Elliptic operators with unbounded drift coefficients and Neumann boundary condition

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Abstract

We study the realization A_N of the operator $\mathcal{A} = \frac{1}{2}\Delta - \langle DU, D\cdot\rangle$ in $L^2(\Omega,\mu)$ with Neumann boundary condition, where Ω is a possibly unbounded convex open set in $\mathbb{R}^N$, U is a convex unbounded function, $DU(x)$ is the element with minimal norm in the subdifferential of U at x, and $\mu(dx) = c\exp(-2U(x))dx$ is a probability measure, infinitesimally invariant for $\mathcal{A}$. We show that A_N is a dissipative self-adjoint operator in $L^2(\Omega,\mu)$. Log-Sobolev and Poincaré inequalities allow then to study smoothing properties and asymptotic behavior of the semigroup generated by A_N.

1 Introduction

Linear elliptic operators with regular and bounded coefficients in $\mathbb{R}^n$ is an old and well studied subject. In particular, unique solvability and estimates for the solutions to equations of the type

$$\lambda u - \mathcal{A}u = f \tag{1.1}$$

in suitable Banach spaces, such as L^p spaces, Sobolev spaces, Hölder spaces etc. has been the object of deep investigation and nowadays a satisfactory theory is available. See e.g. the classical book [10].

If the coefficients of the elliptic operator $\mathcal{A}$ are regular but unbounded one can prove in general existence of a solution for large λ but not uniqueness. For instance, for each $\lambda > 0$ the one dimensional equation

$$\lambda\varphi - \frac{1}{2}\,\varphi_{xx} - x^3\varphi_x = 0,$$

has a bounded non zero classical solution. See [9].

In the last few years an increasing interest has been devoted to elliptic operators with unbounded coefficients in the whole $\mathbb{R}^N$, or in an unbounded open subset Ω of $\mathbb{R}^n$. In the case of regular coefficients we refer to [4, 9, 11, 12, 14].

When the coefficients are not continuous one cannot expect in general existence of any solution in spaces of continuous functions; also the choice of L^p spaces with respect to the

Lebesgue measure is not appropriate and leads to several difficulties. See the book [7], where (1.1) is considered under minimal regularity conditions on the coefficients of $\mathcal{A}$. A more natural choice is the space $L^p(\mathbb{R}^N, \mu)$ where μ is a probability measure such that

$$\int_\Omega \mathcal{A}u(x)\mu(dx) = 0, \quad u \in C_0^\infty(\mathbb{R}^N).$$

Several papers have been devoted to existence and uniqueness of invariant measures μ associated to elliptic operators, and to the properties of the realizations of such operators in $L^p(\mathbb{R}^N, \mu)$, $1 \le p \le \infty$, especially for $p = 2$. See e.g. [7, 5, 6, 16, 17].

In this paper we shall consider elliptic operators of the form

$$\mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle \tag{1.2}$$

This class of operators enjoys nice functional properties. As first realized by Kolmogorov [8], if U is a C^1 function and $\exp(-2U)$ is in $L^1(\mathbb{R}^N)$, the probability measure

$$\nu(dx) = \left(\int_{\mathbb{R}^N} e^{-2U(x)}dx\right)^{-1} e^{-2U(x)}dx \tag{1.3}$$

is an infinitesimally invariant measure for $\mathcal{A}$, and $\mathcal{A}$ is symmetric in $L^2(\mathbb{R}^N, \nu)$ in the sense that

$$\int_{\mathbb{R}^N} \mathcal{A}u(x)\; v(x)\nu(dx) = \int_{\mathbb{R}^N} \mathcal{A}v(x)\; u(x)\nu(dx), \quad u, v \in C_0^\infty(\mathbb{R}^N).$$

The condition that the drift $\langle F, Du\rangle$ has $F = -DU$ looks rather restrictive. However, it is well known (and easy to see) that if an operator $\mathcal{A}$ of the type $\mathcal{A}u = \frac{1}{2}\Delta u + \langle F, Du\rangle$ with regular F admits an infinitesimally invariant probability measure $\nu = \rho(x)dx$, then $\mathcal{A}$ is symmetric in $L^2(R^N, \nu)$ if and only if $2F = D\log\rho$. In this case the corresponding diffusion process, described by the differential stochastic equation

$$dX = F(X)dt + dW(t),$$

is said to be *reversible*.

We shall assume that U is a real valued convex function such that

$$\lim_{|x|\to+\infty} U(x) = +\infty. \tag{1.4}$$

No other growth assumption will be made. If U is differentiable at x, $DU(x)$ is the gradient of U at x; if U is not differentiable at x, $DU(x)$ is meant as the element with minimal norm in the subdifferential of U at x. Since U is real valued and convex, it is continuous and $DU(x)$ is well defined for each $x \in \mathbb{R}^N$, but the function $x \mapsto DU(x)$ may be discontinuous.

After the study of the realization of $\mathcal{A}$ in $L^2(R^N, \nu)$, we shall consider an open convex set $\Omega \subset \mathbb{R}^N$ with C^2 boundary, and we shall study the realization A_N of $\mathcal{A}$ in $L^2(\Omega, \mu)$ with Neumann boundary condition, where

$$\mu(dx) = \left(\int_\Omega e^{-2U(x)}dx\right)^{-1} e^{-2U(x)}dx. \tag{1.5}$$

It is easy to see that if $u \in C^2(\overline{\Omega})$ has compact support and null normal derivative at the boundary, then

$$\int_\Omega \mathcal{A}u(x)\mu(dx) = 0.$$

Therefore, the measure μ is infinitesimally invariant for $\mathcal{A}$. Note that this is not true if u satisfies the Dirichlet boundary condition.

