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DIRICHLET BOUNDARY CONDITIONS FOR ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT

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ABSTRACT. We study the realisation A of the operator $\mathcal{A} = \Delta - \langle D\Phi, D\cdot\rangle$ in $L^2(\Omega,\mu)$ with Dirichlet boundary condition, where Ω is a possibly unbounded open set in $\mathbb{R}^N$, Φ is a semi-convex function and the measure $d\mu(x) = \exp(-\Phi(x))\,dx$ lets $\mathcal{A}$ be formally self-adjoint. The main result is that $A : D(A) = \{u \in H^2(\Omega,\mu) : \langle D\Phi, Du\rangle \in L^2(\Omega,\mu),\ u = 0 \text{ at } \partial\Omega\}$ is a dissipative self-adjoint operator in $L^2(\Omega,\mu)$.

1. Introduction

Second-order elliptic operators with unbounded coefficients in $\mathbb{R}^N$ or in unbounded subsets of $\mathbb{R}^N$ have been the object of several recent papers; see e.g. [2, 3, 8, 1, 9]. Since the very first studies it was apparent that operators of the type $\mathcal{A}u = \operatorname{Tr} Q(x)D^2u(x) + \langle F(x)Du(x)\rangle$, without potential terms, are not well settled in L^p spaces with respect to the Lebesgue measure, unless the matrix Q and the vector F satisfy very severe restrictions, such as global Lipschitz continuity (see [9, 7]). It is much more natural and fruitful to work in suitably weighted L^p spaces; see [3, 8]. This is what we do in this paper. We consider the operator $\mathcal{A}$ defined by

$$\mathcal{A}u = \Delta u - \langle D\Phi, Du\rangle = e^{\Phi}\operatorname{div}(e^{-\Phi}Du), \tag{1}$$

where $\Phi : \mathbb{R}^N \to \mathbb{R}$ is a C^2 semi-convex function, i.e., there is $\alpha \ge 0$ such that

$$\Phi_\alpha(x) := \Phi(x) + \alpha|x|^2/2 \quad \text{is convex,} \tag{2}$$

or, equivalently, the matrix $D^2\Phi(x) + \alpha I$ is nonnegative definite at each x. We emphasize that we do not assume any growth restriction on Φ or on its derivatives. The natural weight is then $\rho(x) = e^{-\Phi(x)}$ because, as it is easy to check, if Ω is any open set in $\mathbb{R}^N$,

$$\int_\Omega \mathcal{A}u\, v\, d\mu = -\int_\Omega \langle Du, Dv\rangle\, d\mu, \quad \forall u,\ v \in C_0^\infty(\Omega),$$

if $\mu(dx) = e^{-\Phi(x)}dx$, so that $\mathcal{A}$ is associated to a nice Dirichlet form and it is formally self-adjoint in $L^2(\Omega,\mu)$. The aim of this paper is to study the realisation of $\mathcal{A}$ in $L^2(\Omega,\mu)$ with Dirichlet boundary condition, i.e., the operator

$$A : D(A) = \{u \in H^2(\Omega,\mu) \cap H^1_0(\Omega,\mu) :\ \mathcal{A}u \in L^2(\Omega,\mu)\} \to L^2(\Omega,\mu); \quad Au = \mathcal{A}u. \tag{3}$$

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Note that for $u \in H^2(\Omega,\mu)$, condition $\mathcal{A}u \in L^2(\Omega,\mu)$ is equivalent to requiring $\langle D\Phi, Du\rangle \in L^2(\Omega,\mu)$. Our main result is that A is self-adjoint and dissipative, provided $\partial\Omega$ is smooth enough and the normal derivative $\partial\Phi/\partial n$ is bounded from above on $\partial\Omega$. A lot of consequences then follow; see Section 3.

A natural approach to the study of $\mathcal{A}$ consists in defining an operator $A_0 : C_0^\infty(\Omega) \to L^2(\Omega,\mu)$, $A_0 u = \mathcal{A}u$, in showing that A_0 is closable, and that its closure is self-adjoint and dissipative. But the problem of the characterisation of the domain of the closure still remains. So, we follow a more direct approach, solving the resolvent equation $\lambda u - \mathcal{A}u = f$ for all $\lambda > 0$ and $f \in C_0^\infty(\Omega)$, which is dense in $L^2(\Omega,\mu)$. Proving the existence of a solution to $\lambda u - \mathcal{A}u = f$ that vanishes on $\partial\Omega$ is not hard, thanks to the regularity of the data. Estimates of its $H^1(\Omega,\mu)$-norm, and uniqueness of the solution in $D(A)$, are easy consequences of the integration formula (5) proved in Lemma 2.2 below. Estimating the second-order derivatives of u is much more delicate, and here the assumptions of semi-convexity and of upper boundedness of $\partial\Phi/\partial n$ are used and play a fundamental role.

This paper is in some sense parallel to the paper [3], where the operator $\mathcal{A}$ was studied in the whole space $\mathbb{R}^N$ and in any convex regular open set Ω with Neumann boundary condition. The conclusions of [3] are similar to the ones of the present paper, but the assumptions on Φ and Ω are a bit different, i.e., Φ is just convex, with no further regularity assumption, and Ω is convex, too.

