Maximal regularity for Kolmogorov operators in L^2 spaces with respect to invariant measures

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Abstract
We prove an optimal embedding result for the domains of Kolmogorov (or degenerate hypoelliptic Ornstein–Uhlenbeck) operators in L^2 spaces with respect to invariant measures. We use an interpolation method together with optimal L^2 estimates for the space derivatives of $T(t)f$ near $t = 0$, where $T(t)$ is the Ornstein–Uhlenbeck semigroup and f is any function in L^2.

Résumé
Nous montrons un résultat d'injection optimal pour les domaines des opérateurs de Kolmogorov (ou les opérateurs d'Ornstein–Uhlenbeck hypoelliptiques dégénérés) sur les espaces L^2 avec une mesure invariante. On utilise une méthode d'interpolation et des estimations optimales pour la norme L^2 de la dérivée spatiale de $T(t)f$ près de $t = 0$. Où $T(t)$ est le semi-groupe d'Ornstein–Uhlenbeck et f est un élément de L^2.
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1 Introduction

This note concerns the differential operator

$$\mathcal{L}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}D_{ij}u(x) + \sum_{i,j=1}^{d} b_{ij}x_jD_iu = \frac{1}{2}\,\mathrm{Tr}(QD^2u(x)) + \langle Bx, Du(x)\rangle, \quad x \in \mathbb{R}^d, \qquad (1)$$

where B and Q are real $d\times d$-matrices, Q is symmetric and nonnegative. Therefore $\mathcal{L}$ is a possibly degenerate elliptic operator that we assume to be hypoelliptic, and that is called Kolmogorov or degenerate Ornstein–Uhlenbeck operator. The hypoellipticity assumption may be stated as follows: the symmetric matrices Q_t defined by

$$Q_t := \int_0^t e^{sB}Qe^{sB^*}\,\mathrm{d}s$$

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have nonzero determinant for some (equivalently, for all) $t > 0$. An obvious assumption that ensures the non-singularity of Q_t is the non-singularity of Q. In this case the operator in (1) is non-degenerate, and this paper gives just an alternative proof to already known results ([10], [14]). So, we emphasize here the degenerate case.
The hypoellipticity condition implies that the Gaussian measures $\mathcal{N}_{e^{tB}x,Q_t}$ with covariance operator Q_t and mean $e^{tB}x$ ($t > 0$, $x \in \mathbb{R}^d$) are all absolutely continuous with respect to the d-dimensional Lebesgue measure. With the aid of such measures the Ornstein–Uhlenbeck semigroup $(T(t))_{t\geq 0}$ is readily defined by

$$(T(t)f)(x) = \int_{\mathbb{R}^d} f \,\mathrm{d}\mathcal{N}_{e^{tB}x,Q_t} := \frac{1}{(2\pi)^{d/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}y,y\rangle} f(e^{tB}x - y)\,\mathrm{d}y,\ x \in \mathbb{R}^d. \quad (2)$$

As is easily seen, the function $u(t,x) := (T(t)f)(x)$ is a classical solution to the Cauchy problem $u_t = \mathcal{L}u$ ($t > 0$, $x \in \mathbb{R}^d$), $u(0,\cdot) = f$, for a wide class of initial data f.
Together with hypoellipticity, the other structural assumption of this paper is existence of an invariant measure for $\mathcal{L}$, i.e., a probability measure μ such that

$$\int_{\mathbb{R}^d} \mathcal{L}u \,\mathrm{d}\mu = 0$$

for all $u \in C_b^2(\mathbb{R}^d)$. It is well known that such a measure exists if and only if the improper integral

$$Q_\infty := \int_0^\infty e^{sB}Qe^{sB^*}\,\mathrm{d}s \quad \text{converges}, \quad (3)$$

(see, e.g., [2, Sec. 6.2.1]) and this happens if and only if all the eigenvalues of B have negative real part. Under this hypothesis the determinant of Q_∞ is positive, the invariant measure is unique, and it is the Gaussian measure $\mu := \mathcal{N}_{0,Q_\infty}$, i.e.,

$$\mathrm{d}\mu(x) = \frac{1}{(2\pi)^{d/2}(\det Q_\infty)^{1/2}} e^{-\frac{1}{2}\langle Q_\infty^{-1}x,x\rangle}\,\mathrm{d}x := \rho(x)\,\mathrm{d}x. \quad (4)$$

The simplest significant example is a Kolmogorov operator in $\mathbb{R}^2$:

$$\mathcal{L}u(x,y) = \frac{1}{2}u_{xx}(x,y) - (y+x)u_x(x,y) + xu_y(x,y), \quad (5)$$

which arises in stochastic perturbations of motions with friction (see, e.g., [6]) and which has the Gaussian measure $\mathcal{N}_{0,I/2}$ as invariant measure (see Section 5).
An important feature of second order elliptic operators in L^2 spaces with respect to invariant measures is their dissipativity. In our case, since $\mathcal{L}(u^2) = 2u\mathcal{L}u + \langle QDu, Du\rangle$ and the integral of $\mathcal{L}(u^2)$ vanishes, we have

$$\langle \mathcal{L}u, u\rangle_{L^2} = \int_{\mathbb{R}^d} u\,\mathcal{L}u\,\mathrm{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^d} \langle QDu, Du\rangle\,\mathrm{d}\mu \leq 0 \quad (6)$$

for all $u \in C_b^2(\mathbb{R}^d)$. Therefore, $\mathcal{L} : D(\mathcal{L}) := C_b^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d,\mu)$ is closable, and we denote by $(L, D(L))$ (or simply by L) its closure. L turns out to be the infinitesimal generator of $T(t)$ in $L^2(\mathbb{R}^d,\mu)$, see, e.g., [4, Sec. 10.2].
Note that L is not symmetric in the degenerate hypoelliptic case, because symmetry is equivalent to $Q^{1/2}e^{sB^*} = e^{sB}Q^{1/2}$ for each $s > 0$ (see again [4, Sec. 10.2]), and this implies that the kernel of each Q_t contains the kernel of $Q^{1/2}$, so that $\det Q_t = 0$.