The main result of this paper is that $A_N : D(A_N) = \{u \in H^2(\Omega,\mu) : \partial u/\partial n = 0,\ \mathcal{A}u \in L^2(\Omega,\mu)\}$ is a self-adjoint dissipative operator.

We use a penalization method, introducing the family of operators in $\mathbb{R}^N$

$$\mathcal{A}_\varepsilon u(x) = \frac{1}{2}\Delta u(x) - \langle DU_\varepsilon(x), Du(x)\rangle, \quad x \in \mathbb{R}^N, \tag{1.6}$$

where

$$U_\varepsilon(x) = U(x) + \frac{1}{2\varepsilon}(\mathrm{dist}(x,\Omega))^2. \tag{1.7}$$

Operators of this type in the whole $\mathbb{R}^N$ have been already considered, for instance in [5], under further assumptions on U. Here we prove that the realizations of the operators $\mathcal{A}_\varepsilon$ in $L^2(\mathbb{R}^N,\nu_\varepsilon)$, with domain $D_\varepsilon = \{u \in H^2(\mathbb{R}^N,\nu_\varepsilon) : \langle DU_\varepsilon, Du\rangle \in L^2(\mathbb{R}^N,\nu_\varepsilon)\}$ and

$$\nu_\varepsilon(dx) = \left(\int_{\mathbb{R}^N} e^{-2U_\varepsilon(x)}dx\right)^{-1} e^{-2U_\varepsilon(x)}dx$$

are self-adjoint and dissipative. Therefore the equation

$$\lambda u - \mathcal{A}_\varepsilon u = \widetilde{f},$$

where $\widetilde{f}$ is the null extension of f to the whole $\mathbb{R}^N$, has a unique solution $u_\varepsilon \in D_\varepsilon$. It is possible to see that $\|u_\varepsilon\|_{H^2(\mathbb{R}^N,\nu_\varepsilon)}$ is bounded by a constant independent of ε. It follows that the restrictions $u_{\varepsilon|\Omega}$ are bounded in $H^2(\Omega,\mu)$ by a constant independent of ε. Therefore a sequence $u_{\varepsilon_k|\Omega}$ converges to a limiting function $u \in H^2(\Omega,\mu)$ which turns out to be a solution to

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega, \\[2mm] \dfrac{\partial u}{\partial n} = 0 & \text{at } \partial\Omega. \end{cases}$$

The solution is unique thanks to an *a priori* estimate, which comes from the integration by parts formula

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle\mu(dx) + \frac{1}{2}\int_{\partial\Omega}\frac{\partial u}{\partial n}(x)\psi(x)\mu(dx). \tag{1.8}$$

Indeed, taking $\psi = u$, where u is any solution, we get immediately

$$\|u\|_{L^2(\Omega,\mu)} \le \frac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \tag{1.9}$$

so that the solution is unique.

Formula (1.8) is crucial in our analysis. It implies also that A_N is symmetric, and, through its consequence (1.9), that A_N is dissipative. Since the resolvent set of A_N is not empty, it follows that A_N is self-adjoint.

The general theory of operators in Hilbert spaces yields that A_N is the infinitesimal generator of an analytic contraction semigroup $T(t)$, which can be naturally extended to an analytic contraction semigroup in $L^p(\Omega,\mu)$ for each $p \in (1,\infty)$.

Under the further assumption that $U - \omega|x|^2/2$ is convex for some $\omega > 0$ we show that μ satisfies Poincaré and log-Sobolev inequalities, that is

$$\int_\Omega |u(x) - \overline{u}|^2\nu(dx) \le \frac{1}{2\omega}\int_\Omega |Du(x)|^2\nu(dx), \quad u \in H^1(\Omega,\mu), \tag{1.10}$$

and

$$\int_\Omega u^2(x)\log(u^2(x))\mu(dx) \le \frac{1}{\omega}\int_\Omega |Du(x)|^2\mu(dx) + \overline{u^2}\log(\overline{u^2}), \quad u \in H^1(\Omega,\mu). \tag{1.11}$$

Here $\overline{u} = \int_\Omega u(x)\mu(dx)$ is the mean value of u, and we adopt the convention $0\log 0 = 0$.

As well known, (1.11) implies that $T(t)$ is hypercontractive, with

$$\|T(t)\varphi\|_{L^{q(t)}(\Omega,\mu)} \leq \|\varphi\|_{L^p(\Omega,\mu)}, \quad t>0,\ p\geq 2,\ \varphi\in L^p(\Omega,\mu),$$

and $q(t) = 1 + (p-1)e^{2\omega t}$.

A more general class of operators, namely $\mathcal{B}u = \mathcal{A}u + \langle G, Du\rangle$ has been recently studied in [16], where suitable assumptions are made in order that the realization B_p of $\mathcal{B}$ in $L^p(\mathbb{R}^N,\nu)$, $1<p<\infty$, generates an analytic semigroup in $L^p(\mathbb{R}^N,\nu)$. In the case $G=0$, i.e. $\mathcal{B}=\mathcal{A}$, the assumptions of [16] are not comparable to ours; however they lead to the characterization of the domain of B_p as the space $W^{2,p}(\mathbb{R}^N,\nu)$, for $1<p<\infty$. Their method seems to be hardly extendable to the case of an unbounded Ω with Neumann boundary condition, and to the case of discontinuous coefficients. In fact, to our knowledge there are no papers about elliptic operators with unbounded coefficients in an unbounded domain and Neumann boundary condition. The paper [2] deals with elliptic operators with possibly unbounded discontinuous coefficients in an unbounded domain, but it is focused on existence and properties of infinitesimally invariant measures, the underlying boundary condition is the Dirichlet condition, and no effort is made towards the characterization of the domain of their realizations.