2. The domain of $\mathcal{A}$ with Dirichlet boundary condition

Throughout the paper we assume that Ω is an open set in $\mathbb{R}^N$ with sufficiently smooth (at least C^2) boundary. By $L^2(\Omega)$ and $H^k(\Omega)$, $k \in \mathbb{N}$, we mean the usual L^2 and Sobolev spaces with respect to the Lebesgue measure. The spaces $H^k(\Omega,\mu)$, $k = 1, 2$, are defined as the set of all $u \in H^k_{loc}(\Omega)$ such that the function u and its partial derivatives up to the order k belong to $L^2(\Omega,\mu)$. They are Hilbert spaces with the standard inner products $\langle u, v\rangle = \int_\Omega \big(uv + \sum_{|\alpha|=1}^k D^\alpha u D^\alpha v\big) e^{-\Phi} dx$. $H^1_0(\Omega,\mu)$ is the subspace of $H^1(\Omega,\mu)$ consisting of the functions with null trace on the boundary. By $C^k_b(\mathbb{R}^N)$ we denote the space of bounded functions with bounded derivatives up to order k. We say that $\partial\Omega$ is *uniformly* C^k if there exist $r > 0$, $m \in \mathbb{N}$ and a (at most countable) family $\{B_j = B_r(x_j), j \in J\}$ of balls covering $\partial\Omega$ with at most m overlapping and C^k-diffeomorphisms $\phi_j : B_j \to B_1(0)$ such that $\phi_j(B_j \cap \Omega) = B_1(0) \cap \{y_N > 0\}$ and $\sup_j \|\phi_j\|_{C^k} + \|\phi_j^{-1}\|_{C^k} < \infty$.

Lemma 2.1. *$C_0^\infty(\Omega)$ is dense in $L^2(\Omega,\mu)$ and in $H^1_0(\Omega,\mu)$.*

Proof. Let $u \in L^2(\Omega,\mu)$, or $u \in H^1_0(\Omega)$. Let $\theta : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that $0 \le \theta(x) \le 1$ for each x, $\theta \equiv 1$ in $B(0,1)$, $\theta \equiv 0$ outside $B(0,2)$, and set $u_n(x) = u(x)\theta(x/n)$. Then $u_n \to u$ in $L^2(\Omega,\mu)$. Indeed,

$$\int_\Omega |u_n - u|^2 \, d\mu \le \int_{\{x\in\Omega,\, |x|\ge n\}} |u|^2 \, d\mu$$

which goes to 0 as $n \to \infty$. If $u \in H^1_0(\Omega)$, then $u_n \to u$ in $H^1(\Omega,\mu)$, because $Du_n(x) = \theta(x/n)Du(x) + D\theta(x/n)u(x)/n$. Since each u_n has bounded support, it may be approximated in $L^2(\Omega)$ (respectively, in $H^1(\Omega)$) by a sequence of $C_0^\infty(\Omega)$ functions. Such a sequence also approximates u_n in $L^2(\Omega,\mu)$ (respectively, in $H^1(\Omega,\mu)$) because μ is equivalent to the Lebesgue measure on each compact subset of $\mathbb{R}^N$. □

The realisation A of $\mathcal{A}$ in $L^2(\Omega,\mu)$ with Dirichlet boundary condition is defined by (3). The following integration formulae will be very useful in what follows.

Lemma 2.2. *Let $\psi \in H^1_0(\Omega,\mu)$, $u \in H^2(\Omega,\mu)$ be such that $\mathcal{A}u \in L^2(\Omega,\mu)$. Then*

$$\int_\Omega \mathcal{A}u\,\psi\,d\mu = -\int_\Omega \langle Du, D\psi\rangle\,d\mu. \tag{4}$$

More generally, if $\psi \in H^1(\Omega,\mu)$ and $u \in H^2(\Omega,\mu)$ is such that $\mathcal{A}u \in L^2(\Omega,\mu)$, then

$$\int_\Omega \mathcal{A}u\,\psi\,d\mu = -\int_\Omega \langle Du, D\psi\rangle\,d\mu + \int_{\partial\Omega} \frac{\partial u}{\partial n}\psi e^{-\Phi} d\sigma, \tag{5}$$

where $d\sigma$ denotes the usual Lebesgue surface measure, the last integral is understood as $\lim_{R\to\infty}\int_{\partial\Omega} \frac{\partial u}{\partial n}\,\psi\theta(x/R)e^{-\Phi}d\sigma$, and θ is the function used in Lemma 2.1.

Proof. The proof of (4) is immediate if $\psi \in C_0^\infty(\Omega)$, and the statement follows by approximation in the general case. Equality (5) is obtained by approximating ψ by $\psi(x)\theta(x/R)$. □

Let us state a consequence of Lemma 2.2.

Lemma 2.3. *If $\partial\Omega$ is uniformly C^2 and $u \in H^2(\Omega,\mu)$ is such that $\mathcal{A}u \in L^2(\Omega,\mu)$, then $\partial u/\partial n$ is in $L^2(\partial\Omega, \exp(-\Phi)\,d\sigma)$. Moreover, there exists $C > 0$ such that for every $\varepsilon \in (0,1)$ the following estimate holds:*

$$\int_{\partial\Omega}\Big(\frac{\partial u}{\partial n}\Big)^2 e^{-\Phi}d\sigma \le \varepsilon\Big(\|\mathcal{A}u\|^2_{L^2(\Omega,\mu)} + \|\,|D^2u|\,\|^2_{L^2(\Omega,\mu)}\Big) + \frac{C}{\varepsilon}\|\,|Du|\,\|^2_{L^2(\Omega,\mu)}.$$

Proof. It is sufficient to take $\psi = \langle Du, \mathcal{N}\rangle$ in (5), where $\mathcal{N}$ is any C^1_b extension to $\mathbb{R}^N$ of the normal vector field n, and then to use the Hölder inequality. □