The main achievement of this paper is a regularity result for the functions in the domain of L. We show that they belong to a non-isotropic Sobolev space "naturally" associated to $\mathcal{L}$. They have first and second order derivatives with respect to some variables in $L^2(\mathbb{R}^d, \mu)$, and they belong to suitable fractional weighted Sobolev spaces with respect to the other variables. In the case of the two-dimensional example (5), the functions $u \in D(L)$ have first and second order derivatives with respect to x in $L^2(\mathbb{R}^2, \mu)$, and they satisfy

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{|u(x, y_1)e^{-y_1^2/2} - u(x, y_2)e^{-y_2^2/2}|^2}{|y_1 - y_2|^{7/3}} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, e^{-x^2} \, \mathrm{d}x < \infty.$$

For a precise statement in the general case, we use an equivalent condition to hypoellipticity, which is known as *Kalman rank condition* and is the following: the block matrix

$$[Q^{1/2}, BQ^{1/2}, B^2Q^{1/2}, \ldots, B^{d-1}Q^{1/2}] \quad \text{has rank } d.$$

This allows to decompose $\mathbb{R}^d$ into the direct sum of n nontrivial subspaces, where n is the minimum integer such that the rank of $[Q^{1/2}, BQ^{1/2}, B^2Q^{1/2}, \ldots, B^{n-1}Q^{1/2}]$ is d. Set $V_h := \text{Range}\, Q^{1/2} + \text{Range}\, BQ^{1/2} + \cdots + \text{Range}\, B^hQ^{1/2}$ for $h = 0, \ldots, n-1$, let P_0 be the orthogonal projection on $W_0 := V_0$ and let P_h be the orthogonal projection onto $W_h := V_h \ominus V_{h-1}$ if $h = 1, \ldots, n-1$. Then $\mathbb{R}^d = \bigoplus_{h=0}^{n-1} W_h$. We fix orthonormal bases in the subspaces W_h, whose union is an orthonormal basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$. For every $h = 0, \ldots, n-1$ we denote by I_h the set of indices i such that the vectors e_i with $i \in I_h$ span W_h. After this change of coordinates the second order derivatives which appear in (1) are only the $D_{ij}u$ with $i, j \in I_0$.
The main theorem of this paper states that the domain of L is continuously embedded in $H^{2,2/3,\ldots,2/(2n-1)}(\mathbb{R}^d, \mu)$. This space is defined in terms of series developments with Hermite polynomials, see Section 3. Its elements u have derivatives D_iu, $D_{ij}u$ in $L^2(\mathbb{R}^2, \mu)$ for every $i, j \in I_0$, and for every index $i \in I_h$, $h = 1, \ldots, n-1$, they satisfy

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^2} \frac{|(u\sqrt{\rho})(x_1, \ldots, x_i^1, \ldots, x_d) - (u\sqrt{\rho})(x_1, \ldots, x_i^2, \ldots, x_d)|^2}{|x_i^1 - x_i^2|^{1+4/(2h+1)}} \, \mathrm{d}x_i^1 \, \mathrm{d}x_i^2 \, \mathrm{d}\hat{x}_i < \infty$$

where ρ is the density of μ given by (4) and $\mathrm{d}\hat{x}_i = \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_d$ is the $(d-1)$-dimensional Lebesgue measure in $\mathbb{R}^{d-1}$.
More generally, we prove that for each positive integer k the domain of L^k is continuously embedded in $H^{2k,2k/3,\ldots,2k/(2n-1)}(\mathbb{R}^d, \mu)$, whose definition for general k is similar to the case $k = 1$.
Since our weighted Lebesgue and Sobolev spaces are locally equivalent to the usual Lebesgue and Sobolev spaces, it follows that for each $u \in D(L)$ there exist the derivatives D_iu, $D_{ij}u$ for $i, j \in I_0$ and they are in $L^2_{loc}(\mathbb{R}^d, \mathrm{d}x)$; moreover $u \in H^{2/(2n-1)}_{loc}(\mathbb{R}^d, \mathrm{d}x)$. The last exponent $2/(2n-1)$ agrees with the general local regularity results of [15]. Concerning local maximal regularity, we mention also the paper [5] where it was proved that the second order derivatives $D_{ij}u$, $i, j \in I_0$, exist and belong to $L^2_{loc}(\mathbb{R}^d, \mathrm{d}x)$. In fact, the papers [5, 15] deal with second order operators of the type $X_0 + \sum_{j=1}^k X_j^2$ in nilpotent Lie groups, such that all the X_j's are left invariant vector fields, homogeneous with respect to suitable families of dilations, and satisfy the Hörmander commutator condition. It can be proved that under suitable assumptions on B, the operator $L - \frac{\mathrm{d}}{\mathrm{d}t}$ belongs to this class of operators, see, e.g., [9].
Global regularity results and estimates in weighted or non-weighted Sobolev spaces seem to be missing from the literature yet. The different regularity degree with respect to different variables should not be surprising, being a typical feature of hypoelliptic operators. A result of this type in non-isotropic Hölder spaces instead of Sobolev spaces has been already proved in [11].

Our result is proved by an interpolation method that uses sharp estimates for the space derivatives of $T(t)f$ for small $t > 0$ and for each $f \in L^2(\mathbb{R}^d, \mu)$. Let us describe it in the case of example (5). For each couple of nonnegative integers k_1, k_2 there is $c > 0$ such that

$$\|D_x^{k_1} D_y^{k_2} T(t) f\|_{L^2(\mathbb{R}^2,\mu)} \le \frac{c}{t^{(k_1+k_2)/2+k_2}} \|f\|_{L^2(\mathbb{R}^2,\mu)}, \qquad t \in (0,1).$$

This implies that for every positive integer k the norm of $T(t)$ as an operator from $L^2(\mathbb{R}^2, \mu)$ to $H^{3k,k}(\mathbb{R}^2, \mu)$ is bounded by $c/t^{3k/2}$ near $t = 0$. An argument from general interpolation/semigroup theory shows now that this estimate with $k = 1$ implies that the real interpolation space $(L^2(\mathbb{R}^2, \mu), D(L^2))_{1/2,2}$ is continuously embedded in $(L^2(\mathbb{R}^2, \mu), H^{3,1}(\mathbb{R}^2, \mu))_{2/3,2}$. On the one hand, the space $(L^2(\mathbb{R}^2, \mu), D(L^2))_{1/2,2}$ coincides with $D(L)$, because L is the infinitesimal generator of a contraction positivity preserving semigroup in a Hilbert space. On the other hand, the interpolation space $(L^2(\mathbb{R}^2, \mu), H^{3,1}(\mathbb{R}^2, \mu))_{2/3,2}$ is contained in $H^{2,2/3}(\mathbb{R}^2, \mu)$, and the embedding follows.

