ASYMPTOTIC BEHAVIOR AND HYPERCONTRACTIVITY IN NONAUTONOMOUS ORNSTEIN-UHLENBECK EQUATIONS

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Abstract. In this paper we investigate a class of nonautonomous linear parabolic problems with time-depending Ornstein-Uhlenbeck operators. We study the asymptotic behavior of the associated evolution operator and evolution semigroup in the periodic and non-periodic situation. Moreover, we show that the associated evolution operator is hypercontractive.

1. Introduction

In this paper we continue the investigations of [DPL06, GL07] on a class of nonautonomous linear parabolic problems with time-depending Ornstein-Uhlenbeck operators. We study asymptotic behavior and hypercontractivity in Cauchy problems,

$$\begin{cases} u_s(s,x) + \mathcal{L}(s)u(s,x) = 0, \;\; s \le t, \; x \in \mathbb{R}^n, \\ \\ u(t) = \varphi(x), \; x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

as well as equations with time in the whole $\mathbb{R}$ and no initial or final data,

$$\lambda u(s,x) - (u_s(s,x) + \mathcal{L}(s)u(s,x)) = h(s,x), \;\; s \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{1.2}$$

Here $(\mathcal{L}(t))_{t\in\mathbb{R}}$ is a family of Ornstein-Uhlenbeck operators,

$$\mathcal{L}(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}\,\left(B(t)B^*(t)\mathrm{D}_x^2\varphi(x)\right) + \langle A(t)x + f(t), \mathrm{D}_x\varphi(x)\rangle, \quad x \in \mathbb{R}^n, \tag{1.3}$$

with continuous and bounded data $A, B : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}^n$. Throughout the paper we assume that the operators $\mathcal{L}$ are uniformly elliptic, i.e. there exists $\mu_0 > 0$ such that

$$\|B(t)x\| \ge \mu_0\|x\|, \quad t \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{1.4}$$

The backward Cauchy problem (1.1) is the Kolmogorov equation of the nonautonomous stochastic ODE

$$\begin{cases} \mathrm{d}X_t = (A(t)X_t + f(t))\mathrm{d}t + B(t)\mathrm{d}W(t), \\ \\ X_s = x, \end{cases} \tag{1.5}$$

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where $W(t)$ is a standard n-dimensional Brownian motion and $s \in \mathbb{R}$, $x \in \mathbb{R}^n$. Indeed, denoting by $X(s,t,x)$ the solution to (1.5), for each $t \in \mathbb{R}$ and $\varphi \in C_b^2(\mathbb{R}^n)$ the function $u(s,x) := \mathbb{E}(\varphi(X(s,t,x)))$ satisfies (1.1). See e.g. [GS72, KS91].

Under our ellipticity assumption, u is in fact a classical solution to (1.1) just for $\varphi \in C_b(\mathbb{R}^n)$. The transition evolution operator $P_{s,t}\varphi(x) := \mathbb{E}[\varphi(X(t,s,x))]$ may be explicitly written as

$$P_{s,t}\varphi(x) = \int_{\mathbb{R}^n} \varphi(y)\mathcal{N}_{m(t,s),Q(t,s)}(dy), \quad \varphi \in C_b(\mathbb{R}^n),\ s \le t. \tag{1.6}$$

Here $\mathcal{N}_{m(t,s),Q(t,s)}$ is the Gaussian measure with mean $m(t,s)$ and covariance $Q(t,s)$ given respectively by

$$m(t,s) := U(t,s)x + \int_s^t U(t,r)f(r)dr, \quad Q(t,s) := \int_s^t U(t,r)B(r)B^*(r)U^*(t,r)dr, \tag{1.7}$$

and U is the evolution operator for $A(\cdot)$, i.e. for each $x \in \mathbb{R}^n$ the function $t \mapsto U(t,s)x$ is the solution to $\xi'(t) = A(t)\xi(t)$, $\xi(s) = x$.

In the autonomous elliptic case $B(t) \equiv B$, $A(t) \equiv A$, $f(t) \equiv 0$, with $\det B \neq 0$, we have $P_{s,t} = T(t-s)$ where $T(t)$ is the Ornstein-Uhlenbeck semigroup. $T(t)$ is a Markov semigroup in $C_b(\mathbb{R}^n)$. Its asymptotic behavior is well understood in the case that all the eigenvalues of A have negative real part, so that $\|e^{tA}\|$ decays exponentially as $t \to \infty$. In this case, for each $x \in \mathbb{R}^n$ $T(t)\varphi(x)$ converges to a constant which is the mean value of φ with respect to the unique invariant measure $\mu = \mathcal{N}_{0,Q_\infty}$ of $T(t)$, i.e. the unique Borel probability measure in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} T(t)\varphi \, d\mu = \int_{\mathbb{R}^n} \varphi \, d\mu, \quad t > 0,\ \varphi \in C_b(\mathbb{R}^n).$$

For each $p \in [1,+\infty)$, $T(t)$ is extended in a standard way to a contraction semigroup (still denoted by $T(t)$) in $L^p(\mathbb{R}^n,\mu)$. If $\varphi \in L^p(\mathbb{R}^n,\mu)$, then $T(t)\varphi$ converges exponentially to the mean value of φ in $L^p(\mathbb{R}^n,\mu)$, and the rate of convergence coincides with the rate of decay of $\|e^{tA}\|$ to zero. Moreover, $T(t)$ is hypercontractive, i.e. for $p > 1$ and $t > 0$ it maps $L^p(\mathbb{R}^n,\mu)$ into $L^{q(t)}(\mathbb{R}^n,\mu)$ for a suitable $q(t) > p$, and with norm ≤ 1.

In our nonautonomous case the assumption that $\|e^{tA}\|$ decays exponentially as $t \to \infty$ is replaced by the assumption that $\|U(t,s)\|$ decays exponentially as $t - s \to \infty$. More precisely we assume that

$$\begin{aligned} \omega_0(U) := \inf\{ \quad & \omega \in \mathbb{R} :\ \exists M = M(\omega)\,\text{such that} \\ & \|U(t,s)\| \le M e^{\omega(t-s)}, \quad -\infty < s \le t < \infty\} < 0. \end{aligned} \tag{1.8}$$

Then there is not a unique invariant measure, but there exist families of Borel probability measures $\{\nu_t :\ t \in \mathbb{R}\}$, called *entrance laws at time* $-\infty$ in [Dyn89] and *evolution systems of measures* in [DPR05], such that

$$\int_{\mathbb{R}^n} P_{s,t}\varphi \, d\nu_s = \int_{\mathbb{R}^n} \varphi \, d\nu_t, \quad \varphi \in C_b(\mathbb{R}^n),\ s \le t. \tag{1.9}$$

Such families are infinitely many, and they were characterized in [GL07]. Among all of them, a distinguished one has a prominent role in the asymptotic behavior of $P_{s,t}$. It is the family of measures ν_t defined by

$$\nu_t = \mathcal{N}_{g(t,-\infty),Q(t,-\infty)}, \quad t \in \mathbb{R}, \tag{1.10}$$

and it is the unique one with uniformly bounded moments of some order, i.e. there exists $\alpha > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^n} |x|^\alpha \nu_t(\mathrm{d}x) < +\infty. \tag{1.11}$$

In fact, it satisfies (1.11) for each $\alpha > 0$. This implies that for each $\varphi \in C_b(\mathbb{R}^n)$ and for each $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ we have

$$\lim_{s\to-\infty} P_{s,t}\varphi(x) = \int_{\mathbb{R}^n} \varphi(y)\mathrm{d}\nu_t.$$

As in the autonomous case, we have a much better behavior if we work in L^p spaces with respect to the measures ν_t. But in this context, the evolution operator $P_{s,t}$ maps $L^p(\mathbb{R}^n, \nu_t)$ into $L^p(\mathbb{R}^n, \nu_s)$, hence it cannot be seen as an evolution operator in a fixed Banach space X. Still, we have the contraction estimate

$$\|P_{s,t}\|_{\mathcal{L}(L^p(\mathbb{R}^n,\nu_t),L^p(\mathbb{R}^n,\nu_s))} \le 1, \quad s < t,$$

as well as smoothing estimates, proved in [GL07], that are optimal both for $t - s$ close to 0 and for $t - s \to \infty$, and that are quite similar to the corresponding estimates in the autonomous case:

$$\|\mathrm{D}^\alpha_x P_{s,t}\|_{\mathcal{L}(L^p(\mathbb{R}^n,\nu_t),L^p(\mathbb{R}^n,\nu_s))} \le \begin{cases} C(t-s)^{-|\alpha|/2}\mathrm{e}^{\omega|\alpha|(t-s)}, & 0 < t-s < 1, \\ C\mathrm{e}^{\omega|\alpha|(t-s)}, & t-s > 1. \end{cases} \tag{1.12}$$

Here α is any multi-index, ω is any number in $(\omega_0(U), 0)$ and $C = C(\alpha, \omega)$.

Such estimates are the starting point for our study of asymptotic behavior in the L^2 setting. As in the theory of ordinary differential equations, we get very precise asymptotic behavior results if the data are time periodic. In this case the asymptotic behavior of the evolution operator $P_{s,t}$ is driven by the spectral properties of $P_{0,T}$, where T is the period. Note that $P_{0,T}$ is a bounded operator in $L^2(\mathbb{R}^n, \nu_0)$ since $\nu_0 = \nu_T$. By estimates (1.12), $P_{0,T}$ is bounded from $L^2(\mathbb{R}^n, \nu_0)$ to $H^1(\mathbb{R}^n, \nu_0)$, which is compactly embedded in $L^2(\mathbb{R}^n, \nu_0)$ since ν_0 is a Gaussian measure with nondegenerate covariance matrix. Therefore, its spectrum consists of 0, plus (at most) a sequence of eigenvalues. We show that the unique eigenvalue of $P_{0,T}$ in the unit circle is 1, that it has eigenvalues with modulus equal to $\exp(\omega_0(U)T)$, and that the modulus of the other eigenvalues does not exceed $\exp(\omega_0(U)T)$.

For any $t \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^n, \nu_t)$ let

$$M_t\varphi := \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu_t$$

be the mean value of φ with respect to ν_t. We know from [DPL06] that the $L^2(\mathbb{R}^n, \nu_s)$-norm of $P_{s,t}(\varphi - M_t\varphi)$ converges exponentially to 0 as $t - s \to \infty$. Using the above spectral properties, we determine the exact convergence rate, proving that for each $\omega \in (\omega_0(U), 0)$ there is $M > 0$ such that

$$\|P_{s,t}(\varphi - M_t\varphi)\|_{L^2(\mathbb{R}^n,\nu_s)} \le Me^{\omega(t-s)}\|\varphi\|_{L^2(\mathbb{R}^n,\nu_t)}, \quad s < t,\ \varphi \in L^2(\mathbb{R}^n, \nu_t), \tag{1.13}$$

and that for each $\omega < \omega_0(U)$ there is no M such that (1.13) holds. Moreover, (1.13) holds also for $\omega = \omega_0(U)$ iff all the eigenvalues of $U(T, 0)$ with modulus equal to $\exp(T\omega_0(U))$ are semisimple.