Lemma 2.2 implies that the operator A is symmetric. In the next theorem we prove that it is self-adjoint if Φ is smooth enough, and

$$\frac{\partial\Phi}{\partial n} \le 0 \quad \text{on } \partial\Omega. \tag{6}$$

Theorem 2.4. *Assume that $\partial\Omega \in C^3$ and that Φ satisfies* (2) *and* (6). *Then $(A, D(A))$ is self-adjoint and dissipative in $L^2(\Omega,\mu)$. Moreover, the map $u \mapsto \langle (D^2\Phi)Du, Du\rangle$ is continuous from $D(A)$ to $L^1(\Omega,\mu)$.*

Proof. We have to show that, for $\lambda > 0$ and $f \in L^2(\Omega,\mu)$, the equation $\lambda u - Au = f$ has a unique solution $u \in D(A)$. Uniqueness is an immediate consequence of Lemma 2.2, taking $\psi = u$ in (5). Concerning existence, we first assume that $f \in C_0^\infty(\Omega)$ and we show that there is a solution $u \in D(A)$ satisfying

$$\begin{cases} \text{(a)} \quad \|u\|_{L^2(\Omega,\mu)} \le \dfrac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \\ \text{(b)} \quad \|\,|Du|\,\|_{L^2(\Omega,\mu)} \le \dfrac{1}{\sqrt{\lambda}}\|f\|_{L^2(\Omega,\mu)}, \\ \text{(c)} \quad \|\,|D^2u|\,\|_{L^2(\Omega,\mu)} + \|\langle (D^2\Phi_\alpha)Du, Du\rangle\|_{L^1(\Omega,\mu)} \le \Big(2 + \dfrac{\alpha}{\lambda}\Big)\|f\|_{L^2(\Omega,\mu)}, \end{cases} \tag{7}$$

where Φ_α is defined in (2). Using the Lax-Milgram lemma, we find $u \in H^1_0(\Omega,\mu)$ such that

$$\lambda\int_\Omega u\psi\,d\mu + \int_\Omega \langle Du, D\psi\rangle\,d\mu = \int_\Omega f\psi\,d\mu, \quad \forall\psi \in H^1_0(\Omega,\mu).$$

By local elliptic regularity, $u \in H^2_{\rm loc}(\Omega)$ and $\lambda u - \mathcal{A}u = f$. In particular, $\mathcal{A}u \in L^2(\Omega,\mu)$. Again, by classical elliptic regularity,

$$u \in C^{2,\beta}(\Omega \cap B(0,R)) \cap H^3(\Omega \cap B(0,R))$$

for every $R > 0$ and $\beta < 1$.

Now we can prove (7). To prove estimates (a) and (b), we multiply the identity $\lambda u - \mathcal{A}u = f$ by u, we integrate over Ω and we use (4) to get

$$\int_\Omega (\lambda u^2 + |Du|^2)\, d\mu = \int_\Omega fu\, d\mu \le \|f\|_{L^2(\Omega,\mu)} \|u\|_{L^2(\Omega,\mu)}$$

which implies that (a) and (b) hold. To prove (c) we differentiate the equation $\lambda u - \mathcal{A}u = f$ with respect to x_h, $h+1,\dots,N$, and we get

$$\lambda D_h u - \Delta(D_h u) + \langle D(D_h\Phi), Du\rangle + \langle D\Phi, D(D_h u)\rangle = D_h f,$$

that is,

$$\lambda D_h u - \mathcal{A} D_h u + \sum_{k=1}^N D_{hk}\Phi\, D_k u = D_h f.$$

Set $\theta_R(x) = \theta(x/R)$. Multiplying by $\theta_R^2 D_h u$, summing over h, and integrating by parts, from (5) we get, since $u \in H^3(\Omega \cap B(0,R))$ for every R,

$$\int_\Omega \Big\{ \theta_R^2(\lambda |Du|^2 + |D^2u|^2 + \langle D^2\Phi Du, Du\rangle) + 2\sum_{h=1}^N \theta_R \langle D(D_h u), D\theta_R\rangle D_h u \Big\} d\mu \tag{8}$$

$$= \int_{\partial\Omega} \theta_R^2 \sum_{h=1}^N \frac{\partial D_h u}{\partial n} D_h\, u\, e^{-\Phi} d\sigma + \int_\Omega \theta_R^2 \langle Df, Du\rangle\, d\mu.$$

Since f has compact support, for R large enough $\theta_R \equiv 1$ on the support of f. Using (4) again in the last integral, we write it as $-\int_\Omega f(\lambda u - f)\, d\mu$. Thanks to the assumption $D^2\Phi \ge -\alpha I$, we obtain

$$\begin{aligned} &\int_\Omega \theta_R^2(\lambda|Du|^2 + |D^2u|^2 + \langle (D^2\Phi_\alpha)Du, Du\rangle)\, d\mu \\ &\le \int_\Omega \left(\alpha\theta_R^2|Du|^2 + CR^{-1}\theta_R|D^2u|\,|Du| + f(\lambda u - f)\right)\, d\mu \\ &\quad + \int_{\partial\Omega} \theta_R^2 \langle (D^2u)n, Du\rangle\, e^{-\Phi} d\sigma, \end{aligned} \tag{9}$$

for a suitable $C > 0$, independent of R. Using (a) and (b) we get

$$\int_\Omega \left(\alpha\theta_R^2|Du|^2 + f(\lambda u - f)\right)\, d\mu \le \left(2 + \frac{\alpha}{\lambda}\right) \|f\|^2_{L^2(\Omega,\mu)}.$$