We remark that, although not very common in the literature about PDE's, L^p and Sobolev spaces with respect to invariant measures are much more suited to Kolmogorov operators than L^p spaces with respect to the Lebesgue measure or other weighted spaces. Apart from their intrinsic interest as nice examples of hypoelliptic operators, the main motivation for the study of Kolmogorov operators is probabilistic: given the stochastic differential equation in $\mathbb{R}^d$

$$\begin{cases} \mathrm{d}X_t = BX_t \,\mathrm{d}t + Q^{1/2}\,\mathrm{d}W_t, \\ X(0) = x, \end{cases}$$

where $W(t)$ is a standard Brownian motion, the Ornstein–Uhlenbeck semigroup is nothing but the transition semigroup of the process, i.e., $T(t)f(x) = \mathbb{E}(f(X_t))$ for each Borel measurable and bounded f, and μ is the invariant measure of the process, i.e., for any $t > 0$ we have $\int_{\mathbb{R}^d} T(t)f \,\mathrm{d}\mu = \int_{\mathbb{R}^d} f \,\mathrm{d}\mu$, again for each Borel measurable and bounded f. So, the invariant measure is associated to a property of conservation of mean values which is widely used in probability and in ergodic theory (see, e.g., the books [2, 8]).

A description of the basic features Ornstein–Uhlenbeck semigroups in L^p spaces with respect to invariant measures, under hypoellipticity conditions, may be found in [2]. A detailed study of the spectral properties of their generators is in [13].

2 The Ornstein–Uhlenbeck semigroup

Throughout this section we write $\|f\|_2$ instead of $\|f\|_{L^2(\mathbb{R}^d,\mu)}$. D_i denotes the partial derivative in the direction e_i, and D denotes the gradient. Moreover P_h, $h = 0, \ldots, n-1$, are the projections associated to the Kalman rank condition, introduced in Section 1.

The Ornstein–Uhlenbeck semigroup is defined on $L^2(\mathbb{R}^d, \mu)$ by formula (2). It is not hard to see that it is a contraction semigroup; indeed, for each $f \in L^2(\mathbb{R}^d, \mu)$ and for all $x \in \mathbb{R}^d$ we have by the Hölder inequality

$$|(T(t)f)(x)|^2 \le \int_{\mathbb{R}^d} |f(e^{tB}x - y)|^2 \,\mathrm{d}\mu_{0,Q_t} = (T(t)f^2)(x),$$

so integrating both sides against the invariant measure μ we obtain $\|T(t)f\|_2 \le \|f\|_2$. The representation formula (2) shows that $T(t)f$ is differentiable for all $f \in L^2(\mathbb{R}^d, \mu)$, and

$$(DT(t)f)(x) = -c_t \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} f(y) e^{tB^*} Q_t^{-1}(e^{tB}x - y)\,\mathrm{d}y$$

with $c_t = (2\pi)^{-d/2}(\det Q_t)^{-1/2}$, so for each $i = 1, \dots, d$ we have

$$|(D_i T(t)f)(x)| \le c_t \int\limits_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} |f(y)| \cdot |\langle e^{tB^*} Q_t^{-1}(e^{tB}x - y), e_i\rangle| \, \mathrm{d}y.$$

By the Cauchy–Schwartz inequality we obtain

$$\begin{aligned}
|(D_i T(t)f)(x)|^2 &\le c_t^2 \Bigg(\int\limits_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} |f(y)| \cdot |\langle e^{tB^*} Q_t^{-1}(e^{tB}x - y), e_i\rangle| \, \mathrm{d}y \Bigg)^2 \le \\
&\le c_t \int\limits_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} f^2(y) \, \mathrm{d}y \cdot \\
&\quad \cdot c_t \int\limits_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} |\langle e^{tB^*} Q_t^{-1}(e^{tB}x - y), e_i\rangle|^2 \, \mathrm{d}y = \\
&= (T(t)f^2)(x) \cdot \tfrac{1}{(2\pi)^{d/2}(\det Q_t)^{1/2}} \int\limits_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(e^{tB}x-y), e^{tB}x-y\rangle} |\langle e^{tB^*} Q_t^{-1}(e^{tB}x - y), e_i\rangle|^2 \, \mathrm{d}y = \\
&= (T(t)f^2)(x) \cdot \tfrac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-\frac{|y|^2}{2}} |\langle e^{tB^*} Q_t^{-1/2} y, e_i\rangle|^2 \, \mathrm{d}y \le \\
&\le \widetilde{c} \, \|Q_t^{-1/2} e^{tB} P_h\|^2 \cdot (T(t)f^2)(x),
\end{aligned}$$

so that integrating with respect to μ and using its invariance we obtain

$$\|D_i T(t)f\|_2^2 \le c \|Q_t^{-1/2} e^{tB} P_h\|^2 \cdot \|f\|_2^2 \quad \text{for some constant } c > 0. \tag{7}$$

This shows that to estimate the derivatives of $T(t)f$ near $t = 0$ the crucial part is a precise estimation of $Q_t^{-1/2} e^{tB}$ for various directions in $\mathbb{R}^d$, according to the decomposition of the space. This was done in [11], where the proof is based on sharp estimates on Q_t near $t = 0$ (see Seidman [16]).

Lemma 1. *Let $\omega > \omega_0(B)$, the growth bound of $(e^{tB})_{t \ge 0}$. Then there exists a constant $c > 0$ such that for all $0 \le h, k \le n - 1$ and $t \ge 0$ the estimates*

$$\|P_h e^{tB} P_k\| = \|P_h e^{tB^*} P_k\| \le \begin{cases} ct^{h-k} e^{\omega t}, & h \ge k, \\ cte^{\omega t}, & h < k. \end{cases} \tag{8}$$

hold. Furthermore, there is a constant $c > 0$ such that

$$\begin{aligned}
\|Q_t^{-1/2} P_h\| &\le \frac{c}{t^{1/2+h}}, & \|P_h Q_t^{1/2}\| &\le ct^{1/2+h}, & 0 < t \le 1, \\
\|Q_t^{-1/2}\| &\le c, & \|Q_t^{1/2}\| &\le c \max(1, e^{\omega t}) & t \ge 1, \\
\|P_h e^{tB^*} Q_t^{-1/2}\| &\le \frac{ce^{\omega t}}{t^{1/2+h}}, & \|Q_t^{1/2} e^{tB} P_h\| &\le ce^{\omega t} t^{1/2+h}, & t > 0.
\end{aligned}$$

Now (7) and the above Lemma 1 yield

$$\|D_i T(t)f\|_2 \le \frac{ce^{\omega t}}{t^{1/2+h}} \|f\|_2, \;\; i \in I_h, \; t > 0. \tag{9}$$

This is the first step in proving the following proposition.

