\alpha/2,2+\alpha}([0, T] \times \overline{\Omega}_0)$.

Now we come back to the original problem (1). Recalling that $s(t, \xi) = w(t, \xi)$ for each $t \in [0, T]$, $\xi \in \partial\Omega$, we can define Γ_t. Of course s has the same regularity of w, i.e. it is in $C^{1+\alpha/2,2+\alpha}([0, T] \times \Gamma_0)$. Then we define $\widetilde{u}$ through (11), where Φ is given by (8). Again, $\widetilde{u}$ has the same regularity of w. As a last step we define u through the change of coordinates, $u(t, x) = \widetilde{u}(t, \xi)$ where $x = \xi + \Phi(t, \xi)$. This leads to loss of regularity: starting with initial data in $C^{3+\alpha}$ we get a local solution with $C^{2+\alpha}$ space regularity. The final result, dealing also with further regularity (see [2]), is the following.

Theorem 1.1 *Let $\Omega_0 \subset \mathbb{R}^N$ be a nonempty bounded open set with $C^{3+\alpha}$ boundary Γ_0, and let $u_0 \in C^{3+\alpha}(\overline{\Omega}_0)$ satisfy the compatibility conditions $u_0 = 0$, $\partial u_0/\partial n = -1$ at Γ_0. Then there is $T > 0$ such that problem (1) has a solution (Ω_t, u) such that the $(N+1)$-dimensional hypersurface $\mathcal{S} = \{(t, x) : \ 0 \leq t \leq T, \ x \in \Gamma_t\}$ and each $\Gamma_t = \partial\Omega_t$ are of class $C^{1+\alpha/2,2+\alpha}$, and the function $u : \{(t, x); \ 0 \leq t \leq \delta, \ x \in \overline{\Omega}_t\} \mapsto \mathbb{R}$ is of class $C^{1+\alpha/2,2+\alpha}$.*

If in addition Γ_0 and u_0 are in $C^{4+\alpha}$, and the further compatibility condition $\mathcal{B}(\Delta u_0) = 0$ at Γ_0 holds, then $\mathcal{S}$ and each Γ_t are of class $C^{3/2+\alpha/2,3+\alpha}$, and the function u is of class $C^{3/2+\alpha/2,3+\alpha}$. Moreover, the couple (Ω_t, u) is the unique solution with such regularity properties.

We are not able to show that the regularity of the initial datum is preserved throughout the evolution. Although this is natural for global weak solutions, as the example of self-similar solutions shows, it not satisfactory for local classical solutions. One could expect that in a small time interval the solution remains at least as regular as the initial datum; indeed this is what happens in other free boundary problems of parabolic type (e.g. [10, 11] with boundary conditions of Stefan type), and in problem (2) for special initial data.

An interesting situation in which there is no loss of regularity, at least in small time intervals, is the case of initial data near special smooth solutions, such as stationary solutions, self-similar solutions, travelling waves. In particular, $C^{2+\alpha}$ initial data near any smooth stationary solution (Ω, U) of (2), with bounded Ω, were considered in the paper [3], where we studied stability of smooth stationary solutions, establishing a linearized stability principle for (2) in the time independent case. Since self-similar solutions to (1) become stationary solutions to a problem of the type (2) after a suitable change of coordinates, we could also consider initial data for (1) near self-similar solutions.

Concerning the unbounded domain case, stability of planar (i.e., depending only on one space variable) travelling waves is of physical interest. However, while stability under one-dimensional perturbations is relatively easy (see e.g. [13, 6]), stability for genuinely multidimensional perturbations comes out to be a rather complicated problem. See [4] for the heat equation, and [7, 5, 17, 18] for other free boundary problems of this type.

Another example without loss of regularity was considered by Andreucci and Gianni in [1], where a two-phase version of (2) was studied in a strip for $C^{2+\alpha}$ initial data far from special solutions, satisfying suitable monotonicity conditions, i.e. u_0 was assumed to be strictly monotonic in the direction orthogonal to the strip. This allowed them to perform the classical procedure (already used in the Stefan problem and also in a two-dimensional version of problem (2), see [21]) of taking u as new independent variable for small t.

We describe here the approach to stability of self-similar solutions to (1) developed in [3]. We introduce self-similar variables,

$$\hat{x} = \frac{x}{(T-t)^{\frac{1}{2}}}, \quad \hat{t} = -\log(T-t), \tag{18}$$

and we set

$$\hat{u}(\hat{x}, \hat{t}) = \frac{u(x,t)}{(T-t)^{\frac{1}{2}}}, \quad \hat{\Omega}_{\hat{t}} = \{\hat{x} : x \in \Omega_t\}. \tag{19}$$

Omitting the hats, we arrive at

$$\begin{cases} u_t = \Delta u - \dfrac{1}{2}\langle x, Du\rangle + \dfrac{1}{2}u, \quad t > 0,\ x \in \Omega_t, \\[2ex] u = 0, \quad \dfrac{\partial u}{\partial n} = 1, \quad t > 0,\ x \in \partial\Omega_t. \end{cases} \tag{20}$$

The self-similar solution defined in (4) is transformed by (18)–(19) into a stationary solution

$$U(x) = f(|x|),\ \Omega = \{x \in \mathbb{R}^N : |x| < r\}, \tag{21}$$

of (20). From now on we proceed as before: we change variables through the isomorphism (9), taking now as reference domain the stationary set Ω instead of the initial set Ω_0, we set $\tilde{u}(t,\xi) = u(t,x) - U(x)$, we define w by the splitting (11) and we arrive at a final equation for w in the fixed domain $\Omega = B(0,r)$,

$$\begin{cases} w_t = \Delta w - \dfrac{1}{2}\langle \xi, Dw\rangle + \dfrac{w}{2} + \phi(w, Dw, D^2 w), \quad t \geq 0,\ \xi \in \overline{\Omega}, \\[2ex] \dfrac{\partial w}{\partial \nu} + \Big(\dfrac{N-1}{r} - \dfrac{r}{2}\Big) w = \psi(w, Dw), \quad t \geq 0,\ \xi \in \partial\Omega, \\[2ex] w(0,\xi) = w_0(\xi), \quad \xi \in \overline{\Omega}, \end{cases} \tag{22}$$

where ϕ and ψ are smooth and quadratic near $w \equiv 0$; in particular, they vanish at $w \equiv 0$. ϕ is still nonlocal in the second order derivatives. The initial datum $w_0 = \tilde{u}_0 - U - \langle DU, \Phi(0,\xi)\rangle$ does not vanish, but it is small if u_0 is close to the stationary U.