Still in the case of T-periodic coefficients, a natural setting for problem (1.2) is the space $L^2_{\#}(\mathbb{R}^{1+n},\nu)$ consisting of the Lebesgue measurable functions h such that $h(s+T,x) = h(s,x)$ a.e. and the norm

$$\|h\|_{L^2_{\#}(\mathbb{R}^{1+n},\nu)} = \left(\frac{1}{T}\int_0^T\int_{\mathbb{R}^n} |h(s,x)|^2 \,\mathrm{d}\nu_s\mathrm{d}s\right)^{1/2}$$

is finite. In the paper [GL07] we showed that if λ is any complex number, $h \in L^2_{\#}(\mathbb{R}^{1+n},\nu)$, and $u \in L^2_{\#}(\mathbb{R}^{1+n},\nu) \cap H^{1,2}_{loc}(\mathbb{R}^{1+n}, \mathrm{d}t\times\mathrm{d}x)$ is a time periodic solution of (1.2), then u belongs to $H^{1,2}_{\#}(\mathbb{R}^{1+n},\nu)$ i.e. u_t and all the space derivatives $u_{x_ix_j}$ belong to $L^2_{\#}(\mathbb{R}^{1+n},\nu)$. The operator

$$\begin{cases} G_{\#} : D(G_{\#}) = H^{1,2}_{\#}(\mathbb{R}^{1+n},\nu) \mapsto L^2_{\#}(\mathbb{R}^{1+n},\nu), \\ \\ G_{\#}u(s,x) = u_s(s,x) + \mathcal{L}(s)u(s,x) \end{cases}$$

may be seen as the infinitesimal generator of the evolution semigroup $\mathcal{P}^{\#}_{\tau}u$ in $L^2_{\#}(\mathbb{R}^{1+n},\nu)$ defined by

$$(\mathcal{P}^{\#}_{\tau}u)(s,x) = (P_{s,s+\tau}u(s+\tau,\cdot))(x), \quad s\in\mathbb{R},\ x\in\mathbb{R}^n,\ \tau\geq 0,\ u\in L^2_{\#}(\mathbb{R}^{1+n},\nu), \tag{1.14}$$

and the measure ν is invariant for the semigroup $(\mathcal{P}^{\#}_{\tau})_{\tau\geq 0}$, see [DPL06]. Although $(\mathcal{P}^{\#}_{\tau})_{\tau\geq 0}$ is not a standard evolution semigroup (since, as we already remarked, $P_{s,s+\tau}$ does not act in a fixed Banach space X but it maps $L^2(\mathbb{R}^n,\nu_{s+\tau})$ into $L^2(\mathbb{R}^n,\nu_s)$), a part of the classical theory of evolution semigroups may be extended to our situation, and the spectral properties of the generator $G_{\#}$ are strongly connected with the asymptotic behavior of $\mathcal{P}^{\#}_{\tau}$. In its turn, the asymptotic behavior of $\mathcal{P}^{\#}_{\tau}$ may be easily deduced from the asymptotic behavior of $P_{s,t}$. In particular, setting

$$(\Pi u)(t,x) := M_t u(t,\cdot), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{1.15}$$

and using (1.13), we see that $\mathcal{P}^{\#}_{\tau}u$ converges exponentially to Πu as $\tau\to\infty$, for each $u\in L^2_{\#}(\mathbb{R}^{1+n},\nu)$, and the growth bound of $(\mathcal{P}^{\#}_{\tau}(I-\Pi))_{\tau\geq 0}$ is $\omega_0(U)$. Π is the spectral projection relative to $\sigma(G_{\#})\cap i\mathbb{R} = 2\pi i\mathbb{Z}/T$, its range is isomorphic to $L^2_{\#}(\mathbb{R};\mathrm{d}t)$. Moreover, $G_{\#}$ has infinitely many isolated eigenvalues on the vertical line $\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda = \omega_0(U)\}$. The real parts of the remaining eigenvalues are less than $\omega_0(U)$. On the other hand, the spectrum of $G_{\#}$ consists of eigenvalues only, because $D(G_{\#})$ is compactly embedded in $L^2_{\#}(\mathbb{R}^{1+n},\nu)$ as we proved in [GL07].

So, $G_{\#}$ has a spectral gap that corresponds precisely to the asymptotic behavior of $(\mathcal{P}^{\#}_{\tau}(I-\Pi))_{\tau\geq 0}$. This implies that for each λ with real part in $(0,+\infty)$, in $(\omega_0(U),0)$, and also for $\lambda\in i\mathbb{R}\setminus 2\pi i\mathbb{Z}/T$, for each $h\in L^2_{\#}(\mathbb{R}^{1+n},\nu)$ equation (1.2) has a unique solution $u\in D(G_{\#})$. For $\lambda=0$, it is easy to see that the range of $G_{\#}$ consists of the functions h such that the mean value $\int_0^T\int_{\mathbb{R}^n} h(t,x)\,\mathrm{d}\nu_t\,\mathrm{d}t$ vanishes, and in this case the solution of (1.2) is unique up to constants.

If the data are not periodic but just bounded, a natural Hilbert setting for problem (1.2) is the space $L^2(\mathbb{R}^{1+n},\nu)$ consisting of the Lebesgue measurable functions h such that the norm

$$\|h\|_{L^2(\mathbb{R}^{1+n},\nu)} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}^n} |h(s,x)|^2\,\mathrm{d}\nu_s\mathrm{d}s\right)^{1/2}$$

is finite. A maximal regularity result similar to the one in the periodic space still holds, namely if $\lambda \in \mathbb{C}$, $h \in L^2(\mathbb{R}^{1+n}, \nu)$ and $u \in L^2(\mathbb{R}^{1+n}, \nu) \cap H^{1,2}_{loc}(\mathbb{R}^{1+n}, \mathrm{d}t \times \mathrm{d}x)$ is a solution of (1.2), then $u \in H^{1,2}(\mathbb{R}^{1+n}, \nu)$ i.e. u_t and all the space derivatives $u_{x_i x_j}$ belong to $L^2(\mathbb{R}^{1+n}, \nu)$. The operator

$$\begin{cases} G : D(G) = H^{1,2}(\mathbb{R}^{1+n}, \nu) \mapsto L^2(\mathbb{R}^{1+n}, \nu), \\ \\ Gu(s, x) = u_s(s, x) + \mathcal{L}(s)u(s, x) \end{cases}$$

is the infinitesimal generator of the evolution semigroup $(\mathcal{P}_\tau)_{\tau \geq 0}$ in $L^2(\mathbb{R}^{1+n}, \nu)$, defined as $\mathcal{P}^{\#}_\tau$ by

$$(\mathcal{P}_\tau u)(s, x) = (P_{s,s+\tau} u(s + \tau, \cdot))(x), \quad s \in \mathbb{R},\ x \in \mathbb{R}^n,\ \tau \geq 0,\ u \in L^2(\mathbb{R}^{1+n}, \nu). \tag{1.16}$$

See [GL07]. A part of the properties of $(\mathcal{P}^{\#}_\tau)_{\tau \geq 0}$ and $G_{\#}$ are enjoyed by $(\mathcal{P}_\tau)_{\tau \geq 0}$ and G. However, without periodicity and compact embeddings, the results are less precise. G has still a spectral gap: its spectrum contains the whole imaginary axis, and it has no elements with real part in $(c_0, 0)$, where $c_0 < 0$ depends on A and B. Therefore, for each λ with real part in $(c_0, 0) \cup (0, +\infty)$ and for each $h \in L^2(\mathbb{R}^{1+n}, \nu)$, equation (1.2) has a unique solution in $D(G)$. For $\lambda = 0$, we show that for $h \in L^2(\mathbb{R}^{1+n}, \nu)$ problem (1.2) has a solution in $D(G)$ iff the function $t \mapsto M_t h$ has a primitive in $L^2(\mathbb{R}; \mathrm{d}t)$, in this case the solution is unique.

The projection Π defined in (1.15) is still the spectral projection relative to the imaginary axis, the range of Π is isomorphic to $L^2(\mathbb{R}; \mathrm{d}t)$, the restriction of $(\mathcal{P}_\tau)_{\tau \geq 0}$ to the range of Π is the translation semigroup in $L^2(\mathbb{R}; \mathrm{d}t)$, and the growth bound of $(\mathcal{P}_\tau(I - \Pi))_{\tau \geq 0}$ does not exceed c_0. So, for each $u \in L^2(\mathbb{R}^{1+n}, \nu)$, $\mathcal{P}_\tau u$ converges exponentially to Πu as $\tau \to \infty$ and we have an estimate for the convergence rate; the optimal convergence rate is still an open problem.

Our procedure is reversed with respect to the periodic setting. As a first result we show that $\mathcal{P}_\tau(I - \Pi)$ converges exponentially to zero through Poincaré type inequalities that hold in $D(G)$. Then from the general theory of semigroups, it follows that the spectrum of the part of G in $(I - \Pi)(L^2(\mathbb{R}^{1+n}, \nu))$ is contained in the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq c_0\}$. Moreover we obtain asymptotic behavior properties of $P_{s,t}$ from the asymptotic behavior properties of $\mathcal{P}_\tau$, adapting to our situation the method used for the standard evolution semigroups and evolution operators. A crucial point in the proof is the continuity of the function $s \mapsto \|P_{s,s+\tau}\varphi\|^2_{L^2(\mathbb{R}^n, \nu_s)}$, for each $\tau > 0$ and for each good φ, say $\varphi \in C^1_b(\mathbb{R}^n)$. Eventually, we obtain

$$\|P_{s,t}(\varphi - M_t\varphi)\|_{L^2(\mathbb{R}^n, \nu_s)} \leq e^{c_0(t-s)} \|\varphi\|_{L^2(\mathbb{R}^n, \nu_t)}, \quad s < t,\ \varphi \in L^2(\mathbb{R}^n, \nu_t),$$

where c_0 is the above constant.

In the last section we show that $(P_{s,t})_{s \leq t}$ is hypercontractive, i.e. $P_{s,t}$ maps $L^q(\mathbb{R}^n, \nu_t)$ into $L^{p(s,t)}(\mathbb{R}^n, \nu_s)$ for suitable $p(s, t) > q$ if $s < t$, $q > 1$, and

$$\|P_{s,t}\varphi\|_{L^{p(s,t)}(\mathbb{R}^n, \nu_s)} \leq \|\varphi\|_{L^q(\mathbb{R}^n, \nu_t)}, \quad \varphi \in L^q(\mathbb{R}^n, \nu_t),\ s \leq t.$$

Moreover, $p(s, t) \geq 1 + (q - 1)e^{2c_0(s-t)}$. Estimates of this type are well-known in the autonomous case, see [CMG96, Fuh98, Gro75]. As far as we know, this is the first hypercontractivity result in the nonautonomous case.

Our approach is based on the ideas used in [Gro75]. More precisely, we differentiate

$$\alpha(s) = \|P_{s,t}\varphi\|_{L^{p(s,t)}(\mathbb{R}^n,\nu_s)}$$

with respect to s for suitable functions φ and we show that $\alpha'(s) \geq 0$ for $s \leq t$ with help of a variant of the classical logarithmic Sobolev inequalities. The difference with the autonomous case is that we have to deal with additional terms since the measure ν_s depends on s as well.

2. Spectral properties and asymptotic behavior

In this section we investigate the spectrum of $(\mathcal{P}_\tau)_{\tau\geq 0}$ and $(\mathcal{P}_\tau^{\#})_{\tau\geq 0}$, and of their generators. This leads to results about asymptotic behavior of such semigroups, and of the evolution operator $P_{s,t}$.

We already remarked that the general theory of parabolic evolution operators in Banach spaces cannot be directly applied to our $P_{s,t}$ because it does not act on a fixed L^2 space but it maps $X(t) = L^2(\mathbb{R}^n, \nu_t)$ into $X(s) = L^2(\mathbb{R}^n, \nu_s)$ and these spaces do not coincide in general. The same difficulty arises for the evolution semigroups $(\mathcal{P}_\tau^{\#})_{\tau\geq 0}$ and $(\mathcal{P}_\tau)_{\tau\geq 0}$, since the general theory (see e.g. the monograph [CL99]) has been developed for evolution semigroups associated to evolution operators in a fixed Banach space X. Therefore, we have to start from the very beginning. However, some results can be extended to our situation with minor modifications. This is the case of the spectral mapping theorems of the next subsection.

2.1. Spectral mapping theorems.

We start with the spectral mapping theorem for $(\mathcal{P}_\tau^{\#})_{\tau\geq 0}$. Next proposition 2.1 is is a variant of [CL99, Theorem 3.13] for time-depending spaces. Its proof is based on the "change-of-variable" trick, see [LMS95].

We need some preparatory remarks.

If X is any Banach space, we define the space $L^2_{\#}(\mathbb{R}, X)$ as the space of all Bochner measurable functions $Z : \mathbb{R} \mapsto X$, such that $Z(\theta + T) = Z(\theta)$ for almost all $\theta \in \mathbb{R}$ and $\|Z\|^2 := \int_0^T \|Z(\theta)\|_X^2 \,\mathrm{d}\theta < \infty$.