Proposition 2. *For any $N \in \mathbb{N}$ there exist a constant c such that*

$$\|D_{i_1} D_{i_2} \cdots D_{i_N} T(t) f\|_2 \le \frac{c}{t^{N/2+h_1+h_2+\cdots+h_N}} \|f\|_2, \qquad t \in (0,1), \tag{10}$$

for all $f \in L^2(\mathbb{R}^d, \mu)$ and $i_j \in I_{h_j}$, $j = 1, \ldots, N$.

Proof. We prove by induction on $N \in \mathbb{N}$. The cases $N = 0, 1$ are already settled. First of all, notice that for any continuously differentiable f

$$DT(t)f = e^{tB^*} T(t) Df \tag{11}$$

holds, hence for each $f \in L^2(\mathbb{R}^d, \mu)$ we have

$$(D_i T(t) f)(x) = \sum_{l=1}^{d} (e^{tB^*})_{il} (T(t) D_l f)(x),$$

and for any $N \in \mathbb{N}$, $i, i_1, i_2, \ldots, i_N \in \mathbb{N}$

$$\begin{aligned}(D_{i_1} D_{i_2} \cdots D_{i_N} D_i T(t) f)(x) &= (D_{i_1} D_{i_2} \cdots D_{i_N} D_i T(t/2) T(t/2) f)(x) = \\ &= \sum_{l=1}^{d} (e^{tB^*/2})_{il} D_{i_1} D_{i_2} \cdots D_{i_N} T(t/2) D_l T(t/2) f(x).\end{aligned}$$

Fix $\omega > \omega_0(B)$, suppose that assertion (10) is true for some $N > 0$, and let $i \in I_h$, $0 \le h \le n-1$. According to the induction hypothesis we can estimate the L_2-norm by using the triangle inequality

$$\begin{aligned}\|D_{i_1} D_{i_2} \cdots D_{i_N} D_i T(t) f\|_2 &\le \frac{c}{(t/2)^{N/2+h_1+h_2+\cdots+h_N}} \sum_{l=1}^{d} \big|(e^{tB^*/2})_{il}\big| \cdot \|D_l T(t/2) f\|_2 \le \\ &\le \frac{c}{(t/2)^{N/2+h_1+h_2+\cdots+h_N}} \sum_{l=1}^{d} \big\|P_h e^{tB^*/2} P_{k(l)}\big\| \cdot \|P_{k(l)} DT(t/2) f\|_2,\end{aligned}$$

where $k(l)$ is such that $l \in I_{k(l)}$. Applying first (8) from Lemma 1 and then inequality (9) we can continue the above estimate and obtain

$$\|D_{i_1} D_{i_2} \cdots D_{i_N} D_i T(t) f\|_2 \le \frac{c}{(t/2)^{N/2+h_1+h_2+\cdots+h_N}} \sum_{k=0}^{n-1} \big\|P_h e^{tB^*/2} P_k\big\| \cdot \|P_k DT(t/2) f\|_2 \le \tag{12}$$

$$\le \frac{c^2 e^{\omega t/2}}{(t/2)^{N/2+h_1+h_2+\cdots+h_N}} \Big(\sum_{k=0}^{h-1} t/2 \|P_k DT(t/2) f\|_2 + \sum_{k=h}^{n-1} (t/2)^{k-h} \|P_k DT(t/2) f\|_2 \Big) \le \tag{13}$$

$$\le \frac{c^2 e^{\omega t/2}}{(t/2)^{N/2+h_1+h_2+\cdots+h_N}} \Big(\sum_{k=0}^{h-1} d_k t/2 \cdot \frac{c e^{\omega t/2}}{(t/2)^{1/2+k}} \|f\|_2 + \sum_{k=h}^{n-1} (t/2)^{k-h} \cdot \frac{d_k c e^{\omega t/2}}{(t/2)^{1/2+k}} \|f\|_2 \Big) \le \tag{14}$$

$$\le \frac{c'}{t^{(N+1)/2+h_1+h_2+\cdots+h_N+h}} \|f\|_2, \qquad t \in (0,1). \tag{15}$$

All the constants in (12) are absolute if N is fixed. This yields the statement. ∎

Remark 3. Let $\omega > \omega_0(B)$. The above proof also shows that for any $N \in \mathbb{N}$ there exist a constant c such that

$$\|D_{i_1} D_{i_2} \cdots D_{i_N} T(t) f\|_2 \le \frac{c e^{\omega t}}{t^{N/2+h_1+h_2+\cdots+h_N}} \|f\|_2 \qquad \text{for all } t \in (0, +\infty),$$

and for all $f \in L^2(\mathbb{R}^d, \mu)$, $i_j \in I_{h_j}$, $j = 1, \ldots, N$.

3 Interpolation for anisotropic, weighted Sobolev spaces

Here and in the following, if R is a $k \times k$ positive definite matrix and m is any positive integer, $H^m(\mathbb{R}^k, \mathcal{N}_{0,R})$ is the Hilbert space of the functions $u \in L^2(\mathbb{R}^k, \mathcal{N}_{0,R})$ such that all the (weak) derivatives $D^\beta f$ exist and belong to $L^2(\mathbb{R}^k, \mathcal{N}_{0,R})$ for $|\beta| \le m$.

3.1 Preliminaries on symmetric Ornstein–Uhlenbeck operators and Hermite polynomials

We recall some well known facts about symmetric Ornstein–Uhlenbeck operators and Hermite polynomials. In dimension 1, the latter are defined by

$$H_n(x) := \frac{(-1)^n}{\sqrt{n!}}\, e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\, e^{-x^2/2}, \ \ n \in \mathbb{N} \cup \{0\}, \ x \in \mathbb{R},$$

and they form an orthonormal basis in the space $L^2(\mathbb{R}, \mathcal{N}_{0,1})$. In general dimension k, for any multi-index β we define the polynomials H_β by

$$H_\beta(x) = \prod_{j=1}^k H_{\beta_j}\left(\frac{x_j}{\sqrt{\lambda_j}}\right), \ \ x \in \mathbb{R}^k, \tag{16}$$

if $R =$ diag $[\lambda_1, \dots, \lambda_k]$, and by

$$H_\beta(x) = \prod_{j=1}^k H_{\beta_j}\left(\frac{(Ux)_j}{\sqrt{\lambda_j}}\right), \ \ x \in \mathbb{R}^d, \tag{17}$$

if R is not diagonal and U is an orthogonal matrix (fixed once and for all) such that URU^{-1} is diagonal.