2 The realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N,\nu)$

In this section we describe the main properties of the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N,\nu)$. We give alternative proofs of the results of [5] that we need here, avoiding some unnecessary assumptions made in [5].

Let $U:\mathbb{R}^N\mapsto\mathbb{R}$ be a convex function, satisfying (1.4). Then U is continuous, and there are $a\in\mathbb{R}$, $b>0$ such that $U(x)\geq a+b|x|$, for each $x\in\mathbb{R}^N$. It follows that the probability measure (1.3) is well defined, and that $\int_{\mathbb{R}^N}|x|^k e^{-2U(x)}dx<\infty$ for each $k>0$.

For each $x\in\mathbb{R}^N$, the subdifferential $\partial U(x)$ of U at x is the set $\{y\in\mathbb{R}^N:\ U(\xi)\geq U(x)+\langle y,\xi-x\rangle,\ \forall\,\xi\in\mathbb{R}^N\}$. Since U is convex, $\partial U(x)$ has a unique element with minimal norm, that we denote by $DU(x)$. Of course if U is differentiable at x, $DU(x)$ is the usual gradient.

Since DU is not continuous in general, and we made no growth assumptions, handling the operator $\mathcal{A}$ is rather delicate. To overcome these difficulties we introduce the *Moreau-Yosida approximations* of U,

$$U_\alpha(x) = \inf\left\{U(y)+\frac{1}{2\alpha}|x-y|^2:\ y\in\mathbb{R}^N\right\},\ \ x\in\mathbb{R}^N,\ \alpha>0,$$

which will be the main technical tool of this section. The functions U_α are convex, differentiable, and for each $x\in\mathbb{R}^N$ we have (see e.g. [3, Pr. 2.6, Pr. 2.11][(1)])

$$U_\alpha(x)\leq U(x), |DU_\alpha(x)|\leq|DU(x)|,\ \lim_{\alpha\to 0}U_\alpha(x)=U(x),\ \lim_{\alpha\to 0}DU_\alpha(x)=DU(x).$$

Moreover DU_α is Lipschitz continuous for each α, with Lipschitz constant $1/\alpha$. This is of much help, because elliptic operators with Lipschitz continuous (although unbounded) coefficients have nice properties, and they are well studied. See e.g. [12, 13, 15].

The space $H^1(\mathbb{R}^N,\nu)$ is naturally defined as the set of all $u\in H^1_{loc}(\mathbb{R}^N)$ such that u, $D_iu\in L^2(\mathbb{R}^N,\nu)$, $i=1,\ldots,N$. It is a Hilbert space with the standard scalar product

$$\langle u,v\rangle = \left(\int_{\mathbb{R}^N}e^{-2U(x)}dx\right)^{-1}\int_{\mathbb{R}^N}(uv+\sum_{i=1}^n D_iuD_iv)e^{-2U(x)}dx.$$

[1]To be precise, what is called "Yosida approximations" in [3] are the functions $DU_\alpha = (I-(I+\alpha\partial U)^{-1})/\alpha$, which approximate DU.

Similarly, the space $H^2(\mathbb{R}^N,\nu)$ is defined as the set of all $u\in H^2_{loc}(\mathbb{R}^N)$ such that u, D_iu, $D_{ij}u\in L^2(\mathbb{R}^N,\nu)$, $i,j=1,\dots,N$. It is a Hilbert space with the scalar product

$$\langle u,v\rangle=\left(\int_{\mathbb{R}^N}e^{-2U(x)}dx\right)^{-1}\int_{\mathbb{R}^N}(uv+\sum_{i=1}^N D_iuD_iv+\sum_{i,j=1}^N D_{ij}uD_{ij}v)e^{-2U(x)}dx.$$

We shall use also the spaces $C^k_b(\mathbb{R}^N)$ ($k\in\mathbb{N}\cup\{\infty\}$), consisting of all C^k functions $u:\mathbb{R}^N\mapsto\mathbb{R}$ with bounded derivatives up to the order k, their subspace $C^\infty_0(\mathbb{R}^N)$ consisting of all smooth compactly supported functions, and the spaces $C^{k+\theta}_b(\mathbb{R}^N)$ ($k\in\mathbb{N}\cup\{0\}$, $\theta\in(0,1)$) consisting of functions in $C^k_b(\mathbb{R}^N)$ with uniformly θ-Hölder continuous k-th order derivatives.

Lemma 2.1 *$C^\infty_0(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N,\nu)$, in $H^1(\mathbb{R}^N,\nu)$ and in $H^2(\mathbb{R}^N,\nu)$.*

Proof — It is well known that every function $u\in X$ with compact support (where $X=L^2(\mathbb{R}^N)$, $X=H^1(\mathbb{R}^N)$, or $X=H^2(\mathbb{R}^N)$) may be approximated in X by a sequence of C^∞_0 functions obtained by convolution with smooth mollifiers. Since u has compact support, such a sequence approximates u also in $L^2(\mathbb{R}^N,\nu)$, in $H^1(\mathbb{R}^N,\nu)$, or in $H^2(\mathbb{R}^N,\nu)$, respectively.