If $X = L^2_{\#}(\mathbb{R}^{1+n})$, then $L^2_{\#}(\mathbb{R}, X)$ may be identified (setting $z(\theta, t, x) = Z(\theta)(t, x)$ for each $Z \in L^2_{\#}(\mathbb{R}, X)$) with the space $L^2_{\#}(\mathbb{R}^{2+n})$ consisting of the Lebesgue measurable functions z defined in $\mathbb{R}^{2+n}$ such that $z(\theta + T, t, x) = z(\theta, t, x)$, $z(\theta, t + T, x) = z(\theta, t, x)$ for almost all θ, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, endowed with the norm

$$\|z\| = \frac{1}{T}\left(\int_0^T \int_0^T \int_{\mathbb{R}^n} |z(\theta, t, x)|^2 \nu_t(\mathrm{d}x)\, \mathrm{d}t\, \mathrm{d}\theta\right)^{1/2}.$$

Proposition 2.1. *If A, B, and f are T-periodic, then*

$$\sigma(\mathcal{P}_\tau^{\#}) \setminus \{0\} = \mathrm{e}^{\tau\sigma(G_\#)}, \quad \tau > 0.$$

Proof. The inclusion $\mathrm{e}^{\tau\sigma(G_\#)} \subset \sigma(\mathcal{P}_\tau^{\#})$ comes from the general theory of semigroups, see e.g. [EN00, §3.6]. We have to prove that $\sigma(\mathcal{P}_\tau^{\#}) \setminus \{0\} \subset \mathrm{e}^{\tau\sigma(G_\#)}$, or, equivalently, that if $\lambda \in \rho(G_\#)$ then $e^{\tau\lambda} \in \rho(\mathcal{P}_\tau^{\#})$.

Set $X := L^2_{\#}(\mathbb{R}^{1+n})$. We define two semigroups in the space $L^2_{\#}(\mathbb{R}, X)$. The first one is the T-periodic evolution semigroup associated to our semigroup $\mathcal{P}_\tau^{\#}$, the second one is the so called multiplication semigroup by $\mathcal{P}_\tau^{\#}$:

$$\begin{aligned}(\widetilde{\mathcal{P}}_\tau Z)(\theta) &= \mathcal{P}_\tau^{\#}(Z(\theta-\tau)), \quad \tau > 0,\\ (\mathcal{E}_\tau Z)(\theta) &= \mathcal{P}_\tau^{\#}(Z(\theta)), \quad \tau > 0.\end{aligned}$$

It is easy to see that the infinitesimal generator A of $(\mathcal{E}_\tau)_{\tau\geq 0}$ is the multiplication operator by $G_{\#}$, that is

$$D(A) = \{Z \in L^2_{\#}(\mathbb{R}, X) :\ Z(\theta) \in D(G_{\#}) \text{ a.e}\}, \quad AZ(\theta) = G_{\#}Z(\theta),$$

the resolvent set $\rho(A)$ of A coincides with $\rho(G_{\#})$, and $(R(\lambda, A)F)(\theta) = R(\lambda, G_{\#})(F(\theta))$ for all $\lambda \in \rho(G_{\#})$, $F \in L^2_{\#}(\mathbb{R}, X)$ and $\theta \in \mathbb{R}$.

Now we prove that $\rho(A) = \rho(\widetilde{G})$, where $\widetilde{G}$ is the infinitesimal generator of $(\widetilde{\mathcal{P}}_\tau)_{\tau\geq 0}$.

Setting as above $z(\theta, t, x) = Z(\theta)(t, x)$, we identify $L^2_{\#}(\mathbb{R}, X)$ with $L^2_{\#}(\mathbb{R}^{2+n})$. Then $\widetilde{\mathcal{P}}_\tau$ and $\mathcal{E}_\tau z$ may be rewritten as semigroups in $L^2_{\#}(\mathbb{R}^{2+n})$,

$$\begin{aligned}(\widetilde{\mathcal{P}}_\tau z)(\theta, t, x) &= \mathcal{P}_\tau^{\#} z(\theta-\tau, \cdot, \cdot)(t, x) &&= P_{t,t+\tau} z(\theta-\tau, t+\tau, \cdot)(x),\\ (\mathcal{E}_\tau z)(\theta, t, x) &= \mathcal{P}_\tau^{\#} z(\theta, \cdot, \cdot)(t, x) &&= P_{t,t+\tau} z(\theta, t+\tau, \cdot)(x).\end{aligned}$$

We define the isometry $J : L^2_{\#}(\mathbb{R}^{2+n}) \mapsto L^2_{\#}(\mathbb{R}^{2+n})$ by

$$(Jz)(\theta, t, x) = z(\theta - t, t, x), \quad (\theta, t, x) \in \mathbb{R}^{2+n}.$$

Then $\mathcal{E}_\tau J = J\widetilde{\mathcal{P}}_\tau$ for each $\tau > 0$, and this implies immediately that $D(\widetilde{G}) = J^{-1}(D(A))$, $\widetilde{G} = J^{-1}AJ$ and $\rho(\widetilde{G}) = \rho(A)$.

So, we have

$$\rho(G_{\#}) = \rho(A) = \rho(\widetilde{G}).$$

Since $(\widetilde{\mathcal{P}}_\tau)_{\tau\geq 0}$ is an evolution semigroup, then by the general theory of evolution semigroups we have $\rho(\mathcal{P}_\tau^{\#}) = \rho(\widetilde{\mathcal{P}}_\tau) = \mathrm{e}^{\tau\rho(\widetilde{G})}$ for each $\tau \geq 0$, see e.g. [CL99, Theorem 2.30]. In particular, if $\lambda \in \rho(G_{\#})$ then $\mathrm{e}^{\tau\lambda} \in \rho(\widetilde{\mathcal{P}}_\tau) = \rho(\mathcal{P}_\tau^{\#})$, and the statement follows. □

We have a corresponding result in the non-periodic case. The proof is the same, with the space $L^2(\mathbb{R}, L^2(\mathbb{R}^{1+n}, \nu))$ instead of $L^2_{\#}(\mathbb{R}, L^2_{\#}(\mathbb{R}^{1+n}))$.

Proposition 2.2. *We have*

$$\sigma(\mathcal{P}_\tau) \setminus \{0\} = \mathrm{e}^{\tau\sigma(G)}, \quad \tau > 0.$$

2.2. **Exponential dichotomy and asymptotic behavior of $P_{s,t}$ in the periodic case.** Throughout this section we assume that A, B, and f are T-periodic. As in the case of a fixed Banach space X (see [Hen81]), the asymptotic behavior of $P_{s,t}$ is determined by the spectral properties of the Poincaré operators,

$$V(t) := P_{t-T,t} \in \mathcal{L}(L^2(\mathbb{R}^n, \nu_t)), \quad t \in \mathbb{R}.$$

In the following proposition we collect the spectral properties of the operators $V(t)$ that will be used in the sequel. An important role is played by the projections on the subspace of constant functions, given by the mean values:

$$M_t\varphi := \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\nu_t, \quad \varphi \in L^2(\mathbb{R}^n, \nu_t). \tag{2.1}$$

We recall that the eigenvalues of $U(t+T,t)$ are independent of t, and that λ is a semisimple eigenvalue of $U(t+T,t)$ iff it is a semisimple eigenvalue of $U(T,0)$ iff it is a semisimple eigenvalue of $U^*(T,0)$. Moreover, denoting by r_0 the spectral radius of all the operators $U(t+T,t)$ we have $\omega_0(U) = \frac{1}{T}\log r_0$, i.e.

$$r_0 = e^{\omega_0(U)T}.$$

Proposition 2.3. *The spectrum of $V(t)$ is independent of t, and it consists of isolated eigenvalues with modulus ≤ 1, plus 0. Moreover,*

(a) *If $\lambda \in \sigma(V(t))$ and $|\lambda| = 1$, then $\lambda = 1$, it is a simple eigenvalue, and the eigenspace consists of the constant functions. The spectral projection is M_t.*
(b) *If $\lambda \in \sigma(V(t))$ and $|\lambda| < 1$, then $|\lambda| \leq r_0$, and the generalized eigenspace consists of polynomials with degree $\leq \frac{\log|\lambda|}{\log r_0}$.*
(c) *For $|\lambda| < 1$, there exists a non-constant polynomial φ of degree 1 satisfying $V(t)\varphi = \lambda\varphi$ if and only if $\lambda \in \sigma(U(T,0))$. In this case,*

$$\varphi(x) = \langle \vec{c}, x\rangle + \frac{1}{\lambda - 1}\langle \vec{c}, g(t, t-T)\rangle,$$

where $\vec{c}$ is an eigenvector of $U^*(t, t-T)$ with eigenvalue λ.
(d) *An eigenvalue of $V(t)$ with modulus equal to r_0 is semisimple iff it is a semisimple eigenvalue of $U(T,0)$.*

Proof. By estimates (1.12), $V(t)$ maps continuously $L^2(\mathbb{R}^n, \nu_t)$ into $H^1(\mathbb{R}^n, \nu_t)$, which is compactly embedded in $L^2(\mathbb{R}^n, \nu_t)$ because ν_t is a Gaussian measure with nondegenerate covariance matrix. Therefore it is a compact operator, and its spectrum consists of 0 and of isolated nonzero eigenvalues.

From the equality

$$P_{s,t}V(t) = V(s)P_{s,t}, \quad s < t,$$

it follows that if φ is an eigenfunction of $V(t)$ with eigenvalue $\lambda \neq 0$, then $P_{s,t}\varphi$ is an eigenfunction of $V(s)$ with eigenvalue λ. It follows that the spectrum of $V(t)$ is independent of t.

Let φ be again an eigenfunction of $V(t)$ with eigenvalue $\lambda \neq 0$. Then $P_{t-nT,t}\varphi = (V(t))^n\varphi = \lambda^n\varphi$ for each $n \in \mathbb{N}$, so that, by estimate (1.12),

$$|\lambda|^n \|D^\alpha\varphi\|_{L^2(\mathbb{R}^n,\nu_t)} \leq C\mathrm{e}^{\omega|\alpha|nT}\|\varphi\|_{L^2(\mathbb{R}^n,\nu_t)}, \quad n \in \mathbb{N}, \tag{2.2}$$

for $\omega \in (\omega_0(U), 0)$ and for each multi-index α. Therefore, $|\lambda| \leq 1$ and $D^\alpha\varphi = 0$ if $|\alpha| > \log|\lambda|/\omega_0(U)T$. This proves that the eigenspace consists of polynomials with degree $\leq \log|\lambda|/\log r_0$.

To complete the proof of statement (b) we argue by recurrence. Assume that for some $r \in \mathbb{N}$ the kernel of $(\lambda I - V(t))^r$ consists of polynomials with degree $\leq \log|\lambda|/\log r_0$, and let $\varphi \in \mathrm{Ker}\,(\lambda I - V(t))^{r+1}$. Then the function $\psi := \lambda\varphi - V(t)\varphi$ is a polynomial with degree $\leq \log|\lambda|/\log r_0$, as well as $V(t)^k\psi$ for each $k \in \mathbb{N}$. Indeed, each $P_{s,t}$ maps polynomials of degree n into polynomials of degree $\leq n$, for each $n \in \mathbb{N}$. Since

$$V(t)^n\varphi = \lambda^n\varphi - \sum_{k=0}^{n-1}\lambda^{n-1-k}V(t)^k\psi, \quad n \in \mathbb{N},$$

then $D^\alpha(V(t)^n\varphi) = \lambda^n D^\alpha\varphi$, for $|\alpha| >\ \log|\lambda|/\log r_0$. Using (2.2) as before we see that φ is a polynomial with degree $\leq \log|\lambda|/\log r_0$. This proves statement (b).

Now we can prove statement (a). Estimate (2.2) shows that if $V(t)\varphi = \lambda\varphi$ and $|\lambda| = 1$, then φ is constant, and since $V(t)$ is the identity on constant functions, we have $\lambda = 1$. By statement (b), also the kernel of $(I - V(t))^2$ consists of the constant functions, so that it coincides with the kernel of $I - V(t)$, and 1 is a simple eigenvalue.

The projection M_t maps $L^2(\mathbb{R}^n, \nu_t)$ onto the kernel of $I - V(t)$. Moreover, it commutes with $V(t)$, since for each $\varphi \in L^2(\mathbb{R}^n, \nu_t)$ we have

$$V(t)M_t\varphi = M_t\varphi = \int_{\mathbb{R}^n} \varphi(x)\nu_t(\mathrm{d}x) = \int_{\mathbb{R}^n} (V(t)\varphi)(x)\nu_{t-T}(\mathrm{d}x)$$
$$= \int_{\mathbb{R}^n} (V(t)\varphi)(x)\nu_t(\mathrm{d}x) = M_tV(t)\varphi.$$

Since 1 is a simple eigenvalue, then M_t is the associated spectral projection.