These polynomials constitute an orthonormal basis in $L^2(\mathbb{R}^d, \mathcal{N}_{0,R})$, being the eigenfunctions of the self-adjoint non-positive Ornstein–Uhlenbeck operator A defined by

$$D(A) = H^2(\mathbb{R}^k, \mathcal{N}_{0,R}), \ \ Au(x) = \frac{1}{2}\,\mathrm{Tr}(RD^2u(x)) - \frac{1}{2}\langle x, Du(x)\rangle,$$

with eigenvalue $-\sum_{j=1}^k \beta_j/2$.

It can be shown that $H^m(\mathbb{R}^k, \mathcal{N}_{0,R})$ is the domain of the operator $(\sqrt{I-A})^m$, and its graph norm is equivalent to the norm associated to the natural scalar product in $H^m(\mathbb{R}^k, \mathcal{N}_{0,R})$,

$$\langle f, g\rangle_{H^m(\mathbb{R}^k, \mathcal{N}_{0,R})} := \sum_{0 \le |\beta| \le m} \langle D^\beta f, D^\beta g\rangle_{L^2(\mathbb{R}^k, \mathcal{N}_{0,R})}. \tag{18}$$

In fact, an extension of this result to L^p spaces with $p \neq 2$ holds even for Ornstein–Uhlenbeck operators in infinitely many variables (see [3]).

This motivates the definition of $H^s(\mathbb{R}^k, \mathcal{N}_{0,R})$ for any $s > 0$ as the domain of $(\sqrt{I-A})^s$, i.e., the set of functions $u \in L^2(\mathbb{R}^k, \mathcal{N}_{0,R})$ such that the series

$$\sum_{|\beta| \ge 0} \left(1 + \sum_{j=1}^k \frac{\beta_j}{2}\right)^{2s} \langle u, H_\beta\rangle^2_{L^2(\mathbb{R}^k, \mathcal{N}_{0,R})} := \|u\|^2_{H^s(\mathbb{R}^k, \mathcal{N}_{0,R})} \tag{19}$$

converges. To be consistent we use the above norm also for $s = m \in \mathbb{N}$, instead of the equivalent norm associated to the scalar product (18).

3.2 Anisotropic Sobolev spaces in dimension d

In this section it will be important that we fix some orthonormal basis $e_1, \dots, e_d$ in the space $\mathbb{R}^d$ and the partial derivatives D_i are understood in these directions. In the next section this will be chosen as the basis coming from the decomposition of the space $\mathbb{R}^d$ in connection with the Kalman rank condition.

Let R be a $d \times d$ symmetric positive definite matrix and let $\nu := \mathcal{N}_{0,R}$ be the associated Gaussian measure. For any multi-index β, let H_β be the Hermite polynomial (in dimension d) defined in Section 3.1.

Take $m \in \mathbb{N}$ and fix a subset $I \subseteq \{1, \dots, d\}$ as well. Denote by Λ_I the set of all multi-indices $\beta \in (\mathbb{N} \cup \{0\})^d$ such that $\beta_j = 0$ for $j \notin I$.

For each $s > 0$, we define the Sobolev space $H^s_I(\mathbb{R}^d, \nu)$ as the space of the functions $u \in L^2(\mathbb{R}^d, \nu)$ such that the series

$$\sum_{|\beta| \geq 0,\, \beta \in \Lambda_I} \left(1 + \sum_{j \in I} \frac{\beta_j}{2}\right)^{2s} \langle u, H_\beta \rangle^2_{L^2(\mathbb{R}^d, \nu)} \tag{20}$$

converges. It is a Hilbert space with the scalar product

$$\langle f, g \rangle_{H^s_I(\mathbb{R}^d, \nu)} := \sum_{|\beta| \geq 0,\, \beta \in \Lambda_I} \left(1 + \sum_{j \in I} \frac{\beta_j}{2}\right)^{2s} \langle f, H_\beta \rangle_{L^2(\mathbb{R}^d, \nu)} \langle g, H_\beta \rangle_{L^2(\mathbb{R}^d, \nu)}.$$

It follows from the considerations in Section 3.1 that if $s = m$ is integer, we have

$$H^m_I(\mathbb{R}^d, \nu) = \left\{ f : \; f \in L^2(\mathbb{R}^d, \nu),\ \exists D^\beta f \in L^2(\mathbb{R}^d),\ \beta \in \Lambda_I,\ |\beta| \leq m \right\},$$

and its scalar product is equivalent to

$$(f, g) \mapsto \sum_{0 \leq |\beta| \leq m,\, \beta \in \Lambda_I} \langle D^\beta f, D^\beta g \rangle_{L^2(\mathbb{R}^d, \nu)}.$$

Now let us partition the set $\{1, \dots, d\}$ into n non-empty subsets I_h ($h = 0, \dots, n-1$); we denote by W_h the subspace of $\mathbb{R}^d$ spanned by $\{e_j : \; j \in I_h\}$. Given n positive numbers $s_0, \dots, s_{n-1}$ we define

$$H^{s_0, s_1, \dots, s_{n-1}}(\mathbb{R}^d, \nu) := \bigcap_{h=0}^{n-1} H^{s_h}_{I_h}(\mathbb{R}^d, \nu), \tag{21}$$

which is still a Hilbert space, with the sum scalar product. The associated norm is

$$\|u\|^2_{H^{s_0, s_1, \dots, s_{n-1}}(\mathbb{R}^d, \nu)} = \sum_{|\beta| \geq 0} \sum_{h=0}^{n-1} \left(1 + \sum_{j \in I_h} \frac{\beta_j}{2}\right)^{2s_h} \langle u, H_\beta \rangle^2_{L^2(\mathbb{R}^d, \nu)}.$$

We are interested in the real interpolation spaces $\big(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \dots, m_{n-1}}(\mathbb{R}^d, \nu)\big)_{\theta, 2}$, when the exponents $m_0, m_1, \dots, m_{n-1}$ are integers.