Therefore it is sufficient to show that every $u\in L^2(\mathbb{R}^N,\nu)$ (respectively, $u\in H^1(\mathbb{R}^N,\nu)$, $u\in H^2(\mathbb{R}^N,\nu)$) may be approximated by a sequence of L^2 (respectively, H^1, H^2) functions with compact support. Let $\theta:\mathbb{R}^N\mapsto\mathbb{R}$ be a smooth function such that $0\le\theta(x)\le 1$ for each x, $\theta\equiv 1$ in $B(0,1)$, $\theta\equiv 0$ outside $B(0,2)$, and set $u_n(x)=u(x)\theta(x/n)$. Then $u_n\to u$ in $L^2(\mathbb{R}^N,\nu)$. Indeed,

$$\int_{\mathbb{R}^N}|u_n-u|^2\nu(dx)\le\int_{|x|\ge n}|u|^2\nu(dx)$$

which goes to 0 as $n\to\infty$. If $u\in H^1(\mathbb{R}^N,\nu)$, then

$$D_iu_n=D_iu(x)\theta(x/n)+u(x)D_i\theta(x/n)/n,$$

where $D_iu\theta(\cdot/n)$ goes to D_iu and $uD_i\theta(\cdot/n)/n$ goes to 0 as $n\to\infty$. If $u\in H^2(\mathbb{R}^N,\nu)$, then

$$D_{ij}u_n(x)=$$

$$=D_{ij}u(x)\theta(x/n)+D_iu(x)D_j\theta(x/n)/n+D_ju(x)D_i\theta(x/n)/n+u(x)D_{ij}\theta(x/n)/n^2,$$

and $D_{ij}u\theta(\cdot/n)$ goes to $D_{ij}u$ while the sequences $uD_i\theta(\cdot/n)/n$, $D_iuD_j\theta(\cdot/n)/n$, D_ju $D_i\theta(\cdot/n)/n$, $uD_{ij}\theta(\cdot/n)/n^2$ go to zero in $L^2(\mathbb{R}^N,\nu)$ as $n\to\infty$. The statement follows. □

Let us define now the realization A of $\mathcal{A}$ in $L^2(\mathbb{R}^N,\nu)$ by

$$\left\{\begin{array}{lcl} D(A) & = & \{u\in H^2(\mathbb{R}^N,\nu):\ \mathcal{A}u\in L^2(\mathbb{R}^N,\nu)\}\\ & = & \{u\in H^2(\mathbb{R}^N,\nu):\ \langle DU,Du\rangle\in L^2(\mathbb{R}^N,\nu)\},\\ (Au)(x)=\mathcal{A}u(x), & & x\in\mathbb{R}^N.\end{array}\right.\tag{2.1}$$

We shall show that A is a self-adjoint dissipative operator. A first important step is next lemma, which yields that A is symmetric.

Lemma 2.2 *For each $u\in D(A)$, $\psi\in H^1(\mathbb{R}^N,\nu)$ we have*

$$\int_{\mathbb{R}^N}(\mathcal{A}u)(x)\psi(x)\nu(dx)=-\frac{1}{2}\int_{\mathbb{R}^N}\langle Du(x),D\psi(x)\rangle\nu(dx).\tag{2.2}$$

Proof — It is sufficient to prove that (2.2) holds for each $\psi \in C_0^\infty(\mathbb{R}^N)$. Indeed, in this case, for $\psi \in H^1(\mathbb{R}^N, \nu)$, (2.2) is obtained approximating ψ by a sequence of C_0^∞ functions.

First we assume that U is continuously differentiable. If $\psi \in C_0^\infty(\mathbb{R}^N)$, then the function $\psi \exp(-2U)$ is continuously differentiable and it has compact support. Integrating by parts $(\Delta u)(x)\psi(x)\exp(-2U(x))$ we get

$$\frac{1}{2}\int_{\mathbb{R}^N}(\Delta u)(x)\psi(x)e^{-2U(x)}dx = -\frac{1}{2}\int_{\mathbb{R}^N}\langle Du(x), D(\psi(x)e^{-2U(x)})\rangle dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^N}\langle Du(x), D\psi(x)\rangle e^{-2U(x)}dx + \frac{1}{2}\int_{\mathbb{R}^N}\langle Du(x), 2DU(x)\rangle\psi(x)e^{-2U(x)}dx$$

so that (2.2) holds.

Let now U be merely continuous. Let U_α be the Yosida approximations defined above, and define accordingly the differential operator $\mathcal{A}_\alpha$, the measure $\nu_\alpha(dx)$, and the operator $A_\alpha : D(A_\alpha) \mapsto L^2(\mathbb{R}^N, \nu_\alpha)$. For each $\psi \in C_0^\infty(\mathbb{R}^N)$ and for each $\alpha > 0$ we have

$$\int_{\mathbb{R}^N}(\mathcal{A}_\alpha u)(x)\psi(x)\exp(-2U_\alpha(x))dx = -\frac{1}{2}\int_{\mathbb{R}^N}\langle Du(x), D\psi(x)\rangle\exp(-2U_\alpha(x))dx,$$

where $\exp(-2U_\alpha)$ goes to $\exp(-2U)$ uniformly in supp ψ, and since DU is locally bounded, $\mathcal{A}_\alpha u \exp(-2U_\alpha)$ goes to $\mathcal{A}u\exp(-2U)$ in $L^2(\text{supp } \psi)$. Letting $\alpha \to 0$ we obtain (2.2). $\square$

Taking $\psi = u$ in (2.2) shows that A is symmetric.