Let us prove statement (c). Let $\varphi(x) = c + \langle \vec{c}, x\rangle$ with $c \in \mathbb{C}$ and $\vec{c} \in \mathbb{C}^n$. Then

$$(V(t)\varphi)(x) = c + \langle \vec{c}, U(t, t-T)x\rangle + \langle \vec{c}, g(t, t-T)\rangle.$$

Hence, $V(t)\varphi = \lambda\varphi$ iff $\lambda \in \sigma(U(t, t-T))$, $\vec{c}$ is an eigenvector of $U^*(t, t-T)$ with eigenvalue λ and $c = \langle \vec{c}, g(t, t-T)\rangle/(\lambda - 1)$.

Note that $U^*(t, t-T)$ has at least one eigenvalue λ with modulus equal to r_0. By statement (b), the corresponding generalized eigenspace of $V(t)$ consists of first order polynomials. Let $\varphi(x) = c + \langle \vec{c}, x\rangle$ be a first order polynomial in the kernel of $\lambda I - V(t)$. The equation $(\lambda I - V(t))\psi = \varphi$ may be solved only by first order polynomials. If $\psi(x) = c_1 + \langle \vec{c}_1, x\rangle$, we have $(\lambda I - V(t))\psi = \varphi$ iff

$$(\lambda - 1)c_1 + \langle \lambda\vec{c}_1, x\rangle - \langle \vec{c}_1, U(t, t-T)x + g(t, t-T)\rangle = c + \langle \vec{c}, x\rangle, \quad x \in \mathbb{R}^n,$$

that is, $(\lambda - 1)c_1 - \langle \vec{c}_1, g(t, t-T)\rangle = c$ and $\lambda\vec{c}_1 - U^*(t, t-T)\vec{c}_1 = \vec{c}$. Since $\vec{c}$ is an eigenvector of $U^*(t, t-T)$ and $c = \langle \vec{c}, g(t, t-T)\rangle/(\lambda - 1)$, we have $(\lambda I - V(t))\psi = \varphi$ iff $\vec{c}_1 \in \mathrm{Ker}\,(\lambda I - U^*(t, t-T))^2 \setminus \mathrm{Ker}\,(\lambda I - U^*(t, t-T))$, and $c_1 = (c + \langle \vec{c}_1, g(t, t-T)\rangle)/(\lambda - 1)$. Statement (d) follows. □

Statements (a) and (b) are a generalization to the periodic nonautonomous case of the results of [MPP02, Proposition 3.2] concerning the spectral properties of elliptic Ornstein-Uhlenbeck operators.

As a consequence of Proposition 2.3 we describe the asymptotic behavior of $P_{s,t}\varphi$ for each $\varphi \in L^2(\mathbb{R}^n, \nu_t)$.

Proposition 2.4. (i) *For each $\omega \in (\omega_0(U), 0)$ there exists $M = M(\omega)$ such that*

$$\|P_{s,t}(\varphi - M_t\varphi)\|_{L^2(\mathbb{R}^n, \nu_s)} \le Me^{\omega(t-s)}\|\varphi\|_{L^2(\mathbb{R}^n, \nu_t)}, \quad s < t,\ \varphi \in L^2(\mathbb{R}^n, \nu_t). \tag{2.3}$$

(ii) *For each $\omega < \omega_0(U)$ there is no M such that* (2.3) *holds.*

(iii) *Estimate* (2.3) *holds for $\omega = \omega_0(U)$ iff all the eigenvalues of $U(T, 0)$ with modulus equal to r_0 are semisimple.*

Proof. (i) Let us split $L^2(\mathbb{R}^n, \nu_t)$ as the direct sum $L^2(\mathbb{R}^n, \nu_t) = X_t \oplus X_c$, where X_t consists of the functions with zero mean value and X_c consists of the constant functions. The orthogonal projection on X_c is M_t, and by Proposition 2.3 (a) it coincides with the spectral projection associated to the eigenvalue 1 of $V(t)$. The spectral radius of the part of $V(t)$ in X_t does not exceed r_0 by Proposition 2.3 (b), but in fact it is equal to r_0, because for each $\lambda \in \sigma(U(T, 0))$ with modulus r_0, λ is also an eigenvalue of $V(t)$ by Proposition 2.3 (c).

From now on we can proceed as in the standard case of constant underlying space (e.g, [Hen81, §7.2]). For $\varphi \in X_t$ and $t - s > 2T$ set $m = [s/T] + 1$, $k = [t/T]$. Since $P_{mT,kT} = V(0)^{k-m}$, then

$$\|P_{s,t}\varphi\|_{L^2(\mathbb{R}^n,\nu_s)} = \|P_{s,mT}V(0)^{k-m}P_{kT,t}\varphi\|_{L^2(\mathbb{R}^n,\nu_s)} \leq \|V(0)^{k-m}\|_{\mathcal{L}(X_0)}\|\varphi\|_{L^2(\mathbb{R}^n,\nu_t)},$$

where $(k-m)T \geq t - s - 2T$. Since $\lim_{h\to\infty} \|V(0)^h\|_{\mathcal{L}(X_0)} = r_0 = e^{\omega_0(U)T}$, it follows that for each $\omega > \omega_0(U)$ there exists $M = M(\omega)$ such that

$$\|P_{s,t}\varphi\|_{L^2(\mathbb{R}^n,\nu_s)} \leq Me^{\omega(t-s)}\|\varphi\|_{L^2(\mathbb{R}^n,\nu_t)}, \quad s < t,$$

which is (2.3) in our case, because $M_t\varphi = 0$.

For general $\varphi \in L^2(\mathbb{R}^n, \nu_t)$, applying the above estimate to $\varphi - M_t\varphi$ gives (2.3).

(ii) By Proposition 2.3, $V(t)$ has some eigenvalue λ with modulus r_0. If φ is an eigenfunction, then it belongs to X_t so that $M_t\varphi = 0$. Moreover, for $s = t - kT$ we have $P_{s,t}\varphi = \lambda^k\varphi$ so that $\|P_{s,t}(\varphi - M_t\varphi)\|_{L^2(\mathbb{R}^n,\nu_s)} = \|P_{s,t}\varphi\|_{L^2(\mathbb{R}^n,\nu_s)} = e^{\omega_0(U)(t-s)}$, and (ii) follows.

(iii) By proposition 2.3(b)(c), the eigenvalues of $V(t)$ with modulus in $(r_0^2, 0)$ coincide with the eigenvalues of $U(t+T,t)$ with modulus in $(r_0^2, 0)$. Therefore, setting $r_1 = \max\{|\lambda| : \lambda \in \sigma(U(T,0)),\ |\lambda| < r_0\}$, $V(t)$ has no eigenvalues with modulus in $(\max\{r_1, r_0^2\}, r_0)$, while the part of the spectrum of $V(t)$ with modulus equal to r_0 consists of eigenvalues of $U(T,0)$. Let Q_t be the associated spectral projection, and let us further decompose X_t as the direct sum $Q_t(X_t) \oplus (I - Q_t)(X_t)$. Note that for $s < t$, $P_{s,t}$ maps $Q_t(X_t)$ into $Q_s(X_s)$ and $(I - Q_t)(X_t)$ into $(I - Q_s)(X_s)$. The spectral radius of $V(0)(I - Q_0 - M_0)$ does not exceed $\max\{r_1, r_0^2\}$, so that arguing as in the proof of statement (i) we obtain that for each $\omega \in (\log\max\{r_1, r_0^2\}, \omega_0)$ there is $M > 0$ such that

$$\|P_{s,t}(I - Q_0 - M_0)\|_{\mathcal{L}(L^2(\mathbb{R}^n,\nu_t),L^2(\mathbb{R}^n,\nu_s))} \leq Me^{\omega(t-s)}, \quad s < t.$$

Assume that all the eigenvalues of $U(T,0)$ with modulus r_0 are semisimple. Then by Proposition 2.3(d) they are semisimple eigenvalues of $V(0)$. Therefore there is $C > 0$ such that

$$\|V(0)^kQ_0\|_{\mathcal{L}(L^2(\mathbb{R}^n,\nu_0))} \leq Cr_0^k, \quad k \in \mathbb{N}.$$

Arguing again as in the proof of statement (i), we obtain that (2.3) holds also with $\omega = \omega_0(U)$.

If one of the eigenvalues λ of $U(T,0)$ with modulus r_0 is not semisimple, again by proposition 2.3(d) it is a non-semisimple eigenvalue of $V(t)$. Then there are nonzero functions φ_0, $\psi_0 \in X_t$ such that $(\lambda I - V(t))\varphi_0 = \psi$, $(\lambda I - V(t))\psi_0 = 0$. It follows that $V(t)^k\varphi_0 = \lambda^k\varphi_0 - k\psi_0$, for each $k \in \mathbb{N}$. Arguing as in the proof of statement (ii) we see that (2.3) cannot hold for $\varphi = \varphi_0$ and $\omega = \omega_0(U)$. □

Proposition 2.4 establishes a sort of exponential dichotomy with any exponent $\omega \in (\omega_0(U), 0)$ for $P_{s,t}$. Indeed, the projections

$$\varphi \mapsto M_t\varphi, \quad t \in \mathbb{R},$$

map each $L^2(\mathbb{R}^n, \nu_t)$ into the common one-dimensional subspace X_c of the constant functions, and satisfy

(a) $M_sP_{s,t} = P_{s,t}M_t$, for $s < t$;
(b) $P_{s,t}$: Range $M_t \mapsto$ Range M_s is invertible (in fact, it is the identity in X_c);

(c) $\|P_{s,t}(I-M_t)\|_{\mathcal{L}(L^2(\mathbb{R}^n,\nu_t),L^2(\mathbb{R}^n,\nu_s))} \le Me^{\omega(t-s)}$, $s<t$.

2.3. **Spectral gap of $G_\#$ and asymptotic behavior of $(\mathcal{P}^\#_\tau)_{\tau\ge 0}$.** Since $D(G_\#)$ is compactly embedded in $L^2_\#((0,T)\times\mathbb{R}^n,\nu)$, see [GL07], the spectrum of $G_\#$ contains eigenvalues only. This allows us to do further investigations of the spectrum of $G_\#$.

The next proposition shows that all the generalized eigenfunctions of $G_\#$ have a special structure.

Proposition 2.5. *Assume that $u\in D(G^r_\#)$ satisfies $(\lambda I - G_\#)^r u = 0$ for some $\lambda\in\mathbb{C}$ and some $r\in\mathbb{N}$. Then*

$$u(t,x)=\sum_{|\alpha|\le K} c_\alpha(t)x^\alpha,$$

where $K\le \frac{\mathrm{Re}\,\lambda}{\omega_0(U)}$ and $c_\alpha\in H^1_\#(0,T)$.

Proof. Let us start with $r=1$. Since $G_\# u=\lambda u$, we have $\mathcal{P}^\#_\tau u = \mathrm{e}^{\lambda\tau}u$ for $\tau\ge 0$. Therefore, by estimates (1.12), for any $\omega>\omega_0(U)$ there exists $C>0$, such that for any multi-index α,

$$\|\mathrm{e}^{\lambda\tau}D^\alpha_x u\|_{L^2_\#(\mathbb{R}^{1+n},\nu)} = \|D^\alpha_x\mathcal{P}^\#_\tau u\|_{L^2_\#(\mathbb{R}^{1+n},\nu)} \le C\mathrm{e}^{\omega|\alpha|\tau}\|u\|_{L^2_\#(\mathbb{R}^{1+n},\nu)},\quad \tau\ge 1.$$

Letting $\tau\to\infty$, we obtain

$$\|D^\alpha_x u\|_{L^2_\#(\mathbb{R}^{1+n},\nu)}=0$$

for Re $\lambda>\omega|\alpha|$. This implies that $u(t,\cdot)$ is a polynomial of degree less than or equal to $|\mathrm{Re}\ \lambda|/\omega$ for any $\omega\in(\omega_0(U),0)$.