Proposition 4. *Fix $m_0, m_1, \dots, m_{n-1} \in \mathbb{N}$ and $0 < \theta < 1$. Then we have*

$$\Big(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \dots, m_{n-1}}(\mathbb{R}^d, \nu)\Big)_{\theta, 2} = H^{\theta m_0, \theta m_1, \dots, \theta m_{n-1}}(\mathbb{R}^d, \nu).$$

Proof. i): First we consider the case of a diagonal matrix R. We introduce the self-adjoint non-positive operators A_h in $L^2(\mathbb{R}^d, \nu)$, $h = 0, \dots, n-1$, defined by

$$D(A_h) := H^2_{I_h}(\mathbb{R}^d, \nu), \quad A_h u(x) = \frac{1}{2} \sum_{j \in I_h} \lambda_j D_j^2 u - \frac{1}{2} \sum_{j \in I_h} x_j D_j u(x). \tag{22}$$

The polynomials H_β with $\beta \in \Lambda_{I_h}$ are the eigenfunctions of A_h, and for each $s > 0$ we have

$$H^s_{I_h}(\mathbb{R}^d, \nu) = D((\sqrt{I - A_{I_h}})^s) = (L^2(\mathbb{R}^d, \nu), D((\sqrt{I - A_{I_h}})^m))_{s/m,2}, \tag{23}$$

where the first equality holds by definition, and the second equality holds because $\sqrt{I - A_h}$ is a positive self-adjoint operator in a Hilbert space ([17, Thm. 1.18.10]).
Therefore,

$$H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \nu) = \bigcap_{h=0}^{n-1} D((\sqrt{I - A_h})^{m_h}), \tag{24}$$

where the positive operators $\sqrt{I - A_h}$ have commutative resolvents. Then we may use Theorem 1.14.1 of [17], which yields

$$\Big(L^2(\mathbb{R}^d, \nu), \bigcap_{h=0}^{n-1} D((\sqrt{I - A_h})^{m_h})\Big)_{\theta,2} = \bigcap_{h=0}^{n-1} (L^2(\mathbb{R}^d, \nu), D((\sqrt{I - A_h})^{m_h}))_{\theta,2}. \tag{25}$$

Formula (23) with $m = m_h$ and $s = \theta m_h$ gives

$$(L^2(\mathbb{R}^d, \nu), D((\sqrt{I - A_h})^{m_h}))_{\theta,2} = H^{\theta m_h}_{I_h}(\mathbb{R}^d, \nu). \tag{26}$$

Now (24), (25), (26) imply the statement in the diagonal case.
ii): If the matrix R is not diagonal we need a further step for the description of our interpolation spaces. We have to introduce the above mentioned orthogonal matrix U such that

$$URU^{-1} = \mathrm{diag}\,[\lambda_1, \dots, \lambda_d].$$

The change of coordinates $y = Ux$ transforms the Gaussian measure $\mathcal{N}_{0,R}$ into the Gaussian measure $\mathcal{N}_{0,URU^{-1}}$, the basis $\{e_1, \dots, e_d\}$ into the basis $\{Ue_1, \dots, Ue_d\}$ and the subspaces W_h into the subspaces $U(W_h)$, spanned by $\{Ue_j : j \in I_h\}$.
The mapping $f \mapsto f \circ U^{-1}$ is an isomorphism between $L^2(\mathbb{R}^d, \nu)$ and $L^2(\mathbb{R}^d, \mathcal{N}_{0,URU^{-1}})$, and between $H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \nu)$ and $H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \mathcal{N}_{0,URU^{-1}})$ (the latter is understood with respect to the splitting associated to the subspaces $U(W_h)$, $h = 0, \dots, n-1$). Thus, $f \mapsto f \circ U^{-1}$ is an isomorphism between the interpolation spaces $(L^2(\mathbb{R}^d, \nu), H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \nu))_{\theta,2}$ and $(L^2(\mathbb{R}^d, \mathcal{N}_{0,URU^{-1}}), H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \mathcal{N}_{0,URU^{-1}}))_{\theta,2}$. Therefore, the interpolation space $(L^2(\mathbb{R}^d, \nu), H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \nu))_{\theta,2}$ consists of the functions $f \in L^2(\mathbb{R}^d, \nu)$ such that $f \circ U^{-1}$ belongs to $H^{\theta m_0,\theta m_1,\dots,\theta m_{n-1}}(\mathbb{R}^d, \mathcal{N}_{0,URU^{-1}})$, and the statement follows. ■

It is important to remark that if θm_h is integer for some h, say $\theta m_h = m \in \mathbb{N}$, then the functions in the interpolation space belong to $H^m_{I_h}(\mathbb{R}^d, \nu)$, so that they have weak derivatives up to the order m with respect to the variables x_j, $j \in I_h$, and these derivatives belong to $L^2(\mathbb{R}^d, \nu)$. On the other hand, if θm_h is not integer, the regularity properties with respect to the variables x_j, $j \in I_h$, are not obvious. To describe them better, we consider another transformation, the mapping $f \mapsto \sqrt{\rho} f$, where ρ is the density kernel of ν,

$$\rho(x) = \frac{1}{(2\pi)^{d/2} \det R^{1/2}} e^{-\frac{1}{2}\langle R^{-1}x, x\rangle}.$$

This mapping is an isometric isomorphism between $L^2(\mathbb{R}^d, \nu)$ and $L^2(\mathbb{R}^d, \mathrm{d}x)$, but it is *not* an isomorphism between our Sobolev spaces $H^{m_0,m_1,\dots,m_{n-1}}(\mathbb{R}^d, \nu)$ and the corresponding anisotropic Sobolev spaces with respect to the Lebesgue measure. Nevertheless we have the following embedding.

Proposition 5. *Let $I \subset \{1, \dots, d\}$ and $m \in \mathbb{N}$. Then*

$$H_I^m(\mathbb{R}^d, \nu) \subset \Big\{ f : \ f \in L^2(\mathbb{R}^d, \nu), \ \exists (\sqrt{\rho} f) \in L^2(\mathbb{R}^d, \ \mathrm{d}x), \ \beta \in \Lambda_I, \ |\beta| \le m \Big\}$$

and there is a constant C such that for each $f \in H_I^m(\mathbb{R}^d, \nu)$ we have

$$\sum_{|\beta| \le m, \ \beta \in \Lambda_I} \|(\sqrt{\rho} f)\|_{L^2(\mathbb{R}^d, \ \mathrm{d}x)} \le C \|f\|_{H_I^m(\mathbb{R}^d, \nu)}.$$