Moreover,

$$\int_\Omega CR^{-1}\theta_R|D^2u|\,|Du|\, d\mu \le \frac{C}{2R}\int_\Omega \theta_R^2|D^2u|^2\, d\mu + \frac{C}{2R}\int_\Omega |Du|^2\, d\mu.$$

Let us now show that the boundary integral in (9) is negative. Since $u = 0$ on $\partial\Omega$, we have $\langle Du, \tau\rangle = 0$ and $\langle (D^2u)\tau, \tau\rangle = 0$ for every tangent vector τ to $\partial\Omega$. Then $Du = (\partial u/\partial n)n$ and $\langle (D^2u)n, Du\rangle = \langle (D^2u)n, n\rangle \partial u/\partial n$ at $\partial\Omega$. Therefore

$\Delta u = \operatorname{trace} D^2 u = \langle (D^2 u)n, n\rangle$ at $\partial\Omega$, and the equality $\lambda u - \mathcal{A}u = f$ (which is satisfied also at $\partial\Omega$, since $u \in C^2(\overline{\Omega} \cap B(0,R))$ for every R) yields

$$\Delta u = \frac{\partial \Phi}{\partial n}\frac{\partial u}{\partial n}, \qquad \langle (D^2 u)n, Du\rangle = \frac{\partial \Phi}{\partial n}\left(\frac{\partial u}{\partial n}\right)^2, \qquad \text{at } \partial\Omega,$$

hence

$$\int_{\partial\Omega} \theta_R^2 \langle (D^2 u)n, Du\rangle\, e^{-\Phi}\, d\sigma = \int_{\partial\Omega} \theta_R^2 \frac{\partial \Phi}{\partial n}\Big(\frac{\partial u}{\partial n}\Big)^2 e^{-\Phi}\, d\sigma \le 0,$$

thanks to (6). Thus, we have proved that

$$\int_\Omega \left(1 - \frac{C}{2R}\right)\theta_R^2 |D^2 u|^2 + \theta_R^2 \langle (D^2\Phi_\alpha)Du, Du\rangle\, d\mu \le \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda}\right)\|f\|^2_{L^2(\Omega,\mu)},$$

and statement (c) follows by letting $R \to \infty$.

The general case $f \in L^2(\Omega,\mu)$ is easily handled by approximation. Let $(f_n) \subset C_0^\infty(\Omega)$ be such that $f_n \to f$ in $L^2(\Omega,\mu)$ and let $u_n \in D(A)$ be such that $\lambda u_n - Au_n = f_n$. The above estimates imply that the sequence (u_n) converges to a function u in $H^2(\Omega,\mu)$ and it is readily seen that $u \in D(A)$, $\lambda u - Au = f$ and that (a), (b), and (c) hold. □

Condition (6) can be relaxed assuming some more regularity on $\partial\Omega$.

Theorem 2.5. *Assume that $\partial\Omega \in C^3$ and that it is uniformly C^2. Let Φ be a C^2 function satisfying* (2) *and*

$$\frac{\partial \Phi}{\partial n} \le k \ \text{ at } \partial\Omega, \tag{10}$$

for some $k \in \mathbb{R}$. Then $(A, D(A))$ is self-adjoint and dissipative in $L^2(\Omega,\mu)$. Moreover, the map $u \mapsto \langle (D^2\Phi)Du, Du\rangle$ is continuous from $D(A)$ to $L^1(\Omega,\mu)$.

Proof. The proof is similar to the proof of Theorem 2.4. For $f \in C_0^\infty(\Omega)$, $\lambda > 0$, let $u \in H_0^1(\Omega,\mu)$ be the variational solution of the equation $\lambda u - \mathcal{A}u = f$. As in Theorem 2.4 we get estimates (7)(a), (b) and

$$\begin{aligned} &\int_\Omega \left(1 - \frac{C}{2R}\right)\theta_R^2 |D^2 u|^2\, d\mu + \int_\Omega \theta_R^2 \langle (D^2\Phi_\alpha)Du, Du\rangle\, d\mu \\ &\le \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda}\right)\|f\|^2_{L^2(\Omega,\mu)} + \int_{\partial\Omega} \theta_R^2 \frac{\partial\Phi}{\partial n}\Big(\frac{\partial u}{\partial n}\Big)^2 e^{-\Phi}\, d\sigma. \end{aligned} \tag{11}$$

The boundary integral does not exceed

$$k \int_{\partial\Omega} \theta_R^2 \Big(\frac{\partial u}{\partial n}\Big)^2 e^{-\Phi}\, d\sigma,$$

and it can be estimated as follows (see also Lemma 2.3).

Let us take $\psi = \theta_R^2 \langle Du, \mathcal{N}\rangle$ in (5), where $\mathcal{N}$ is any C_b^1 extension to $\mathbb{R}^N$ of the normal vector field n, so that, using Hölder inequality, we obtain for every $0 < \varepsilon < 1$

$$\int_{\partial\Omega} \theta_R^2 \Big(\frac{\partial u}{\partial n}\Big)^2 e^{-\Phi} d\sigma \le \varepsilon(\|\mathcal{A}u\|^2_{L^2(\Omega,\mu)} + \|\theta_R |D^2 u|\|^2_{L^2(\Omega,\mu)}) + \frac{C}{\varepsilon}\||Du|\|^2_{L^2(\Omega,\mu)}.$$