Suppose now that the assertion holds for $r=1,\dots,r_0$ and assume that $u\in D(G^{r_0+1}_\#)$ satisfies $(\lambda I-G_\#)^{r_0+1}u=0$ for some $\lambda\in\mathbb{C}$. Then,

$$\mathcal{P}^\#_\tau u = \mathrm{e}^{\lambda\tau}\sum_{j=0}^{r_0}\frac{\tau^j}{j!}(\lambda-G_\#)^j u,\quad \tau\ge 1.$$

By the induction hypothesis, $(\lambda I-G_\#)^j u$ is a polynomial of degree $\le$ Re $\lambda/\omega_0(U)$, so that $D^\alpha(\lambda I-G_\#)^j u=0$ for $1\le j\le r_0$ and $|\alpha|>$ Re $\lambda/\omega_0(U)$. So, we obtain

$$D^\alpha_x\mathcal{P}^\#_\tau u = \mathrm{e}^{\lambda\tau}D^\alpha_x u,\quad \tau\ge 1,$$

and the assertion for r_0+1 follows as above. □

Proposition 2.5 implies that the eigenfunctions with eigenvalues λ such that $\mathrm{Re}\,\lambda\in(2\omega_0(U),0]$ are first or zero order polynomials with respect to x, with coefficients possibly depending on t. In the next proposition we characterize the eigenvalues that have eigenfunctions of this type.

Proposition 2.6. *Assume that $u(t,x)=c(t)+\sum_{i=1}^n c_i(t)x_i$ with c, $c_i\in H^1_\#(0,T)\setminus\{0\}$ satisfies $G_\# u=\lambda u$ for some $\lambda\in\mathbb{C}$. Then*

$$\lambda\in\left(\frac{1}{T}\log\sigma(U(T,0))+\frac{2\pi i}{T}\mathbb{Z}\right)\cup\frac{2\pi i}{T}\mathbb{Z}. \tag{2.4}$$

Conversely, for each λ satisfying (2.4) there is a function $u\neq 0$ as above such that $G_\# u=\lambda u$.

Proof. Since u satisfies $G_{\#}u = \lambda u$, we have

$$c'(t) = \lambda c(t) - \langle f(t), \vec{c}(t)\rangle, \quad t \in \mathbb{R}, \tag{2.5}$$

$$c(0) = c(T) \tag{2.6}$$

$$\vec{c}\,'(t) = (\lambda - A^*(t))\vec{c}(t), \quad t \in \mathbb{R} \tag{2.7}$$

$$\vec{c}(0) = \vec{c}(T) \tag{2.8}$$

where $\vec{c} = (c_1, \ldots, c_n)^T$. Note that every solution of (2.7) is of the form

$$\vec{c}(t) = \mathrm{e}^{\lambda t} U^*(0,t)\vec{c}_0 \text{ with } \vec{c}_0 \in \mathbb{C}^n. \tag{2.9}$$

If $\vec{c}_0 = 0$ we have $\vec{c}(t) \equiv 0$ for $t \in \mathbb{R}$, Hence, the solutions of (2.5) are given by $c(t) = \mathrm{e}^{\lambda t} c_0$ with any $c_0 \in \mathbb{C}$, and equation (2.6) can be satisfied iff $\lambda \in \frac{2\pi i}{T}\mathbb{Z}$.

If $\vec{c}_0 \neq 0$, $\vec{c}(t)$ satisfies (2.8) iff $\vec{c}_0$ is an eigenvector of $V^*(0)$ with eigenvalue $e^{-\lambda T}$, i.e. iff

$$\lambda \in -\frac{1}{T}\log\sigma(U^*(0,T)) + \frac{2\pi i}{T}\mathbb{Z} = \frac{1}{T}\log\sigma(U(T,0)) + \frac{2\pi i}{T}\mathbb{Z}. \tag{2.10}$$

Moreover, since all the solutions of (2.5) are given by

$$c(t) = \mathrm{e}^{\lambda t} c_0 - \int_0^t \mathrm{e}^{\lambda(t-s)}\langle f(s), \vec{c}(s)\rangle\, \mathrm{d}s \text{ with } c_0 \in \mathbb{C}, \tag{2.11}$$

and $\mathrm{e}^{\lambda t} \neq 1$ for $\lambda \in -\frac{1}{T}\log\sigma(U^*(0,T)) + \frac{2\pi i}{T}\mathbb{Z}$, we can find $c_0 \in \mathbb{C}$ such that the function given by (2.11) is a solution to (2.6). □

Corollary 2.7. (i) *$\sigma(G_{\#}) \cup i\mathbb{R} = \frac{2\pi i}{T}\mathbb{Z}$; for each $k \in \mathbb{Z}$ the eigenvalue $\frac{2\pi ik}{T}$ is simple and the eigenspace is spanned by $u(t,x) := e^{2\pi ikt/T}$.*

(ii) *The strips $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \in (\omega_0(U), 0)\}$ and $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda \in (a, \omega_0(U))\}$ are contained in $\rho(G_{\#})$. Here $a = \max\{2\omega_0(U), \frac{1}{T}\log|\mu| : \ \mu \in \sigma(U(T,0)), |\mu| < e^{\omega_0(U)T}\}$.*

(iii) *$\lambda \in \sigma(G_{\#})$ and $\mathrm{Re}\ \lambda = \omega_0(U)$ iff $\mu := e^{\lambda T} \in \sigma(U(T,0))$ and $|\mu| = \omega_0(U)$; for each $k \in \mathbb{Z}$ the eigenvalue $\lambda + \frac{2\pi ik}{T}$ is semisimple iff $e^{\lambda T}$ is a semisimple eigenvalue of $U(T,0)$.*

Proof. All the claims are immediate consequences of Propositions 2.5 and 2.6, except the statements about semi-simplicity.

Let $\lambda = 2\pi ik/T$, and let $\varphi \in \mathrm{Ker}\ (\lambda I - G_{\#})^2$, i.e. $(\lambda I - G_{\#})\varphi(t,x) = ce^{2\pi ikt/T}$ for some $c \in \mathbb{R}$. By proposition 2.5, $\varphi = \varphi(t)$ is independent of x, and $G_{\#}\varphi(t,x) = \varphi'(t)$, so that $\varphi(t) = e^{2\pi ikt/T}(\varphi(0) - ct)$; since φ is T-periodic then $c = 0$. Therefore, the kernel of $(\lambda I - G_{\#})^2$ is equal to the kernel of $\lambda I - G_{\#}$.

Let now λ be an eigenvalue with real part equal to $\omega_0(U)$. By Proposition 2.5, all the generalized eigenfunctions v are first order polynomials with respect to x.

So, let $v(t,x) = c_1(t) + \langle \vec{c}_1(t), x\rangle$ satisfy $(\lambda I - G_{\#})v = u$, where $u(t,x) = c_2(t) + \langle \vec{c}_2(t), x\rangle$ is an eigenfunction with eigenvalue λ. Note that $\vec{c}_2 \neq 0$. As in the proof of Proposition

2.6, we obtain

$$c_1'(t) = \lambda c_1(t) - < f(t), \vec{c}_1(t) > - c_2(t), \quad t \in \mathbb{R}, \tag{2.12}$$

$$c_1(0) = c_1(T) \tag{2.13}$$

$$\vec{c}_1{}'(t) = (\lambda - A^*(t))\vec{c}_1 - \vec{c}_2(t), \quad t \in \mathbb{R} \tag{2.14}$$

$$\vec{c}_1(0) = \vec{c}_1(T) \tag{2.15}$$

All the solutions of (2.14) are of the form

$$\vec{c}_1(t) = \mathrm{e}^{\lambda t} U^*(0,t)\vec{c}_{1,0} - \int_0^t \mathrm{e}^{\lambda(t-s)} U^*(s,t)\vec{c}_2(s)\,\mathrm{d}s$$

with some $\vec{c}_{1,0} \in \mathbb{C}^n$. Since u is an eigenfunction, the proof of Proposition 2.6 yields $\vec{c}_2(s) = \mathrm{e}^{\lambda s} U^*(0,s)\vec{c}_{2,0}$ where $\vec{c}_{2,0}$ is some eigenvector of $\mathrm{e}^{\lambda T} U^*(0,T)$ with eigenvalue 1. Hence,

$$\vec{c}_1(t) = \mathrm{e}^{\lambda t} U^*(0,t)\vec{c}_{1,0} - t\mathrm{e}^{\lambda t} U^*(0,t)\vec{c}_{2,0}, \quad t \in \mathbb{R},$$

Therefore, (2.15) is satisfied iff $\vec{c}_{1,0} = \mathrm{e}^{\lambda T} U^*(0,T)\vec{c}_{1,0} - T\vec{c}_{2,0}$, that is

$$(1 - \mathrm{e}^{\lambda T} U^*(0,T))\vec{c}_{1,0} = -T\vec{c}_{2,0}, \tag{2.16}$$

so that $\vec{c}_{1,0}$ belongs to the kernel of $(1 - \mathrm{e}^{\lambda T} U^*(0,T))^2$. If $\mathrm{e}^{\lambda T}$ is a semisimple eigenvalue of $U(T,0)$, then 1 is a semisimple eigenvalue of $\mathrm{e}^{\lambda T} U^*(0,T)$, and the only couple $(\vec{c}_{1,0}, \vec{c}_{2,0})$ that satisfies (2.16) is $(0,0)$, so that $v = u \equiv 0$. If $\mathrm{e}^{\lambda T}$ is not semisimple, there are nonzero couples $(\vec{c}_{1,0}, \vec{c}_{2,0})$ that satisfy (2.16). Using such couples, nonzero solutions $c_1(t)$, $\vec{c}_1(t)$ of (2.12), ..., (2.15) may be found, and the corresponding functions $v(t) = c_1(t) + \langle \vec{c}_1(t), x\rangle$ satisfy $(\lambda I - G_\#)^2 v = 0$, $(\lambda I - G_\#)v \neq 0$. □

Remark 2.8. The spectral projection of $G_\#$ corresponding to the eigenvalue 0 is

$$u \mapsto \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} u(t,x)\mathrm{d}\nu_t \mathrm{d}t.$$

Indeed, it maps $L^2_\#(\mathbb{R}^{1+n}, \nu)$ onto the kernel X_c of $G_\#$ and it commutes with $G_\#$. This implies that for $h \in L^2_\#(\mathbb{R}^{1+n}, \nu)$ the equation

$$G_\# u = h$$

has a solution $u \in D(G_\#)$ iff the mean value $\int_{(0,T)\times\mathbb{R}^n} h(t,x)\mathrm{d}\nu$ vanishes, and in this case the solution is unique up to constants.

Remark 2.9. In the autonomous case $A(t) \equiv A$, $f(t) \equiv 0$, $B(t) \equiv B$ we have a complete characterization of the spectrum of $G_\#$,

$$\sigma(G_\#) = \left\{ \lambda \in \mathbb{C} : \ \lambda = \frac{2k\pi i}{T} + \sum_{j=1}^r n_j \lambda_j; \ k \in \mathbb{Z}, \ \ n_j \in \mathbb{N} \cup \{0\} \right\}$$

where λ_j, $j = 1, \ldots, r$ are the eigenvalues of A.

Indeed, in this case our evolution system of measures consists of a unique measure ν independent of t, which is the invariant measure of the Ornstein-Uhlenbeck semigroup $T(t)$, and $G_\#$ may be seen as the closure of the sum of the resolvent-commuting operators

$$\begin{cases} G_1 : D(G_1) := \{u \in L^2_\#(\mathbb{R}^{1+n},\nu) :\ \exists u_t \in L^2_\#(\mathbb{R}^{1+n},\nu)\} \mapsto L^2_\#(\mathbb{R}^{1+n},\nu)\}, \\ \\ G_1 u = u_t, \end{cases}$$

$$\begin{cases} G_2 : D(G_2) := \{u \in L^2_\#(\mathbb{R}^{1+n},\nu) :\ \exists u_{x_i},\, u_{x_i x_j} \in L^2_\#(\mathbb{R}^{1+n},\nu)\} \mapsto L^2_\#(\mathbb{R}^{1+n},\nu)\}, \\ \\ (G_2 u)(t,x) = \mathcal{L}u(t,\cdot)(x), \end{cases}$$

hence its spectrum is the sum of the spectra of G_1 and of G_2. The spectrum of G_1 is easily seen to be $\frac{2\pi i}{T}\mathbb{Z}$, while the spectrum of G_2 is equal to the spectrum of the Ornstein-Uhlenbeck operator $\mathcal{L}$ in $L^2(\mathbb{R}^n,\nu)$, that was characterized in [MPP02] as the set of all the complex numbers of the type $\sum_{i=1}^r n_i\lambda_i$, where λ_i, $i = 1,\dots,r$ are the eigenvalues of A and $n_i \in \mathbb{N}\cup\{0\}$.