Proof. We prove only for $m = 1$, then the case of general m follows by induction. If f is a polynomial and $j \in I$, then $D_j(\sqrt{\rho} f) = \sqrt{\rho} D_j f - \langle R^{-1} x, e_j \rangle \sqrt{\rho} f / 2$. The first term is fine, while the second term may be treated as in the isotropic case (e.g., [10, Lemma 2.2]):

$$\int_{\mathbb{R}^d} (\langle R^{-1} x, e_j \rangle f(x))^2 \ \mathrm{d}\nu = - \int_{\mathbb{R}^d} (D_j \rho(x)) \langle R^{-1} x, e_j \rangle f(x)^2 \ \mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \rho(x) (\langle R^{-1} e_j, e_j \rangle f(x)^2 + 2 \langle R^{-1} x, e_j \rangle f(x) D_j f(x)) \ \mathrm{d}x$$

$$\le \langle R^{-1} e_j, e_j \rangle \|f\|^2_{L^2(\mathbb{R}^d, \nu)} + 2 \bigg(\int_{\mathbb{R}^d} (\langle R^{-1} x, e_j \rangle f(x))^2 \rho(x) \ \mathrm{d}x \bigg)^{1/2} \|D_j f\|_{L^2(\mathbb{R}^d, \nu)}$$

$$\le \langle R^{-1} e_j, e_j \rangle \|f\|^2_{L^2(\mathbb{R}^d, \nu)} + \frac{1}{2} \int_{\mathbb{R}^d} (\langle R^{-1} x, e_j \rangle f(x))^2 \rho(x) \ \mathrm{d}x + 2 \|D_j f\|_{L^2(\mathbb{R}^d, \nu)}.$$

Therefore,

$$\|\langle R^{-1} \cdot, e_j \rangle f\|^2_{L^2(\mathbb{R}^d, \nu)} \le 2 \langle R^{-1} e_j, e_j \rangle \|f\|^2_{L^2(\mathbb{R}^d, \nu)} + 4 \|D_j f\|^2_{L^2(\mathbb{R}^d, \nu)},$$

and the statement follows because polynomials are dense in $H_I^1(\mathbb{R}^d, \nu)$. ■

Remark 6. Notice that the two spaces in the above proposition are not equal. Take for example $d = 1$, $\nu = \mathcal{N}_{0,1}$ so that $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Some calculation gives that the function $f(x) := e^{\frac{x^2}{4}} (1 + x^2)^{-1/2}$ is such that $\sqrt{\rho} f \in H^1(\mathbb{R}, \ \mathrm{d}x)$ but f does not belong to $H^1(\mathbb{R}, \nu)$.

Now the embedding of the interpolation spaces is easy:

Proposition 7. *Let $m_0, m_1, \dots, m_{n-1} \in \mathbb{N}$ and $0 < \theta < 1$. Then for each $h = 0, \dots, n-1$ we have*

$$\Big(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \dots, m_{n-1}}(\mathbb{R}^d, \nu) \Big)_{\theta, 2} \subset \Big\{ f \in L^2(\mathbb{R}^d, \nu) : \ \sqrt{\rho} f \in H^{\theta m_h}_{I_h}(\mathbb{R}^d, \ \mathrm{d}x) \Big\},$$

and there exists $C > 0$ such that

$$\sum_{h=0}^{n-1} \|\sqrt{\rho} f\|_{H^{\theta m_h}_{I_h}(\mathbb{R}^d, \ \mathrm{d}x)} \le C \|f\|_{\left(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \dots, m_{n-1}}(\mathbb{R}^d, \nu)\right)_{\theta, 2}},$$

for each $f \in \big(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \dots, m_{n-1}}(\mathbb{R}^d, \nu) \big)_{\theta, 2}$.

Here the anisotropic Sobolev spaces with respect to the Lebesgue measure are defined as one can expect: a function $f \in L^2(\mathbb{R}^d,\, \mathrm{d}x)$ belongs to $H^s_{I_h}(\mathbb{R}^d,\, \mathrm{d}x)$ if it has derivatives up to the order $[s]$ with respect to the variables x_j, $j \in I_h$, belonging to $L^2(\mathbb{R}^d,\, \mathrm{d}x)$, and the derivatives $D^\beta f$ of order $[s]$ and $\beta \in \Lambda_h$ have finite seminorm

$$[D^\beta f]^2_{H^s_{I_h}(\mathbb{R}^d,\, \mathrm{d}x)} := \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}^2} \frac{|D^\beta f(x_1, \ldots, x_j^1, \ldots, x_d) - D^\beta f(x_1, \ldots, x_j^2, \ldots, x_d)|^2}{|x_j^1 - x_j^2|^{1+2\{s\}}} \,\mathrm{d}x_i^1 \,\mathrm{d}x_i^2 \,\mathrm{d}\hat{x}_i,$$

where $\{s\} = s - [s]$. The norm is

$$\|f\|^2_{H^s_{I_h}(\mathbb{R}^d,\, \mathrm{d}x)} := \sum_{|\beta| \geq 0,\, \beta \in \Lambda_h} \|D^\beta f\|^2_{L^2(\mathbb{R}^d,\, \mathrm{d}x)} + \sum_{|\beta| = [s],\, \beta \in \Lambda_h} [D^\beta f]^2_{H^s_{I_h}(\mathbb{R}^d,\, \mathrm{d}x)}.$$

Note that, since ρ and all its derivatives are locally bounded, then each f in the interpolation space $(L^2(\mathbb{R}^d, \nu), H^{m_0, m_1, \ldots, m_{n-1}}(\mathbb{R}^d, \nu))_{\theta,2}$ is locally $H^{\theta m_h}$ with respect to the variables x_j, $j \in I_h$.

4 The main result

Recall the decomposition of $\mathbb{R}^d = \bigoplus_{h=0}^{n-1} W_h$ and the corresponding basis $\{e_1, \ldots, e_d\}$ together with the grouping I_h, $h = 0, \ldots, n-1$ of the indices as given in Section 1. We introduce the following abbreviation (μ is the invariant measure for the Ornstein–Uhlenbeck semigroup, see Section 1), for $s > 0$,

$$\mathcal{H}^s(\mathbb{R}^d, \mu) := H^{s, s/3, s/5, \ldots, s/(2n-1)}(\mathbb{R}^d, \mu).$$

Let $(L, D(L))$ be the infinitesimal generator of the the Ornstein–Uhlenbeck semigroup $(T(t))_{t \geq 0}$. Our main result is the following inclusion of $D(L^k)$ into the fractional Sobolev space $\mathcal{H}^{2k}(\mathbb{R}^d, \mu)$.