Since $\mathcal{A}u = \lambda u - f$, writing the last inequality with $\varepsilon k \le 1/2$ and combining it with (11) and with estimates (a), (b), we arrive at

$$\int_\Omega \left(\frac{1}{2} - \frac{C}{2R}\right)\theta_R^2 |D^2 u|^2\, d\mu \quad + \quad \int_\Omega \theta_R^2 \langle (D^2\Phi_\alpha)Du, Du\rangle\, d\mu$$
$$\le \quad \left(2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} + C_1\right)\|f\|^2_{L^2(\Omega,\mu)},$$

with C_1 independent of R. Letting $R \to \infty$ we obtain estimate (7)(c) of Theorem 2.4 (with different constants), and from now on the proof follows the same lines as in Theorem 2.4. □

Remark 2.6. If $D^2\Phi$ is bounded from above, then the mapping $u \mapsto \langle (D^2\Phi)Du, Du\rangle$ is bounded from $H^1(\Omega,\mu)$ to $L^1(\Omega,\mu)$ and the last statement of Theorems 2.4 and 2.5 is obvious. But, if $D^2\Phi$ is not bounded, the statement is not obvious, and it will be used in the next section to obtain a quantitative Poincaré inequality.

We end this section by showing that $D(A)$ can be strictly contained in $H^2(\Omega,\mu) \cap H^1_0(\Omega,\mu)$.

Example 2.7. We construct a convex function $\phi : [0,\infty) \to \mathbb{R}$ such that $e^{-\phi}$ and $x^2 e^{-\phi}$ are in $L^1(0,\infty)$ but $\phi'^2 e^{-\phi} \notin L^1(0,\infty)$. Then $u(x) = x$ belongs to $H^2(\mu) \cap H^1_0(\mu)$ but not to $D(A)$. For simplicity, ϕ will be nonsmooth. However, smooth versions are easily obtained using straightforward arguments.

Let $0 = a_1 < b_1 < a_2 < b_2 < \cdots$ be points in $[0,\infty)$ such that $b_j - a_j = 1$. Set $1/l_j = a_{j+1} - b_j$, $l_1 = 1$ and define $\phi' = 1$ in (a_1, b_1), $\phi' = l_j$ in (b_j, a_{j+1}) and $\phi' = l_{j-1}$ in (a_j, b_j). We have to choose $1 = l_1 < l_2 < \cdots$ in such a way that ϕ satisfies the properties above. First observe that ϕ is convex, $\phi' \ge 1$, hence $\phi(x) \ge x$ and then $e^{-\phi}, x^2 e^{-\phi} \in L^1(0,\infty)$. Moreover, if $x \in (b_j, a_{j+1})$, then $\phi(x) \le j + 1 + \sum_{i=1}^{j-1} l_i$ eand therefore

$$\int_{b_j}^{a_{j+1}} \phi\prime^2 e^{-\phi} dx \ge l_j^2 \exp(-(j+1+\sum_{i=1}^{j-1} l_i)) \ge l_j \exp(-(j+1+\sum_{i=1}^{j-1} l_i)).$$

Choosing (inductively) $l_j = e^{(j+1+\sum_{i=1}^{j-1} l_i)}$ the above integral is bigger than 1, hence, summing over j, ϕ'^2 does not belong to $L^1(\mu)$.

3. Further properties of A

Under the assumptions of either Theorem 2.4 or Theorem 2.5, since the operator A is self-adjoint and dissipative in $L^2(\Omega,\mu)$, it is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega,\mu)$. In this section we prove further properties of $T(t)$ and of A.

The characterisation of the domain of $(-A)^{1/2}$ is a standard consequence of the integration formula (4), as the following proposition shows. Recall that the norm in $H^1_0(\Omega,\mu)$ is given by $\|u\|_{H^1_0(\Omega,\mu)} = \|u\|_{L^2(\Omega,\mu)} + \|Du\|_{L^2(\Omega,\mu)}$.

Proposition 3.1. *The domain of $(-A)^{1/2}$ is $H^1_0(\Omega,\mu)$. Therefore, the restriction of $T(t)$ to $H^1_0(\Omega,\mu)$ is an analytic semigroup in $H^1_0(\Omega,\mu)$.*

Proof. Any $u \in D((-A)^{1/2})$ is the $L^2(\Omega,\mu)$-limit of a sequence of functions $u_n \in D(A) \subset H^1_0(\Omega,\mu)$ which is a Cauchy sequence with respect to the norm $\|u\|_{L^2} + \langle -Au, u\rangle_{L^2}$. From (4) it follows that (Du_n) is a Cauchy sequence in $L^2(\Omega,\mu)$, hence $u \in H^1_0(\Omega,\mu)$. Conversely, let $u \in H^1_0(\Omega,\mu)$ and let $u_n \in C_0^\infty(\Omega) \subset D(A)$ converge to u in $H^1(\Omega,\mu)$. Formula (4) implies that (u_n) is a Cauchy sequence in $D((-A)^{1/2})$, hence $u \in D((-A)^{1/2})$. □

Corollary 3.2. *Under the assumptions of either Theorem* 2.4 *or Theorem* 2.5, $T(t)$ *is a symmetric Markov semigroup, that is, a semigroup of self-adjoint positivity preserving operators in* $L^2(\Omega,\mu)$ *that satisfy* $\|T(t)f\|_\infty \le \|f\|_\infty$ *for each* $f \in L^2(\Omega,\mu)\cap L^\infty(\Omega,\mu)$ *and* $t>0$.

Proof. Since A is self-adjoint, each $T(t)$ is self-adjoint. To prove that each $T(t)$ preserves positivity and that it is a contraction in L^∞, we use the Beurling-Deny criteria; see e.g. [4, Theorems 1.3.2, 1.3.3].