Proposition 2.10. *We have*

$$\mathrm{Ker}(I - \mathcal{P}_T^\#) = L^2_\#(0,T) = \mathrm{Ker}(I - \mathcal{P}_T^\#)^2,$$

so that 1 *is a semisimple isolated eigenvalue of* $\mathcal{P}_T^\#$*. The spectral projection* Π *is given by*

$$\Pi u(t,x) := M_t u(t,\cdot), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Proof. [EN00, Corollary IV.3.8] yields

$$\mathrm{Ker}(I - \mathcal{P}_T^\#) = \overline{\mathrm{Ker}(2\pi i\mathbb{Z}/T - G_\#)}^{L^2_\#((0,T)\times\mathbb{R}^n,\nu)}.$$

Since $\mathrm{Ker}(2\pi ik/T - G_\#)$ is spanned by the function $u \mapsto \mathrm{e}^{2\pi ik/T}$ for any $k \in \mathbb{Z}$ (see the proof of Proposition 2.6), the first equality follows.

Assume that $u \in \mathrm{Ker}(I - \mathcal{P}_T^\#)^2$, i.e.

$$\left((I - P_{t,t+T})u(t+T,\cdot)\right)(x) = f(t), \quad \text{a.a. } t \in \mathbb{R}$$

for some $f \in L^2_\#(0,T)$. By Proposition 2.3, $u(t)$ is independent of x for a.a. $t \in \mathbb{R}$. Therefore, $u \in L^2_\#(0,T) = \mathrm{Ker}(Id - \mathcal{P}_T)$. This means that 1 is a semisimple eigenvalue of $\mathcal{P}_T^\#$.

By Corollary 2.7 and Proposition 2.1, there are no other eigenvalues with modulus greater than $e^{\omega_0(U)T}$, so that 1 is isolated. The projection $u \mapsto \Pi u$ maps $L^2_\#((0,T)\times\mathbb{R}^n,\nu)$ onto $L^2_\#(0,T)$ and it commutes with $\mathcal{P}_T$. Since 1 is a semisimple eigenvalue, it is the spectral projection. □

Corollary 2.11. *The growth bound of* $(\mathcal{P}_\tau^\#(I-\Pi)_{\tau>0})$ *is* $\omega_0(U)$*. In other words,*

(a) *for $\omega > \omega_0(U)$ there exists $M > 0$ such that*

$$\int_0^T \int_{\mathbb{R}^n} \Big((\mathcal{P}_\tau^{\#}(u - \Pi u))(t,x) \Big)^2 \nu_t(\,\mathrm{d}x)\,\mathrm{d}t \le M \mathrm{e}^{2\omega\tau} \int_0^T \int_{\mathbb{R}^n} ((u - \Pi u)(t,x))^2 \nu_t(\mathrm{d}x)\,\mathrm{d}t, \tag{2.17}$$

$u \in L^2_{\#}(\mathbb{R}^{n+1}, \nu),\ \tau \ge 0;$

(b) *for $\omega < \omega_0(U)$ there does not exist any $M > 0$ such that* (2.17) *holds.*

Moreover, estimate (2.17) *holds for $\omega = \omega_0(U)$ iff all the eigenvalues of $U(T,0)$ with modulus equal to r_0 are semisimple.*

Proof. Since $\Pi u(t,x) = M_t u(t,\cdot)$, $t \in \mathbb{R}$, then the first assertion immediately follows from Proposition 2.4(i) and from the definition of $\mathcal{P}_\tau^{\#}$.

By Proposition 2.6, $\log\sigma(U(T,0))/T \subset \sigma_p(G_{\#})$, so that for any $\mu \in \sigma(U(T,0))$ with modulus equal to $\mathrm{e}^{\omega_0(U)T}$, there is a nonzero eigenfunction u of G with eigenvalue $\lambda = \log\mu/T$, such that $\|\mathcal{P}_\tau^{\#} u\|_{L^2_{\#}((0,T)\times\mathbb{R}^n,\nu)} = \mathrm{e}^{\omega_0(U)\tau}\|u\|_{L^2_{\#}((0,T)\times\mathbb{R}^n,\nu)}$ for each $\tau > 0$. Hence, statement (b) holds.

If all the eigenvalues of $U(T,0)$ with modulus equal to r_0 are semisimple, then estimate (2.3) holds with $\omega = \omega_0(U)$ and consequently (2.17) holds with $\omega = \omega_0(U)$. If some of such eigenvalues μ is not semisimple, the eigenvalue $\lambda = \log\mu/T$ of $G_{\#}$ is not semisimple by Corollary 2.7, and for every $v \in \mathrm{Ker}\ (\lambda I - G_{\#})^2$ such that $\lambda v - G_{\#} v = u \in \mathrm{Ker}\ G_{\#} \setminus \{0\}$ we have $\mathcal{P}_\tau^{\#} v = \mathrm{e}^{\lambda\tau} v - \tau \mathrm{e}^{\lambda\tau} u$ for each $\tau > 0$, so that for $\omega = \omega_0(U)$ there does not exist any $M > 0$ such that (2.17) holds. □

Formula (2.17) improves the convergence result of [DPL06, Prop. 6.4], obtained by different methods.

2.4. **Spectral gap of G and asymptotic behavior of $(\mathcal{P}_\tau)_{\tau\ge 0}$.** In this section the functions A, B, f are not necessarily periodic but just bounded. Although our results are not as precise as in the periodic case, still the Poincaré type inequality of the next theorem yields information on the asymptotic behavior of $(\mathcal{P}_\tau)_{\tau\ge 0}$.

We use the notation of §2.3, setting again for each $u \in L^2(\mathbb{R}^{1+n}, \nu)$

$$(\Pi u)(t,x) = M_t u(t,\cdot) = \int_{\mathbb{R}^n} u(t,x)\,\mathrm{d}\nu_t, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Π is still an orthogonal projection, that maps $L^2(\mathbb{R}^{1+n}, \nu)$ into its subspace of the functions independent of x, isomorphic to $L^2(\mathbb{R}, \mathrm{d}t)$.

Theorem 2.12. *For each $\omega \in (\omega_0(U), 0)$ let $M = M(\omega)$ be given by* (1.8)*. Set moreover $C := \sup_{t\in\mathbb{R}} \|B(t)\|$. Then for each $u \in D(G)$ we have*

$$\int_{\mathbb{R}^{1+n}} (u(t,x) - \Pi u(t))^2\,\mathrm{d}\nu \le \frac{M^2C^2}{2\omega} \int_{\mathbb{R}^{1+n}} |D_x u(t,x)|^2\,\mathrm{d}\nu. \tag{2.18}$$

A similar inequality was proved in [DPL06, Thm. 6.3] in the periodic case for functions in $D(G_{\#})$, but the proof is the same for functions in $D(G)$; one has just to replace the core used in [DPL06] by $D(G_0)$ and the integrals over $(0,T)\times\mathbb{R}^n$ by integrals over $\mathbb{R}^{1+n}$. So, we omit the proof.

Once estimate (2.18) is available, a convergence result follows in a more or less standard way.

Corollary 2.13. *Let ω, M, C be as in Theorem 2.12, and let μ_0 be the constant in* (1.4). *For each $u \in L^2(\mathbb{R}^{1+n},\nu)$ we have*

$$\|\mathcal{P}_\tau(u-\Pi u)\|_{L^2(\mathbb{R}^{1+n},\nu)} \le e^{\omega\mu_0^2\tau/M^2C^2}\|u-\Pi u\|_{L^2(\mathbb{R}^{1+n},\nu)}, \quad \tau>0. \tag{2.19}$$

Again, the proof is the same of [DPL06, Prop. 6.4], and it is omitted.

Corollary 2.13 shows that the growth bound of $\mathcal{P}_\tau(I-\Pi)$ does not exceed the number c_0 defined by

$$c_0 = \inf\left\{\frac{\omega\mu_0^2}{M(\omega)^2C^2} \,:\, \omega \in (\omega_0(U),0)\right\}. \tag{2.20}$$

But c_0 does not seem to be optimal. By estimates (1.12) the asymptotic behavior of the space derivatives of $\mathcal{P}_\tau u$ is the same of the periodic case, and this suggests that the growth bound of $\mathcal{P}_\tau(I-\Pi)$ should be equal to $\omega_0(U)$.

Now we can prove some spectral properties of G.

Proposition 2.14. *The following statements hold true.*

(i) *The spectrum of G is invariant under translations along $i\mathbb{R}$.*
(ii) *$i\mathbb{R} \subset \sigma(G)$, and $\lambda I - G$ is one to one for each $\lambda \in i\mathbb{R}$. The associated spectral projection is Π.*
(iii) $\sigma(G)\cap\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda\in(c_0,0)\}=\varnothing$.
(iv) *If the data A, B, f are T-periodic, then $\frac{1}{T}\log\sigma(U(T,0))+i\mathbb{R}\subset\sigma(G)$.*

Proof. For every $\xi\in\mathbb{R}$ let us consider the unitary operator T_ξ in $L^2(\mathbb{R}^{1+n},\nu)$ defined by $T_\xi u(t,x)=e^{it\xi}u(t,x)$. Since the spectrum of G is equal to the spectrum of $(T_\xi)^{-1}GT_\xi = G+i\xi I$, statement (i) follows.

Let us split $L^2(\mathbb{R}^{1+n},\nu)$ in the direct sum

$$L^2(\mathbb{R}^{1+n},\nu) = (I-\Pi)(L^2(\mathbb{R}^{1+n},\nu))\oplus\Pi(L^2(\mathbb{R}^{1+n},\nu)).$$

The semigroup $\mathcal{P}_\tau$ maps $(I-\Pi)(L^2(\mathbb{R}^{1+n},\nu))$ into itself (the proof is the same of the periodic case), and the growth bound of $\mathcal{P}_\tau(I-\Pi)$ is less or equal to c_0, by corollary 2.13. It follows that the spectrum of the part of G in $(I-\Pi)(L^2(\mathbb{R}^{1+n},\nu))$ is contained in the halfplane $\{\lambda\in\mathbb{C}:\mathrm{Re}\,\lambda\le c_0\}$.

The part of G in $\Pi(L^2(\mathbb{R}^{1+n},\nu))$ is just the time derivative, with domain isomorphic to $H^1(\mathbb{R},dt)$. Its spectrum is $i\mathbb{R}$, and it has no eigenvalues. Statements (ii) and (iii) follow.

In the periodic case, let $\mu\in\sigma(U(T,0))$. By Proposition 2.6, $\lambda:=\log\mu/T$ is an eigenvalue of $G_\#$. Let u be an eigenfunction. Fix a function $\theta\in C^\infty(\mathbb{R})$ such that $\theta(t)\equiv 1$ in $(-\infty,0]$, $\theta\equiv 0$ in $[T,+\infty)$, and define $\theta_k(t)=\theta(t-kT)$ for $t\ge 0$, $\theta_k(t)=\theta(-t-kT)$ for $t\le 0$.

Then the functions $u_k(t,x):=u(t,x)\theta_k(t)$ belong to $D(G)$ and satisfy $(\lambda I-G)u_k(t,x)$ $=\theta_k'(t)u(t,x)$, so that $\|(\lambda I-G)u_k\|_{L^2(\mathbb{R}^{1+n},\nu)}$ is bounded by a constant independent of k, while $\|u_k\|_{L^2(\mathbb{R}^{1+n},\nu)^2}\ge\int_{-kT}^{kT}\int_{\mathbb{R}^n}|u(t,x)|^2\,\mathrm{d}\nu = 2k\|u\|_{L^2_\#((0,T)\times\mathbb{R}^n,\nu)}$ goes to ∞ as $k\to\infty$. This shows that $\lambda I-G$ cannot have a bounded inverse, so that $\lambda\in\sigma(G)$. □

Remark 2.15. For $h \in L^2(\mathbb{R}^{1+n}, \nu)$ consider the equation

$$Gu = h.$$

It is equivalent to the system

$$\begin{cases} (i) \quad G(I-\Pi)u = (I-\Pi)h, \\ \\ (ii) \quad G\Pi u = \Pi h. \end{cases}$$

Equation (i) is uniquely solvable with respect to $(I-\Pi)u$, because 0 is in the resolvent set of the part of G in $(I-\Pi)(L^2(\mathbb{R}^{1+n}, \nu))$. Equation (ii) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi u = \Pi h,$$

and it is solvable iff Πh has a primitive ξ in $L^2(\mathbb{R}, \mathrm{d}t)$, in this case the solution is unique.