Theorem 8. *Let $k \in \mathbb{N}$. For the domain of the Ornstein–Uhlenbeck operator L we have*

$$D(L^k) \subseteq \mathcal{H}^{2k}(\mathbb{R}^d, \mu) = H^{2k,\, 2k/3,\, 2k/5, \ldots,\, 2k/(2n-1)}(\mathbb{R}^d, \mu).$$

The proof relies on the abstract interpolation result given below (for a proof see [12]). Recall that whenever $Y \subseteq E \subseteq X$ are Banach spaces and $0 < \beta < 1$, E is said to belong to the class $J_\beta(X, Y)$ if there exists a constant $c > 0$ such that for all $y \in Y$ the norm inequality $\|y\|_E \leq c \|y\|_X^{1-\theta} \cdot \|y\|_Y^\theta$ holds ([17, Sec. 1.10.1]).

Theorem 9. *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup in a Banach space X with generator $(A, D(A))$. Suppose that there is a Banach space $E \subseteq X$ and some constants $m \in \mathbb{N}$, $0 < \beta < 1$, $\omega \in \mathbb{R}$, $c > 0$ such that*

$$\|T(t)\|_{\mathcal{L}(X,E)} \leq \frac{c e^{\omega t}}{t^{m\beta}} \qquad \text{for } t > 0,$$

and for each $x \in X$ the function $(0, \infty) \ni t \mapsto T(t)x \in E$ is measurable. Then E belongs to the class $J_\beta(X, D(A^m))$, so by reiteration

$$(X, D(A^m))_{\theta,p} \subset (X, E)_{\theta/\beta, p} \qquad \text{for all } \theta \in (0, \beta) \text{ and } 1 \leq p \leq \infty.$$

The combination of this theorem with estimates of the derivatives of $(T(t))_{t \geq 0}$ and with the characterization of real interpolation spaces between $L^2(\mathbb{R}^d, \mu)$ and $\mathcal{H}^m(\mathbb{R}^d, \mu)$ yields the

Proof of Theorem 8. We apply Theorem 9 to the Ornstein–Uhlenbeck semigroup T by setting $X = L^2(\mathbb{R}^d, \mu)$ and $E = \mathcal{H}^{k(2n-1)!}(\mathbb{R}^d, \mu)$. The measurability, actually the continuity, assumption is obtained from Lemma 2. Also this lemma implies the estimate for the semigroup

$$\|T(t)\|_{\mathcal{L}(L^2(\mathbb{R}^d, \mu), \mathcal{H}^{k(2n-1)!}(\mathbb{R}^d, \mu))} \leq \frac{c e^{\omega t}}{t^{k(2n-1)!/2}}.$$

Let $m \in \mathbb{N}$ be so large that $\beta := (2n-1)!/2m$ belongs to $(0,1)$. Taking further $\theta = 1/m$, we see that the assumptions of Theorem 9 are fulfilled. Whence we conclude the inclusion

$$\Big(L^2(\mathbb{R}^d,\mu), D(L^{km})\Big)_{1/m,2} \subseteq \Big(L^2(\mathbb{R}^d,\mu), \mathcal{H}^{k(2n-1)!}(\mathbb{R}^d,\mu)\Big)_{2/(2n-1)!,2} = \mathcal{H}^{2k}(\mathbb{R}^d,\mu).$$

Next, we show that the domain of L^k can be obtained as

$$D(L^k) = \Big(L^2(\mathbb{R}^d,\mu), D(L^{km})\Big)_{1/m,2}.$$

The following argument easily proves this equality. Since L is m-dissipative, so are $L - \lambda I$ for all $\lambda > 0$. A classical theorem of Kato [7] tells us that $\lambda I - L$ has bounded imaginary powers for all $\lambda > 0$. Then by complex interpolation (see Triebel [17, Thm. 1.15.3.]), we obtain

$$D(L^k) = \Big[L^2(\mathbb{R}^d,\mu), D(L^{km})\Big]_{1/m}.$$

And now the nice feature of Hilbert spaces enters the picture, namely we have the equality of real an complex interpolation spaces (see Triebel [17, Sec.1.18.10, p. 143])

$$\Big[L^2(\mathbb{R}^d,\mu), D(L^{km})\Big]_{1/m} = \Big(L^2(\mathbb{R}^d,\mu), D(L^{km})\Big)_{1/m,2}.$$

The proof is hence complete. ■

5 An example

Consider the following operator in $\mathbb{R}^{2d}$:

$$\mathcal{L}f(x,y) = \frac{1}{2}\Delta_x f - \langle My + x, D_x f\rangle + \langle x, D_y f\rangle, \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $M \in M_{d\times d}(\mathbb{R})$ is positive. The corresponding matrices Q, B are

$$B = \begin{pmatrix} -\mathrm{Id} & -M \\ \mathrm{Id} & 0 \end{pmatrix}, \qquad \text{and} \quad Q = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

With $n = 1$ the Kalman rank condition is satisfied:

$$\operatorname{rank}[Q^{1/2}, BQ^{1/2}] = \operatorname{rank}\begin{pmatrix} \mathrm{Id} & 0 & -\mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} & 0 \end{pmatrix} = 2d.$$

The corresponding decomposition of the space $\mathbb{R}^{2d}$ are given by the projections

$$P_0 = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

One can determine the matrix Q_∞ as well,

$$Q_\infty = \frac{1}{2}\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & M^{-1} \end{pmatrix},$$

so that, with obvious notation,

$$\mathrm{d}\mu(x,y) = \mathrm{d}\mathcal{N}_{0,Q_\infty}(x,y) = \frac{\det M^{1/2}}{\pi^d} \exp\big(-|x|^2 - \langle My, y\rangle\big)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence Theorem 8 gives that for $g \in L^2(\mathbb{R}^d,\mu)$ and $\lambda > 0$ the solution f to

$$\lambda f - Lf = g$$

lies in $H^{2,2/3}(\mathbb{R}^d,\mu)$. Therefore, it has derivatives up to the 2^{nd} order with respect to the variables in the x-subspace in $L^2(\mathbb{R}^d,\mu)$ and it satisfies

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x,y_1)e^{-\frac{\langle My_1,y_1\rangle}{2}} - f(x,y_2)e^{-\frac{\langle My_2,y_2\rangle}{2}}|^2}{|y_1-y_2|^{d+4/3}}\,\mathrm{d}y_1\,\mathrm{d}y_2\,e^{-|x|^2}\,\mathrm{d}x < \infty.$$

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