As $D((-A)^{1/2}) = H^1_0(\Omega,\mu)$, then $u \in D((-A)^{1/2})$ implies $|u| \in D((-A)^{1/2})$, and

$$\|(-A)^{1/2}(|u|)\|^2 = \int_\Omega |D(|u|)|^2\, d\mu \le \int_\Omega |Du|^2\, d\mu = \|(-A)^{1/2}u\|^2,$$

so that $T(t)$ is positivity-preserving for all $t>0$. Again, since $D((-A)^{1/2}) = H^1_0(\Omega,\mu)$, if $0 \le u \in D((-A)^{1/2})$, then $u\wedge 1 \in D((-A)^{1/2})$, and

$$\|(-A)^{1/2}(u\wedge 1)\|^2 = \int_\Omega |D(u\wedge 1)|^2\, d\mu \le \int_\Omega |Du|^2\, d\mu = \|(-A)^{1/2}u\|^2.$$

This implies that $\|T(t)f\|_\infty \le \|f\|_\infty$ for each $f \in L^2(\Omega,\mu)\cap L^\infty(\Omega,\mu)$. □

Another immediate consequence of the integration formula (4) is that A is injective: if $u \in D(A)$ and $Au = 0$, then $Au\cdot u = 0$, and integrating over Ω we obtain $Du = 0$ so that u is constant on each connected component of Ω; since u vanishes at $\partial\Omega$, then $u = 0$.

A natural question is now whether 0 is in the resolvent set of A. This is true if $D(A)$ is compactly embedded in $L^2(\Omega,\mu)$, because in this case the spectrum of A consists of a sequence of isolated eigenvalues. But in general $D(A)$ is not compactly embedded in $L^2(\Omega,\mu)$, as the following counterexample shows.

Example 3.3. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be any convex C^2 function such that $\varphi(x) = x$ for $x \ge 0$. Set $\Phi(x,y) = \varphi(x) + y^2$, and let Ω be the half-plane $\{(x,y)\in\mathbb{R}^2 : y>0\}$. Then $D(A)$ is not compactly embedded in $L^2(\Omega,\mu)$.

Proof. Let $\theta \in C_0^\infty(0,\infty)$ be such that $\int_0^\infty (\theta(y))^2\exp(-y^2)dy = 1$, and set for each $n\in\mathbb{N}$, $n\ge 3$,

$$u_n(x,y) = \frac{x^n}{\sqrt{(2n)!}}\,\theta(y), \quad x,\ y \ge 0, \quad u_n(x,y) = 0 \text{ otherwise}.$$

Since $d\mu = \exp(-\varphi(x) - y^2)dx\,dy$, then $\|u_n\|_{L^2(\Omega,\mu)} = 1$ for each n. Moreover,

$$D_x u_n(x,y) = \frac{nx^{n-1}}{\sqrt{(2n)!}}\theta(y), \quad D_y u_n(x,y) = \frac{x^n}{\sqrt{(2n)!}}\theta'(y), \quad x>0,$$

$$D_{xx} u_n(x,y) = \frac{n(n-1)x^{n-2}}{\sqrt{(2n)!}}\,\theta(y), \quad D_{yy} u_n(x,y) = \frac{x^n}{\sqrt{(2n)!}}\,\theta''(y), \quad x>0,$$

and every derivative vanishes for $x\le 0$. Therefore, $u_n \in D(A)$ and $\|Au_n\|_{L^2(\Omega,\mu)}$ is bounded by a constant independent of n. But no subsequence may converge in

$L^2(\Omega,\mu)$ because

$$\|u_n - u_m\|^2_{L^2(\Omega,\mu)} = \int_0^\infty \Big(\frac{x^{2n}}{(2n)!} + \frac{x^{2m}}{(2m)!} - 2\frac{x^{n+m}}{\sqrt{(2n)!(2m)!}}\Big) e^{-x}\,dx = 2 - 2\frac{(n+m)!}{\sqrt{(2n)!(2m)!}}$$

and for any fixed n we have

$$\lim_{m\to\infty} \frac{(n+m)!}{\sqrt{(2n)!(2m)!}} = 0, \text{ so that } \lim_{m\to\infty} \|u_n - u_m\|^2_{L^2(\Omega,\mu)} = 2. \qquad \square$$

In the above example $D_x\Phi(x,y)$ is bounded for $x > 0$, and the question of whether $D(A)$ is compactly embedded in $L^2(\Omega,\mu)$ if $|D\Phi|$ goes to ∞ as $|x| \to \infty$ remains open. In the next proposition we show that the answer is positive if Φ satisfies an additional (mild) nonoscillation condition.

Proposition 3.4. *Assume that $\Phi \in C^2(\mathbb{R}^N)$ satisfies $\Delta\Phi \le a|D\Phi|^2 + b$ for some $a < 1$, $b \in \mathbb{R}$. Then the map $u \mapsto |D\Phi|u$ is bounded from $H^1_0(\Omega,\mu)$ to $L^2(\Omega,\mu)$. If, in addition, $|D\Phi| \to \infty$ at infinity, the embedding of $H^1_0(\Omega,\mu)$ (hence that of $D(A)$) in $L^2(\Omega,\mu)$ is compact.*