In the next proposition we give good *a priori* estimates for regular functions, when U has Lipschitz continuous derivatives. This is enough for our aims because we will apply it to the operators $\mathcal{A}_\alpha$.

Proposition 2.3 *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ satisfy (1.4) and have Lipschitz continuous derivatives. For every $u \in C_b^2(\mathbb{R}^N)$ we have*

$$\|\,|Du|\,\|^2_{L^2(\mathbb{R}^N,\nu)} \le 2\|Au\|_{L^2(\mathbb{R}^N,\nu)}\|u\|_{L^2(\mathbb{R}^N,\nu)}, \tag{2.3}$$

and

$$\|\,|D^2u|\,\|_{L^2(\mathbb{R}^N,\nu)} \le 4\|Au\|_{L^2(\mathbb{R}^N,\nu)}. \tag{2.4}$$

Proof — Since DU is Lipschitz continuous, it has at most linear growth as $|x| \to \infty$, so that $|DU| \in L^2(\mathbb{R}^N, \nu)$. It follows that $C_b^2(\mathbb{R}^N)$ is contained in $D(A)$. Then estimate (2.3) follows immediately from (2.2) taking $\psi = u$. To prove (2.4) we first consider a function $u \in C_0^\infty(\mathbb{R}^N)$. In this case the functions $f = \mathcal{A}u$ and D_hU, $h = 1, \dots, N$, are Lipschitz continuous; hence they belong to $H^1_{loc}(\mathbb{R}^N)$ and we have

$$\mathcal{A}D_hu - \sum_{k=1}^N D_{hk}U\, D_ku = D_hf,$$

where the equality is meant in $L^2_{loc}(\mathbb{R}^N)$. Multiplying by D_hu and summing up we get

$$\sum_{h=1}^N \mathcal{A}D_huD_hu \ge \sum_{h=1}^N D_hfD_hu \text{ a.e.},$$

because $D^2U \ge 0$ almost everywhere. Since $D_hu \in D(A)$ and $f \in H^1(\mathbb{R}^N, \nu)$, we may integrate both sides with respect to $\nu(dx)$, and applying (2.2) to both sides we get

$$-\frac{1}{2}\int_{\mathbb{R}^N}|D^2u|^2\nu(dx) \ge -2\int_{\mathbb{R}^N}Au \cdot f\,\nu(dx),$$

which yields (2.4).

Let now $u \in C_b^2(\mathbb{R}^N)$, and let u_k be a sequence in $C_0^\infty(\mathbb{R}^N)$ that goes to u in $H^2(\mathbb{R}^N, \nu)$ as $k \to \infty$. We may assume that $\|Du_k\|_\infty$ is bounded by a constant C independent of k (see the proof of lemma 2.1). Then $D_{ij}u_k$ goes to $D_{ij}u$ and Δu_k go to Δu in $L^2(\mathbb{R}^N, \nu)$. Moreover, up to a subsequence $\langle DU, Du_k - Du\rangle$ goes to 0 almost everywhere and it is bounded by $(C + \|Du\|_\infty)|DU| \in L^2(\mathbb{R}^N, \nu)$. Therefore $\langle DU, Du_k\rangle$ goes to $\langle DU, Du\rangle$ in $L^2(\mathbb{R}^N, \nu)$, so that $\mathcal{A}u_k$ goes to $\mathcal{A}u$ in $L^2(\mathbb{R}^N, \nu)$. Since (2.4) holds for each u_k, letting $k \to \infty$ we obtain that (2.4) holds for u. $\square$

Now we are ready to solve the resolvent equation,

$$\lambda u - Au = f \tag{2.5}$$

for each $\lambda > 0$ and $f \in L^2(\mathbb{R}^N, \nu)$.

Theorem 2.4 *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ satisfy assumption (1.4). Then the resolvent set of A contains $(0, +\infty)$ and*

$$\begin{cases} (i) \quad \|R(\lambda, A)f\|_{L^2(\mathbb{R}^N,\nu)} \le \dfrac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \\[2ex] (ii) \quad \|\,|DR(\lambda, A)f|\,\|_{L^2(\mathbb{R}^N,\nu)} \le \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \\[2ex] (iii) \quad \|\,|D^2R(\lambda, A)f|\,\|_{L^2(\mathbb{R}^N,\nu)} \le 4\|f\|_{L^2(\mathbb{R}^N,\nu)}. \end{cases} \tag{2.6}$$

Proof — To begin with, we note that (2.5) has at most one solution in $D(A)$. Indeed, if $u \in D(A)$ satisfies $\lambda u = Au$ then by (2.2) we have

$$\int_{\mathbb{R}^N} \lambda(u(x))^2 \nu(dx) = \int_{\mathbb{R}^N} (\mathcal{A}u)(x)u(x)\nu(dx) = -\frac{1}{2}\int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx) \le 0,$$

so that $u = 0$.

To find a solution to (2.5), first we consider the case where $f \in C_b^\theta(\mathbb{R}^N)$ for some $\theta \in (0, 1)$.