So, the range of G consists of the functions h such that Πh has a primitive ξ in $L^2(\mathbb{R}, \mathrm{d}t)$. Therefore, G is not a Fredholm operator.

Remark 2.16. Arguing as in Remark 2.9, we obtain that in the autonomous case $A(t) \equiv A$, $f(t) \equiv 0$, $B(t) \equiv B$, the spectrum of G consists of a sequence of vertical lines, and precisely

$$\sigma(G) = \left\{ \lambda \in \mathbb{C} : \text{ Re } \lambda = \sum_{j=1}^{r} n_j \text{Re } \lambda_j ;\ n_j \in \mathbb{N} \cup \{0\} \right\}$$

where λ_j, $j = 1, \dots, r$ are the eigenvalues of A. Since in this case $\omega_0(U)$ is equal to the biggest real part of the eigenvalues of A, then the spectrum of G does not contain elements with real part in $(\omega_0(U), 0)$. So, we have the same spectral gap as in the time periodic context.

In the previous section we deduced asymptotic behavior results for $\mathcal{P}^{\#}_{\tau}$ from asymptotic behavior of $P_{s,t}$. Now we reverse the procedure, deducing asymptotic behavior of $P_{s,t}$ from Corollary 2.13.

Theorem 2.17. *Let c_0 be defined by* (2.20). *For each $s < t \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^n, \nu_t)$ we have*

$$\|P_{s,t}(\varphi - M_t\varphi)\|_{L^2(\mathbb{R}^n,\nu_s)} \leq e^{c_0(t-s)} \|\varphi\|_{L^2(\mathbb{R}^n,\nu_t)}. \tag{2.21}$$

Proof. The starting point is the continuity of the function $s \mapsto \|P_{s,s+\tau}\varphi\|^2_{L^2(\mathbb{R}^n,\nu_s)}$, for each $\tau > 0$ and for each $\varphi \in C^1_b(\mathbb{R}^n)$. Once it is established, we get estimate (2.21) for $\varphi \in C^1_b(\mathbb{R}^n)$, arguing as in the case of evolution semigroups in a fixed Banach space. Since $C^1_b(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \nu_t)$, estimate (2.21) follows for each $\varphi \in L^2(\mathbb{R}^n, \nu_t)$.

Step 1: continuity of $s \mapsto \|P_{s,s+\tau}\varphi\|^2_{L^2(\mathbb{R}^n,\nu_s)}$.

Fix s, $s_0 \in \mathbb{R}$. Changing variables in an obvious way, we write

$$\|P_{s,s+\tau}\varphi\|^2_{L^2(\mathbb{R}^n,\nu_s)} - \|P_{s_0,s_0+\tau}\varphi\|^2_{L^2(\mathbb{R}^n,\nu_{s_0})} = \int_{\mathbb{R}^n} (u(s,x)^2 - u(s_0,x)^2)\mathcal{N}_{0,I}(\mathrm{d}x), \tag{2.22}$$

where

$$u(s,x) := P_{s,s+\tau}\varphi(Q(s,-\infty)^{1/2}x + g(s,-\infty)).$$

Since $\|u\|_\infty \le \|\varphi\|_\infty$, then $|u(s,x)^2 - u(s_0,x)^2| \le 2\|\varphi\|_\infty |u(s,x) - u(s_0,x)|$. We estimate $|u(s,x) - u(s_0,x)|$ changing again variables, as follows:

$$|u(s,x) - u(s_0,x)| \le$$

$$\int_{\mathbb{R}^n} \big|\varphi(Q(s+\tau,s)^{1/2}y + U(s+\tau,s)(Q(s,-\infty)^{1/2}x + g(s,-\infty)) + g(s+\tau,s))$$
$$-\varphi(Q(s_0+\tau,s_0)^{1/2}y + U(s_0+\tau,s_0)(Q(s_0,-\infty)^{1/2}x + g(s_0,-\infty)) + g(s_0+\tau,s_0))\big|$$
$$\mathcal{N}_{0,I}(\mathrm{d}y)$$

$$\le \|\,|D\varphi|\,\|_\infty \Big(\frac{2^{n/2}}{\pi^{n/2}} \|Q(s+\tau,s)^{1/2} - Q(s_0+\tau,s_0)^{1/2}\| +$$
$$\|U(s+\tau,s)Q(s,-\infty)^{1/2} - U(s_0+\tau,s_0)Q(s_0+\tau,s_0)^{1/2}\|\,|x| +$$
$$|g(s+\tau,s) - g(s_0+\tau,s_0)| \Big)$$

Using this estimate, we see that the integral in (2.22) goes to 0 as $s \to s_0$ by dominated convergence.

Step 2: conclusion.
Fix $t \in \mathbb{R}$ and $\xi \in C_c^\infty(\mathbb{R})$ such that $\xi(t) = 1$. Set

$$u(s,x) := \xi(s)\varphi(x), \quad s \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Then $u \in L^2(\mathbb{R}^{1+n}, \nu)$. We recall that

$$(\mathcal{P}_\tau(u - \Pi u))(s,x) = P_{s,s+\tau} u(s+\tau,\cdot)(x) - M_{s+\tau} u(s+\tau,\cdot) = \xi(s+\tau)(P_{s,s+\tau}\varphi(x) - M_{s+\tau}\varphi),$$

so that

$$\|\mathcal{P}_\tau(u - \Pi u)(s,\cdot)\|^2_{L^2(\mathbb{R}^n,\nu_s)} = \xi(s+\tau)^2 \Big(\int_{\mathbb{R}^n} (P_{s,s+\tau}\varphi(x))^2 \nu_s(\mathrm{d}x) - \Big(\int_{\mathbb{R}^n} \varphi(x)\nu_{s+\tau}(\mathrm{d}x) \Big)^2 \Big).$$

Therefore, for each $\tau > 0$ the function $s \mapsto \|\mathcal{P}_\tau(u - \Pi u)(s,\cdot)\|^2_{L^2(\mathbb{R}^n,\nu_s)}$ is continuous. This is true also at $\tau = 0$, since

$$\|(u - \Pi u)(s,\cdot)\|^2_{L^2(\mathbb{R}^n,\nu_s)} = \|\xi(s)(\varphi - M_s\varphi)\|^2_{L^2(\mathbb{R}^n,\nu_s)}$$
$$= |\xi(s)|^2 \Big(\int_{\mathbb{R}^n} \varphi(x)^2 \nu_s(\mathrm{d}x) - \Big(\int_{\mathbb{R}^n} \varphi(x)\nu_s(\mathrm{d}x) \Big)^2 \Big).$$

Hence, we have

$$\begin{aligned}\|P_{s,t}(I-M_t)\varphi\|^2_{L^2(\mathbb{R}^n,\nu_s)} &= \|\mathcal{P}_{t-s}(u-\Pi u)(s,\cdot)\|^2_{L^2(\mathbb{R}^n,\nu_s)}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int\limits_s^{s+\varepsilon}\|\mathcal{P}_{t-s}(u-\Pi u)(\eta,\cdot)\|^2_{L^2(\mathbb{R}^n,\nu_\eta)}\,\mathrm{d}\eta\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\|\chi_{[s,s+\varepsilon]}\mathcal{P}_{t-s}(u-\Pi u)\|^2_{L^2(\mathbb{R}^{n+1},\nu)}\\ &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\|\mathcal{P}_{t-s}(\chi_{[t,t+\varepsilon]}(u-\Pi u))\|^2_{L^2(\mathbb{R}^{n+1},\nu)}\\ &\le \mathrm{e}^{2c_0(t-s)}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\|\chi_{[t,t+\varepsilon]}(u-\Pi u)\|^2_{L^2(\mathbb{R}^{n+1},\nu)}\\ &= \mathrm{e}^{2c_0(t-s)}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\xi(\eta)^2\|(\varphi-M_\eta\varphi)\|^2_{L^2(\mathbb{R}^n,\nu_\eta)}\\ &= \mathrm{e}^{2c_0(t-s)}\|\varphi-M_t\varphi\|^2_{L^2(\mathbb{R}^n,\nu_t)}\end{aligned}$$

and (2.21) follows. □

3. Hypercontractivity

In this section the data A, B, f are bounded but not necessarily periodic.

Since $\mathcal{P}_\tau$ acts as a translation semigroup in the time variable, it cannot improve ν-summability. Thus, it seems hard to get hypercontractivity estimates for $P_{s,t}$ from properties of $\mathcal{P}_\tau$. In fact, we follow the ideas of [Gro75], adapting his procedure to the time depending case: fixed any $t\in\mathbb{R}$ and $q>1$, we look for a differentiable function $p:(-\infty,t]\mapsto[q,+\infty)$ such that $p(t)=q$ and

$$\frac{\partial}{\partial s}\|P_{s,t}\varphi\|_{L^{p(s)}(\mathbb{R}^n,\nu_s)}\ge 0,\quad s\le t$$

for all good (e.g., exponential) functions φ. If such a p exists, we get $\|P_{s,t}\varphi\|_{L^{p(s)}(\mathbb{R}^n,\nu_s)}\le \|\varphi\|_{L^q(\mathbb{R}^n,\nu_t)}$ for all exponential functions, and hence, by density, for all $\varphi\in L^q(\mathbb{R}^n,\nu_t)$.

In the time independent case, hypercontractivity of a semigroup is equivalent to the occurrence of a logarithmic Sobolev inequality for its invariant measure ([Gro75]). Since our measures ν_t are Gaussian, they satisfy logarithmic Sobolev inequalities, which are the starting point of the procedure. As in the autonomous case, what we need are log-Sobolev inequalities expressed in terms of the quadratic forms associated to the operators $L(t)$. Dealing with the nonautonomous case, an additional term appears in the quadratic form, i.e. we have

(3.1)
$$\int\limits_{\mathbb{R}^n}\varphi\,L(t)\varphi\,\nu_t(\mathrm{d}x) = -\frac{1}{2}\int\limits_{\mathbb{R}^n}|B^*(t)\nabla\varphi|^2\nu_t(\mathrm{d}x) - \frac{1}{2}\int\limits_{\mathbb{R}^n}\varphi^2\,\partial_t\rho(x,t)\,\mathrm{d}x,\quad \varphi\in H^2(\mathbb{R}^n,\nu_t),$$

as a consequence of [GL07, Lemma 2.4], and this produces an additional term in the log-Sobolev inequalities. More precisely, the following lemma holds.