Proof. Since $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega,\mu)$ it is sufficient to show that

$$\||D\Phi|u\|_{L^2(\Omega,\mu)} \le C\|u\|_{H^1(\Omega,\mu)}$$

for some $C > 0$ and every $u \in C_0^\infty(\Omega,\mu)$. Integrating by parts and using Young's inequality we get for every $\varepsilon > 0$ and for a suitable C_ε

$$\begin{aligned}
\int_\Omega |u|^2|D\Phi|^2\,d\mu &= -\int_\Omega |u|^2\langle D\Phi, De^{-\Phi}\rangle\,dx \\
&= \int_\Omega |u|^2\Delta\Phi e^{-\Phi}\,dx + 2\int_\Omega u\langle D\Phi, Du\rangle e^{-\Phi}\,dx \\
&\le (a+\varepsilon)\int_\Omega |u|^2|D\Phi|^2\,d\mu + C_\varepsilon\int_\Omega |Du|^2\,d\mu + b\int_\Omega |u|^2\,d\mu.
\end{aligned}$$

Choosing ε such that $a + \varepsilon < 1$, the first statement follows. Concerning the second one, we observe that for each $\varepsilon > 0$ there is $R > 0$ such that $|D\Phi| \ge 1/\varepsilon$ in $\Omega \setminus B(0,R)$. Hence for every u in the unit ball B of $H^1_0(\Omega)$ we have

$$\frac{1}{\varepsilon^2}\int_{\Omega\setminus B(0,R)} |u|^2\,d\mu \le \int_{\Omega\setminus B(0,R)} |u|^2|D\Phi|^2\,d\mu \le C^2.$$

Since the embedding of $H^1(\Omega\cap B(0,R))$ into $L^2(\Omega\cap B(0,R))$ is compact, we can find $\{f_1,\dots,f_k\} \subset L^2(\Omega\cap B(0,R))$ such that the balls $B(f_i,\varepsilon) \subset L^2(\Omega\cap B(0,R))$ cover the restrictions of the functions of B to $\Omega\cap B(0,R)$. Denoting by $\tilde f_i$ the zero-extension of f_i to the whole of Ω, it follows that $B \subset \bigcup_{i=1}^k B(\tilde f_i, (C+1)\varepsilon)$, and the proof is complete. $\square$

The compactness of the resolvent is a consequence of the logarithmic Sobolev inequality

$$\int_\Omega u^2\log(|u|)\,d\mu \le \frac{1}{\omega}\int_\Omega |Du|^2\,d\mu + \|u\|^2_{L^2(\Omega,\mu)}\log(\|u\|_{L^2(\Omega,\mu)}), \tag{12}$$

for all $u \in H^1_0(\Omega,\mu)$ and some $\omega > 0$ (where we set $0\log 0 = 0$).

In what follows we give sufficient conditions for the validity of (12).

Proposition 3.5. *Let us denote by* $\lambda(x)$ *the smallest eigenvalue of the matrix* $D^2\Phi(x)$*. Then:*

(i) *if* $\lambda(x) \geq \omega_0$ *for all* $x \in \mathbb{R}^N$ *then* (12) *holds with* $\omega = \omega_0$*;*
(ii) *if* $\liminf_{|x|\to\infty} \lambda(x) > 0$*, then* (12) *holds for some* $\omega > 0$*.*

Proof. (i) Let $u \in H^1_0(\Omega,\mu)$ and extend u outside Ω by setting $u(x) = 0$ for $x \notin \Omega$. Then the extension is in $H^1(\mathbb{R}^N,\nu)$, where $d\nu(x) = c\exp(-\Phi(x))\,dx$, $c^{-1} = \int_{\mathbb{R}^N} \exp(-\Phi)\,dx \geq 1$. By [3], for each $u \in H^1(\mathbb{R}^N,\nu)$ we have

$$\int_{\mathbb{R}^N} |u|^2 \log|u|\,d\nu \leq \frac{1}{\omega_0}\int_{\mathbb{R}^N} |Du|^2\,d\nu + \|u\|^2_{L^2(\mathbb{R}^N,\nu)} \log(\|u\|_{L^2(\mathbb{R}^N,\nu)}).$$

Since u vanishes outside Ω we easily get

$$\int_\Omega |u|^2 \log|u|\,d\mu \leq \frac{1}{\omega_0}\int_\Omega |Du|^2\,d\mu + \|u\|^2_{L^2(\Omega,\mu)}\left(\frac{1}{2}\log c + \log(\|u\|_{L^2(\Omega,\mu)})\right)$$

and (12) follows since $c \leq 1$.

(ii) The proof is similar to (i), using [11, Theorem 1.3] instead of [3]. □

Corollary 3.6. *Under the assumptions of Proposition* 3.5*,* $H^1_0(\Omega,\mu)$ *is compactly embedded in* $L^2(\Omega,\mu)$*. Therefore,* $\sup\sigma(A) < 0$*. Moreover* $T(t)$ *maps* $L^2(\Omega,\mu)$ *into* $L^{q(t)}(\Omega,\mu)$ *with* $q(t) = 1 + e^{\omega t}$*, and*

$$\|T(t)f\|_{L^{q(t)}(\Omega,\mu)} \leq \|f\|_{L^2(\Omega,\mu)}, \;\; t > 0, \; f \in L^2(\Omega,\mu). \tag{13}$$