Let U_α be the Yosida approximations of U defined above, and let the differential operator $\mathcal{A}_\alpha$, the measure $\nu_\alpha(dx)$, and the operator $A_\alpha : D(A_\alpha) \mapsto L^2(\mathbb{R}^N, \nu_\alpha)$ be defined accordingly. Since DU_α is Lipschitz continuous, by [12, thm. 1] the problem

$$\lambda u_\alpha - \mathcal{A}_\alpha u_\alpha = f$$

has a unique solution $u_\alpha \in C_b^{2+\theta}(\mathbb{R}^N)$. In fact [12, thm. 1] deals with large λ's, but a standard application of the maximum principle (see e.g. [12, lemma 2.4]) and of the Schauder estimates of [12, thm. 1] show that (2.5) is uniquely solvable in $C_b^{2+\theta}(\mathbb{R}^N)$ for each $\lambda > 0$. The integration by parts formula (2.2) gives

$$\int_{\mathbb{R}^N} (\lambda u_\alpha - f)u_\alpha \nu_\alpha(dx) = -\frac{1}{2}\int_{\mathbb{R}^N} |Du_\alpha|^2 \nu_\alpha(dx) \le 0,$$

so that

$$\|u_\alpha\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \le \frac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}. \tag{2.7}$$

Estimates (2.3) and (2.4) applied to u_α give

$$\|\,|Du_\alpha|\,\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \le (2\|\lambda u_\alpha - f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}\|u_\alpha\|_{L^2(\mathbb{R}^N,\nu_\alpha)})^{1/2} \le \frac{2}{\sqrt{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \tag{2.8}$$

and

$$\| \, |D^2 u_\alpha| \, \|_{L^2(\mathbb{R}^N,\nu_\alpha)} \leq 2\|\lambda u_\alpha - f\|_{L^2(\mathbb{R}^N,\nu_\alpha)} \leq 4\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}. \tag{2.9}$$

Since $U_\alpha(x)$ goes to $U(x)$ monotonically as $\alpha \to 0$, then $\exp(-2U_\alpha(x))$ goes to $\exp(-2U(x))$ monotonically, and $(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx)^{-1}$ goes to $(\int_{\mathbb{R}^N} e^{-2U(x)}dx)^{-1}$, $\|f\|_{L^2(\mathbb{R}^N,\nu_\alpha)}$ goes to $\|f\|_{L^2(\mathbb{R}^N,\nu)}$ as $\alpha \to 0$. It follows that $\|u_\alpha\|_{H^2(\mathbb{R}^N,\nu_\alpha)}$ is bounded by a constant independent of α. Since $\|\cdot\|_{H^2(\mathbb{R}^N,\nu)} \leq c(\alpha)\|\cdot\|_{H^2(\mathbb{R}^N,\nu_\alpha)}$, with $\lim_{\alpha\to 0} c(\alpha) = 1$, also $\|u_\alpha\|_{H^2(\mathbb{R}^N,\nu)}$ is bounded by a constant independent of α, for α small. Therefore there is a sequence u_{α_n} that converges weakly in $H^2(\mathbb{R}^N,\nu)$ to a function $u \in H^2(\mathbb{R}^N,\nu)$, and converges to u pointwise a.e. and in $H^1(B(0,R))$ for each $R > 0$. This implies easily that u solves (2.5). Indeed, let $\phi \in C_0^\infty(\mathbb{R}^N)$. For each $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^N} (\lambda u_{\alpha_n} - \frac{1}{2}\Delta u_{\alpha_n} + \langle DU_{\alpha_n}, Du_{\alpha_n}\rangle - f)\phi \, e^{-2U} dx = 0.$$

Letting $n \to \infty$, we get immediately that $\int_{\mathbb{R}^N} (\lambda u_{\alpha_n} - \frac{1}{2}\Delta u_{\alpha_n})\phi e^{-2U(x)}dx$ goes to $\int_{\mathbb{R}^N} (\lambda u - \frac{1}{2}\Delta u)\phi e^{-2U(x)}dx$. Moreover $\int_{\mathbb{R}^N} \langle DU_{\alpha_n}, Du_{\alpha_n}\rangle \phi e^{-2U(x)}dx$ goes to $\int_{\mathbb{R}^N} \langle DU, Du\rangle \phi e^{-2U(x)}dx$ because DU_{α_n} goes to DU in $L^2(\mathrm{supp}\,\phi)$. Therefore letting $n \to \infty$ we get

$$\int_{\mathbb{R}^N} (\lambda u - \mathcal{A}u - f)\phi \, e^{-2U} dx = 0$$

for each $\phi \in C_0^\infty(\mathbb{R}^N)$, and hence $\lambda u - \mathcal{A}u = f$ almost everywhere. So, $u \in D(A)$ is the solution of the resolvent equation, and letting $\alpha \to 0$ in (2.7), (2.8), (2.9), we get

$$\begin{cases} \|u\|_{L^2(\mathbb{R}^N,\nu)} \leq \dfrac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \quad \||Du|\|_{L^2(\mathbb{R}^N,\nu)} \leq \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\nu)}, \\ \\ \||D^2u|\|_{L^2(\mathbb{R}^N,\nu)} \leq 4\|f\|_{L^2(\mathbb{R}^N,\nu)}. \end{cases} \tag{2.10}$$

Let now $f \in L^2(\mathbb{R}^N,\nu)$ and let f_n be a sequence of $C_b^\theta(\mathbb{R}^N)$ functions going to f in $L^2(\mathbb{R}^N,\nu)$ as $n \to \infty$. Because of estimates (2.10), the solutions u_n of

$$\lambda u_n - Au_n = f_n$$

are a Cauchy sequence in $H^2(\mathbb{R}^N,\nu)$, and converge to a solution $u \in H^2(\mathbb{R}^N,\nu)$ of (2.5). Due again to estimates (2.10), u satisfies (2.6).