Lemma 3.1. *For $p \in (1,\infty)$, $t \in \mathbb{R}$ and $\varphi \in W^{2,p}(\mathbb{R}^n,\nu_t)$, we have*

$$\begin{aligned}\int_{\mathbb{R}^n} |\varphi(x)|^p \log(|\varphi(x)|)\nu_t(\mathrm{d}x) \le \|\varphi\|^p_{L^p(\mathbb{R}^n,\nu_t)} \log(\|\varphi\|_{L^p(\mathbb{R}^n,\nu_t)}) \\ +c(p,t)\Big(\mathrm{Re}\ \langle -L(t)\varphi,\varphi_p\rangle_{L^2(\mathbb{R}^n,\nu_t)} + \frac{1}{p}\int_{\mathbb{R}^n} |\varphi(x)|^p \partial_t\rho\ \mathrm{d}x\Big).\end{aligned} \tag{3.2}$$

Here, $\varphi_p = |\varphi|^{p-2}\varphi$ and

$$c(p,t) = \frac{p}{p-1}\|Q^{1/2}(t,-\infty)B^{*-1}(t)\|^2. \tag{3.3}$$

Proof. The starting point is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} |\psi(x)|^2 \log(|\psi(x)|)\nu_t(\mathrm{d}x) \le \|Q^{1/2}(t,-\infty)\nabla\psi\|^2_{L^2(\mathbb{R}^n,\nu_t)} + \|\psi\|^2_{L^2(\mathbb{R}^n,\nu_t)} \log\|\psi\|_{L^2(\mathbb{R}^n,\nu_t)},$$

valid for any $t \in \mathbb{R}$ and $\psi \in H^1(\mathbb{R}^n,\nu_t)$, that follows from the well known logarithmic Sobolev inequality for the Gaussian measure $\mathcal{N}(0,I)$ (e.g., [Gro75, formula (1.2)]) via the standard change of variables already used in the proof of Theorem 2.17. Since $B^*(t)$ is invertible, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\psi(x)|^2 \log(|\psi(x)|)\nu_t(\mathrm{d}x) \ &\le\ \|Q^{1/2}(t,-\infty)B^{*-1}(t)\|^2 \int_{\mathbb{R}^n} |B^*(t)\nabla\psi(x)|^2\nu_t(\mathrm{d}x) \\ &\quad +\|\psi\|^2_{L^2(\mathbb{R}^n,\nu_t)} \log\|\psi\|_{L^2(\mathbb{R}^n,\nu_t)}.\end{aligned} \tag{3.4}$$

The statement will be obtained applying (3.4) to the functions $\varphi_\varepsilon := (|\varphi|^2+\varepsilon)^{\frac{p}{4}}$, and then letting $\varepsilon \to 0^+$. To this aim, we have to estimate the integrals $\int_{\mathbb{R}^n} |B^*(t)\nabla\varphi_\varepsilon|^2\nu_t(\mathrm{d}x)$. Here and in the following, we suppress the dependency of φ and φ_ε on x. An easy calculation shows that

$$\partial_j\varphi_\varepsilon = \frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\partial_j|\varphi|^2 \tag{3.5}$$

$$(B^*(t)\nabla\varphi_\varepsilon)^2 = \frac{p^2}{16}(|\varphi|^2+\varepsilon)^{\frac{p}{2}-2}\left(B^*(t)\nabla|\varphi|^2\right)^2 \tag{3.6}$$

$$\partial_{ij}\varphi_\varepsilon = \frac{p}{4}\left(\frac{p}{4}-1\right)(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\partial_i|\varphi|^2\cdot\partial_j|\varphi|^2 + \frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\partial_{ij}|\varphi|^2. \tag{3.7}$$

It follows from (3.7) and from the identity $L(t)(\varphi\overline{\varphi}) = 2\,\mathrm{Re}\overline{\varphi}\,L(t)\varphi + |B(t)^*\nabla\varphi|^2$ that

$$\begin{aligned}L(t)\varphi_\varepsilon =&\frac{p}{8}\left(\frac{p}{4}-1\right)(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\left(B^*(t)\nabla|\varphi|^2\right)^2 + \frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}L(t)|\varphi|^2 \\ =&\frac{p}{8}\left(\frac{p}{4}-1\right)(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\left(B^*(t)\nabla|\varphi|^2\right)^2 + \frac{p}{2}\mathrm{Re}\ (|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\overline{\varphi}L(t)\varphi \\ &+\frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}|B(t)^*\nabla\varphi|^2.\end{aligned}$$

Since $|\varphi|^2|B^*(t)\nabla\varphi|^2 \geq \frac{1}{4}(B^*(t)\nabla|\varphi|^2)^2$, we obtain

$$\begin{aligned}L(t)\varphi_\varepsilon =&\frac{p}{8}\left(\frac{p}{4}-1\right)(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\left(B^*(t)\nabla|\varphi|^2\right)^2+\frac{p}{2}\mathrm{Re}\,(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\overline{\varphi}L(t)\varphi\\&+\frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}|\varphi|^2|B(t)^*\nabla\varphi|^2+\frac{p}{4}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\varepsilon|B(t)^*\nabla\varphi|^2\\ \geq&\frac{p^2-2p}{32}(|\varphi|^2+\varepsilon)^{\frac{p}{4}-2}\left(B^*(t)\nabla|\varphi|^2\right)^2+\frac{p}{2}\mathrm{Re}\,(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\overline{\varphi}L(t)\varphi.\end{aligned}$$

Finally, (3.6) yields

$$L(t)\varphi_\varepsilon \geq \frac{p-2}{2p}\varphi_\varepsilon^{-1}\left(B^*(t)\nabla\varphi_\varepsilon\right)^2+\frac{p}{2}\,\mathrm{Re}\,(|\varphi|^2+\varepsilon)^{\frac{p}{4}-1}\overline{\varphi}L(t)\varphi.$$

Applying the identity (3.1) to φ_ε we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|B^*\nabla\varphi_\varepsilon|^2\nu_t(\mathrm{d}x) \leq& \int_{\mathbb{R}^n}\varphi_\varepsilon^2\,\partial_t\rho(x,t)\,\mathrm{d}x-\frac{p-2}{p}\int_{\mathbb{R}^n}(B^*(t)\nabla\varphi_\varepsilon)^2\nu_t(\mathrm{d}x)\\&-p\,\mathrm{Re}\int_{\mathbb{R}^n}(|\varphi|^2+\varepsilon)^{\frac{p}{2}-1}\overline{\varphi}L(t)\varphi\,\nu_t(\mathrm{d}x).\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^n}|B^*(t)\nabla\varphi_\varepsilon|^2\nu_t(\mathrm{d}x)\\&\leq -\frac{p^2}{2(p-1)}\left(\mathrm{Re}\int_{\mathbb{R}^n}(\varphi^2+\varepsilon)^{\frac{p}{2}-1}\overline{\varphi}\,L(t)\varphi\,\nu_t(\mathrm{d}x)-\frac{1}{p}\int_{\mathbb{R}^n}\varphi_\varepsilon^2\,\partial_t\rho(x,t)\,\mathrm{d}x\right).\end{aligned}$$

Replacing this estimate in (3.4) and letting ε tend to 0, the lemma follows. □

Next, we prove a variant of [Gro75, Lemma 1.1]. Again, we have to deal with an additional term.

Lemma 3.2. *Let $t\in\mathbb{R}$, $a\in(0,+\infty]$ and $I=(t-a,t]$. Assume that $p\in C^1(I)$ with $p(s)>1$ for $s\in I$, $u(\cdot,x)\in C^1(I)$ for all $x\in\mathbb{R}^n$ and $u(s,\cdot)\not\equiv 0$ for $s\in I$. Moreover, assume that there are $C,k>0$ such that*

$$\max\{|u(s,x)|,|\partial_s u(s,x)|\}\leq C|x|^k,\quad s\in I,\quad x\in\mathbb{R}.$$

Then the function $\alpha: I\to\mathbb{R}$ defined by $\alpha(s)=\|u(s,\cdot)\|_{L^{p(s)}(\mathbb{R}^n,\nu_s)}$ is differentiable in I and

$$\begin{aligned}\alpha'(s)=&\alpha(s)^{1-p(s)}\left\{\mathrm{Re}\,\langle\partial_s u(s,\cdot),u_{p(s)}(s,\cdot)\rangle_{L^2(\mathbb{R}^n,\nu_s)}+\frac{1}{p(s)}\int_{\mathbb{R}^n}|u(s,x)|^{p(s)}\partial_s\rho\,\mathrm{d}x\right.\\&\left.+\frac{p'(s)}{p(s)}\left(\int_{\mathbb{R}^n}|u(s,x)|^{p(s)}\log(|u(s,x)|)\nu_s(\mathrm{d}x)-\alpha(s)^{p(s)}\log(\alpha(s))\right)\right\}.\end{aligned}$$

Proof. We calculate

$$\frac{\partial}{\partial s}\left(|u(s,x)|^{p(s)}\rho(s,x)\right)$$

$$= \left(p'(s)\log(|u(s,x)|)|u(s,x)|^{p(s)} + p(s)\frac{\partial}{\partial s}u(s,x)|u(s,x)|^{p(s)-2}u(s,x)\right)\rho(s,x)$$

$$+|u(s,x)|^{p(s)}\frac{\partial}{\partial s}\rho(s,x), \quad s\in I.$$

By assumption, there exists $h \in L^1(\mathbb{R}^n)$ such that

$$\max\left\{|u(s,x)|^{p(s)}\rho(s,x), \frac{\partial}{\partial s}\left(|u(s,x)|^{p(s)}\rho(s,x)\right)\right\} \le h(x), \quad s\in I,\ x\in\mathbb{R}^n.$$

Hence, the assertion follows from Lebesgue's dominated convergence theorem and the chain rule. □

Now we are able to prove the hypercontractivity of $(P_{s,t})_{s\le t}$.

Theorem 3.3. *Let $q\in(1,\infty)$, $t\in\mathbb{R}$ and let $p(s,t)$ be the solution of*

$$p'(s) = -\frac{p(s)}{c(p,s)},\ s\le t; \quad p(t) = q.$$

Then for $s<t$, $P_{s,t}$ maps $L^q(\mathbb{R}^n,\nu_t)$ into $L^{p(s,t)}(\mathbb{R}^n,\nu_s)$ and

$$\|P_{s,t}\varphi\|_{L^{p(s,t)}(\mathbb{R}^n,\nu_s)} \le \|\varphi\|_{L^q(\mathbb{R}^n,\nu_t)}, \quad \varphi\in L^q(\mathbb{R}^n,\nu_t).$$

Proof. Fix $t\in\mathbb{R}$ and let $\varphi\in span\ \{e^{i\langle k,x\rangle} : k\in\mathbb{R}^n\}$. Set $p(s)=p(s,t)$, $u(s,\cdot)=P_{s,t}\varphi$ and $\alpha(s)=\|P_{s,t}\varphi\|_{L^{p(s)}(\mathbb{R}^n,\nu_s)}$. Since

$$P_{s,t}\varphi_k(x) = e^{i\langle g(t,s)+U(t,s)x,k\rangle - \frac{1}{2}\langle Q(t,s)k,k\rangle},$$

for $\varphi_k(x)=e^{i\langle k,x\rangle}$, then the functions α and p satisfy the assumptions of Lemma 3.2. Using Lemma 3.2, we get

$$\alpha'(s) = \alpha(s)^{1-p(s)}\Bigg\{\mathrm{Re}\ \langle -L(s)u(s,\cdot), u_{p(s)}(s,\cdot)\rangle_{L^2(\mathbb{R}^n,\nu_s)} + \frac{1}{p(s)}\int_{\mathbb{R}^n}|u(s,x)|^{p(s)}\partial_s\rho(s,x)\,dx$$

$$+\frac{p'(s)}{p(s)}\left(\int_{\mathbb{R}^n}|u(s,\cdot)|^{p(s)}\log(|u(s,\cdot)|)\nu_t(dx) - \|u(s,\cdot)\|^{p(s)}_{L^{p(s)}(\mathbb{R}^n,\nu_s)}\log(\|u(s,\cdot)\|_{L^{p(s)}(\mathbb{R}^n,\nu_s)})\right)\Bigg\}.$$

The choice $p'(s) = -\frac{p(s)}{c(p,s)}$ and inequality (3.2) thus yield $\frac{d\alpha(s)}{ds}\ge 0$, which implies

$$\|P_{s,t}\varphi\|_{L^{p(s,t)}(\mathbb{R}^n,\nu_s)} = \alpha(s) \le \alpha(t) = \|\varphi\|_{L^q(\mathbb{R}^n,\nu_t)}, \quad s\le t.$$

Since $span\ \{e^{i\langle k,x\rangle} : k\in\mathbb{R}^n\}$ is dense in $L^q(\mathbb{R}^n,\nu_t)$, the proof is complete. □

Remark 3.4. The solution $p(s,t)$ of

$$p'(s) = -\frac{p(s)}{c(p,s)},\ s\le t; \quad p(t) = q$$

is given by

$$p(s,t) = 1 + (q-1)\exp\left(\int_s^t \|Q^{\frac{1}{2}}(r,-\infty)B^{*-1}(r)\|^{-2}\,\mathrm{d}r\right), \quad s < t.$$

Since $\|Q^{1/2}(r,-\infty)B^{*-1}(r)\|^2 \le \int_{-\infty}^r \|B^*(\sigma)U^*(\sigma,r)B^{*-1}(r)\|^2 d\sigma$, then for each $\omega \in (\omega_0, 0)$ we have

$$\|Q^{\frac{1}{2}}(r,-\infty)B^{*-1}(r)\|^2 \le \frac{C^2(M(\omega))^2}{2\mu_0^2|\omega|}$$

with $C = \sup_{t\in\mathbb{R}} \|B(t)\|$. Hence,

$$p(s,t) \ge 1 + (q-1)e^{2c_0(s-t)}, \quad s \le t,$$

where c_0 is the constant defined in (2.20).

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