Proof. Let B be the unit ball of $H^1_0(\Omega,\mu)$. Inequality (12) yields the existence of a positive constant C such that $\int_\Omega |u|^2\,d\mu \leq C$ for every $u \in B$. Given $t \geq 1$, let $E = \{|u| < t\}$. Then for $R > 0$

$$\begin{aligned}\int_{\Omega\setminus B(0,R)} |u|^2\,d\mu &\leq \int_{(\Omega\setminus B(0,R))\cap E} t^2\,d\mu + \frac{1}{\log t}\int_{(\Omega\setminus B(0,R))\setminus E} |u|^2\log|u|\,d\mu \\ &\leq t^2\mu(\Omega\setminus B(0,R)) + \frac{C}{\log t}\end{aligned}$$

hence, given $\varepsilon > 0$, there exists $R > 0$ such that $\int_{\Omega\setminus B(0,R)} |u|^2\,d\mu \leq \varepsilon$ for every $u \in B$. As in Proposition 3.4, this proves that $H^1_0(\Omega,\mu)$ is compactly embedded in $L^2(\Omega,\mu)$. The fact that $T(t)$ maps $L^2(\Omega,\mu)$ into $L^{q(t)}(\Omega,\mu)$, as well as estimate (13), follow from [5, 6]. □

A necessary and sufficient condition in order that 0 be in the resolvent of A is that the Poincaré inequality holds, i.e.,

$$\int_\Omega |u|^2\,d\mu \leq \frac{1}{\omega}\int_\Omega |Du|^2\,d\mu, \quad u \in H^1_0(\Omega,\mu), \tag{14}$$

for some $\omega > 0$. More precisely, since A is self-adjoint, then $\langle(-A-\omega I)u,u\rangle \geq 0$ for each $u \in D(A)$ if and only if $\sigma(A+\omega I) \subset (-\infty,0]$. In other words, (14) holds for each $u \in D(A)$ (or, equivalently, for each $u \in H^1_0(\Omega,\mu) = D((-A)^{1/2})$) if and only if $\sigma(A) \subset (-\infty,-\omega]$. In this case we have

$$\|T(t)f\|_{L^2(\Omega,\mu)} \leq e^{-\omega t}\|f\|_{L^2(\Omega,\mu)}, \;\; t > 0, \; f \in L^2(\Omega,\mu). \tag{15}$$

Indeed, for each $t > 0$ and $f \in L^2(\Omega,\mu)$,

$$\frac{d}{dt}\|T(t)f\|^2 = \int_\Omega 2AT(t)f\cdot T(t)f\,d\mu = -2\|DT(t)f\|^2 \leq -2\omega\|T(t)f\|^2.$$

If $\Omega = \mathbb{R}^N$, the Poincaré inequality for functions having zero mean is a consequence of the logarithmic Sobolev inequality (in which case $D(A)$ is compactly embedded in $L^2(\Omega,\mu)$) and the constant ω in (14) is the same as in (12); see [10]. This is not true in our setting; see Example 3.9 below. However, in the next proposition we show how to get an explicit estimate of ω in (14) when (6) holds.

Proposition 3.7. *Assume that* (6) *holds and that there exists* $\omega_0 > 0$ *such that the map* $x \mapsto \Phi(x) - \omega_0|x|^2/2$ *is convex. Then* (14) *holds with* $\omega = \omega_0$.

Proof. We have only to show that $\sigma(A) \subset (-\infty, -\omega_0]$. Corollary 3.6 yields that the resolvent of A is compact, hence $\sigma(A)$ consists of eigenvalues. If $\lambda u - Au = 0$ for some $\lambda \in \mathbb{R}$ and $0 \neq u \in D(A)$, we write (8) with $f = 0$ and let $R \to \infty$. Since the boundary integral is nonpositive and $D^2\Phi \geq \omega_0 I$ we get $(\lambda + \omega_0)\int_\Omega |Du|^2 \leq 0$. Since u is not a constant, then $Du \neq 0$ and $\lambda \leq -\omega_0$. This concludes the proof. $\square$

Let us again consider Example 3.3 and show that, in general, the Poincaré inequality does imply that the embedding $D(A) \subset L^2(\Omega,\mu)$ is compact.

Example 3.8. We use the same notation as in Example 3.3. Proposition 3.7, applied to the one-dimensional function $y \mapsto y^2$, $y > 0$, yields

$$\int_0^\infty |u(x,y)|^2\, e^{-y^2} dy \leq \frac{1}{2}\int_0^\infty |D_y u(x,y)|^2 e^{-y^2}\, dy, \qquad \text{a.e. } x \in \mathbb{R},\ u \in H^1_0(\Omega,\mu).$$

Multiplying by $e^{-\phi(x)}$ and integrating with respect to $x \in \mathbb{R}$, we deduce

$$\int_\Omega |u(x,y)|^2\, d\mu \leq \frac{1}{2}\int_\Omega |D_y u(x,y)|^2\, d\mu$$

so that the Poincaré inequality holds, even if $D(A)$ is not compactly embedded in $L^2(\Omega,\mu)$, as we have shown in Example 3.3.

If assumption (6) is replaced by the boundedness of $\partial\Phi/\partial n$ at $\partial\Omega$ and still $\Phi(x) - \omega^2|x|^2$ is convex, the constant ω in (14) may also depend on the constant k in (10), as we show in the following example.

Example 3.9. Let $N = 1$ and let $\mathcal{A}u = u'' - xu'$ be the Ornstein-Uhlenbeck operator. Here $\Phi(x) = x^2/2$, hence $D^2\Phi \equiv 1$ and (12) holds with $\omega = 1$. Let $\Omega_a = (-\infty, a)$ and set $u(x) = a - x$. Then $u \in D(A)$ and

$$\int_{-\infty}^a |u'|^2 d\mu \left(\int_{-\infty}^a |u|^2\, d\mu\right)^{-1} \to 0$$

as $a \to \infty$. This shows that the spectrum of A in $L^2(\Omega_a,\mu)$ is not contained in $(-\infty,-1]$ for large a, hence the constant ω in (14) is smaller than 1.

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