□

Corollary 2.5 *The operator A is self-adjoint and dissipative. $H^1(\mathbb{R}^N,\nu)$ is the domain of $\sqrt{-A}$.*

3 The operator A_N

In this section we describe the main properties of the realization A_N of $\mathcal{A}$ in $L^2(\Omega,\mu)$. We recall that Ω is a convex open set in $\mathbb{R}^N$ with C^2 boundary, and $U : \mathbb{R}^N \mapsto \mathbb{R}$ is a convex function satisfying (1.4). The functional spaces $H^1(\Omega,\mu)$, $H^2(\Omega,\mu)$, $C_b^k(\overline{\Omega},\mu)$, are defined as in the case $\Omega = \mathbb{R}^N$.

Arguing as in the proof of lemma 2.1 we obtain a similar result.

Lemma 3.1 *Let μ be the measure defined in (1.5). Then the functions in $C^\infty(\overline{\Omega})$ with compact support are dense in $L^2(\Omega,\mu)$, in $H^1(\Omega,\mu)$, and in $H^2(\Omega,\mu)$.*

Proof — First we consider a compactly supported function u, belonging to $L^2(\Omega,\mu)$, to $H^1(\Omega,\mu)$, or to $H^2(\Omega,\mu)$. Since the boundary of Ω is C^2, then u may be extended to the whole $\mathbb{R}^N$ in such a way that the extension is in $L^2(\mathbb{R}^N)$, in $H^1(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$, respectively. The extension is then approximated by a sequence u_n of C_0^∞ functions obtained by convolution with smooth mollifiers. The restrictions of u_n to Ω converge to u in $L^2(\Omega,\mu)$, in $H^1(\Omega,\mu)$, or in $H^2(\Omega,\mu)$, respectively.

If u has not compact support, let θ be the function defined in the proof of lemma 2.1. The sequence $u_n(x) = u(x)\theta(x/n)$ consists of L^2 (respectively, H^1, H^2) functions with compact support, and converges to u in $L^2(\Omega,\mu)$, in $H^1(\Omega,\mu)$, or in $H^2(\Omega,\mu)$, respectively (the proof is the same of lemma 2.1). The statement follows. □

The realization A_N of $\mathcal{A}$ in $L^2(\Omega,\mu)$ with Neumann boundary condition is defined by

$$\begin{cases} D(A_N) &= \{u \in H^2(\Omega,\mu) :\ \mathcal{A}u \in L^2(\Omega,\mu),\ \partial u/\partial n = 0 \text{ at } \partial\Omega\} \\[2mm] &= \{u \in H^2(\Omega,\mu) :\ \langle DU, Du\rangle \in L^2(\Omega,\mu),\ \partial u/\partial n = 0 \text{ at } \partial\Omega\}, \\[2mm] (A_N u)(x) = \mathcal{A}u(x), \quad x \in \Omega. & \end{cases} \tag{3.1}$$

A formula similar to (2.2) holds if $\mathbb{R}^N$ is replaced by Ω. The proof is the same as in lemma 2.2, and it is omitted.

Lemma 3.2 *Let $\psi \in H^1(\Omega,\mu)$ and let $u \in H^2(\Omega,\mu)$ be such that $\mathcal{A}u \in L^2(\Omega,\mu)$. Then we have*

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \langle Du(x), D\psi(x)\rangle\mu(dx) + \frac{1}{2}\int_{\partial\Omega} \frac{\partial u}{\partial n}(x)\psi(x)\mu(dx). \tag{3.2}$$

Theorem 3.3 *The resolvent set of A_N contains $(0,+\infty)$. For every $\lambda > 0$ we have*

$$\begin{cases} (i) & \|R(\lambda,A_N)f\|_{L^2(\Omega,\mu)} \le \dfrac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \\[3mm] (ii) & \|\,|DR(\lambda,A_N)f|\,\|_{L^2(\Omega,\mu)} \le \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\Omega,\mu)}, \\[3mm] (iii) & \|\,|D^2R(\lambda,A_N)f|\,\|_{L^2(\Omega,\mu)} \le 4\|f\|_{L^2(\Omega,\mu)}. \end{cases} \tag{3.3}$$

Proof — Let $\lambda > 0$, let $f \in L^2(\Omega,\mu)$, and consider the resolvent equation

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega, \\[2mm] \dfrac{\partial u}{\partial n} = 0 & \text{at } \partial\Omega. \end{cases} \tag{3.4}$$

Uniqueness of the solution to (3.4) in $H^2(\Omega,\mu)$ is easy. Indeed, if $u \in H^2(\Omega,\mu)$, $\partial u/\partial n = 0$ and $\lambda u - \mathcal{A}u = 0$, then taking $\psi = u$ in (3.2) we get $\lambda\|u\|^2_{L^2(\Omega,\mu)} \le 0$, and hence $u = 0$.