Again, techniques of fully nonlinear parabolic problems may be used in problem (22), showing a local existence result, and precisely

Theorem 1.2 *For every $T > 0$ and $\alpha \in (0,1)$ there are R, $\rho > 0$ such that (22) has a solution $w \in C^{1+\alpha/2, 2+\alpha}([0,T] \times \overline{\Omega})$ provided $\|w_0\|_{C^{2+\alpha}(\overline{\Omega})} \leq \rho$ and $\partial w_0/\partial\nu + ((N-1)/r) - (r/2))w_0 = \psi(w_0, Dw_0)$. Moreover w is the unique solution in $B(0,R) \subset C^{1+\alpha/2, 2+\alpha}([0,T] \times \overline{\Omega})$.*

We may go on in the analysis, applying the principle of linearized stability theorem proved in [3] for problems of the type

$$\begin{cases} w_t = \mathcal{A}w + \phi(w, Dw, D^2w), \quad t \geq 0, \ \xi \in \overline{\Omega}, \\ \\ \mathcal{B}w = \psi(w, Dw), \quad t \geq 0, \ \xi \in \partial\Omega, \end{cases} \tag{23}$$

where Ω is a bounded open set with regular boundary, $\mathcal{A}$ is any second order elliptic operator and $\mathcal{B}$ is any nontangential first order differential operator with good coefficients, ϕ, ψ are smooth enough and quadratic at $w \equiv 0$.

Denoting by A the realization of $\mathcal{A}$ in $C^\alpha(\overline{\Omega})$ with homogeneous boundary condition $\mathcal{B}w = 0$, if all the eigenvalues of A have negative real part then the stationary solution $w = 0$ of problem (23) is stable with respect to the $C^{2+\alpha}(\overline{\Omega})$ norm. If A has some eigenvalues with positive real part, then the null solution is unstable in $C^{2+\alpha}(\overline{\Omega})$. In this case there exists a finite dimensional local unstable manifold, consisting of all the small (in the $C^{2+\alpha}(\overline{\Omega})$ norm) initial data w_0 satisfying the compatibility condition $\mathcal{B}w_0 = \psi(w_0, Dw_0)$ and such that (23) has a backward solution going to 0 in $C^{2+\alpha}(\overline{\Omega})$ as $t \to -\infty$. Denoting by P the spectral projection associated to the eigenvalues of A with positive real part, the dimension of the unstable manifold is equal to the dimension of $P(C^\alpha(\overline{\Omega}))$.

In our case, we have

$$\mathcal{A}v = \Delta v - \frac{1}{2}\langle x, Dv\rangle + \frac{1}{2}v,$$

$$\mathcal{B}v = \frac{\partial v}{\partial \nu} + \Big(\frac{N-1}{r} - \frac{r}{2}\Big)v,$$

and the spectrum of A consists of the semisimple eigenvalues 1, 1/2 plus a sequence of negative eigenvalues; moreover, the eigenspace with eigenvalue 1 is one-dimensional, the eigenspace wih eigenvalue 1/2 has dimension N.

It follows that the null solution of (22) is unstable in $C^{2+\alpha}(\overline{\Omega})$, and therefore the self-similar solution of the original problem (1) is unstable. This is not surprising, because the original problem is invariant under translations in x and t; if we apply a small shift to (21), we obtain another self-similar solution which is transformed by (18)–(19) into a solution which starts close to (21) but moves far from it. Therefore, the local unstable manifold of (21) must contain the images under (18) of shifts in space and time of (21), that are given by

$$\sqrt{1+\epsilon_2 e^t}\, U\Big(\frac{x - \epsilon_1 e^{\frac{1}{2}t}}{\sqrt{\epsilon_2 e^t + 1}}\Big), \tag{24}$$

with $\epsilon_1 \in \mathbb{R}^N$ and $\epsilon_2 \in \mathbb{R}$. Since the local unstable manifold has to be $(N+1)$-dimensional, then it consists only of the images of (24) under the transformation (18). However, all the orbits in the unstable manifold have the same selfsimilar profile, so that the equilibrium (21) looks stable even if it unstable. Roughly speaking, the profile itself is stable.

Together with loss of regularity, the other big question about problem (2) is uniqueness of the classical solution. Indeed, the uniqueness results available up to now concern only particular situations, such as radially symmetric solutions of (1), studied in [12], and solutions in cylinders or strips, for initial data which are monotonic in the direction of the axis of the cylinder (see [21] in dimension $N = 2$ and [15] in any dimension), or in the direction orthogonal to the strip (see [1], for the two-phase case). The above mentioned papers [3, 4] give also uniqueness results in the parabolic Hölder space $C^{1+\alpha/2,2+\alpha}$, but only for solutions close to the special solutions considered. The paper [2] gives uniqueness of very regular solutions, in the space $C^{3/2+\alpha/2,3+\alpha}$.

A detailed account of the theory up to 2002 may be found in the lecture notes [20].

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