Now we show that (3.4) has a solution $u \in D(A_N)$. For each $\varepsilon > 0$ let

$$V_\varepsilon(x) = U(x) + \frac{1}{2\varepsilon}(\text{dist}(x,\Omega))^2, \quad x \in \mathbb{R}^N,$$

and let the differential operator $\mathcal{L}_\varepsilon$ be defined by

$$(\mathcal{L}_\varepsilon u)(x) = \frac{1}{2}\Delta u(x) - \langle DV_\varepsilon(x), Du(x)\rangle, \quad x \in \mathbb{R}^N.$$

The function V_ε satisfies obviously (1.4); moreover since Ω is convex, then V_ε is convex. Set

$$Z_\varepsilon = \int_{\mathbb{R}^N} \exp(-2V_\varepsilon(x))dx, \;\; \nu_\varepsilon(dx) = \frac{1}{Z_\varepsilon}\exp(-2V_\varepsilon(x))dx, \tag{3.5}$$

and let A_ε be the realization of $\mathcal{L}_\varepsilon$ in $L^2(\mathbb{R}^N, \nu_{\ e})$ defined by

$$D(A_\varepsilon) = \{u \in H^2(\mathbb{R}^N, \nu_\varepsilon) : \langle DV_\varepsilon, Du\rangle \in L^2(\mathbb{R}^N, \nu_\varepsilon)\}.$$

Let $\widetilde{f}$ be defined by $\widetilde{f}(x) = f(x)$ for $x \in \Omega$, $\widetilde{f}(x) = 0$ for x outside Ω. By theorem 2.4, the problem

$$\lambda u - \mathcal{L}_\varepsilon u = \widetilde{f}, \;\; x \in \mathbb{R}^N, \tag{3.6}$$

has a unique solution $u_\varepsilon \in D(A_\varepsilon)$, which satisfies the estimates

$$\begin{cases} \|u_\varepsilon\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)} \le \dfrac{1}{\lambda}\|\widetilde{f}\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)}, \\[2ex] \|\,|Du_\varepsilon|\,\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)} \le \dfrac{2}{\sqrt{\lambda}}\|\widetilde{f}\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)}, \\[2ex] \|\,|D^2u_\varepsilon|\,\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)} \le 4\|\widetilde{f}\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)} \end{cases} \tag{3.7}$$

due to (2.6). If in addition $f(x) \ge 0$ a.e., then $u_\varepsilon(x) \ge 0$ for each x. Since

$$\|\widetilde{f}\|_{L^2(\mathbb{R}^N,\nu_\varepsilon)} = \left(\frac{1}{Z_\varepsilon}\int_\Omega f^2 e^{-2U}dx\right)^{1/2} = \left(\frac{\int_\Omega e^{-2U}dx}{\int_{\mathbb{R}^N} e^{-2V_\varepsilon}dx}\right)^{1/2} \|f\|_{L^2(\Omega,\mu)}$$

remains bounded as $\varepsilon \to 0$, then u_ε is bounded in $H^2(\mathbb{R}^N, \nu_\varepsilon)$ and the restriction $u_{\varepsilon|\Omega}$ is bounded in $H^2(\Omega,\mu)$. Up to a sequence, $u_{\varepsilon|\Omega}$ converges weakly in $H^2(\Omega,\mu)$ to a function $u \in H^2(\Omega,\mu)$ and it converges to u pointwise a.e. and in $H^{3/2}(\Omega \cap B(0,R), dx)$ for every $R > 0$. Since $\lambda u_\varepsilon - \mathcal{A}u_\varepsilon = f$ in Ω, then u satisfies $\lambda u - \mathcal{A}u = f$ almost everywhere in Ω. Since λu, Δu, f belong to $L^2(\Omega,\mu)$, then also $\langle DU, Du\rangle$ does.

Let us prove that $\partial u/\partial n = 0$ at the boundary. For each $\psi \in C_0^\infty(\mathbb{R}^N)$ we have, by (2.2)

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx) = -\frac{1}{2}\int_{\mathbb{R}^N} \langle Du_\varepsilon, D\psi\rangle \nu_\varepsilon(dx).$$

On the other hand,

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx) = \int_{\mathbb{R}^N\setminus\Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx) + \int_\Omega \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx),$$

where

$$\int_{\mathbb{R}^N\setminus\Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx) = \lambda \int_{\mathbb{R}^N\setminus\Omega} u_\varepsilon \psi\, \nu_\varepsilon(dx),$$

because $\lambda u_\varepsilon - \mathcal{L}_\varepsilon u_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega$, and

$$\int_\Omega \mathcal{L}_\varepsilon u_\varepsilon \psi\, \nu_\varepsilon(dx) = -\frac{1}{2}\int_\Omega \langle Du_\varepsilon, D\psi\rangle \nu_\varepsilon(dx) + \frac{1}{2}\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}(x)\psi(x)\, \nu_\varepsilon(dx)$$

because of (3.2). It follows that

$$\frac{1}{2}\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}(x)\psi(x)\, \nu_\varepsilon(dx) = -\frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega} \langle Du_\varepsilon, D\psi\rangle \nu_\varepsilon(dx) - \lambda \int_{\mathbb{R}^N\setminus\Omega} u_\varepsilon \psi\, \nu_\varepsilon(dx).$$

Note that $\|\psi\|_{H^1(\mathbb{R}^N\setminus\Omega,\nu_\varepsilon)}$ goes to 0 as $\varepsilon \to 0$. Since both u_ε and $|Du_\varepsilon|$ are bounded in $L^2(\mathbb{R}^N,\nu_\varepsilon)$, the right hand side goes to 0 as $\varepsilon \to 0$. On the other hand, since u_ε goes to u in $H^{3/2}(\Omega \cap \operatorname{supp}\psi)$, then $\partial u_\varepsilon/\partial n$ goes to $\partial u/\partial n$ in $L^2(\partial\Omega \cap \operatorname{supp}\psi)$. So we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}(x)\psi(x)\, e^{-2U(x)}dx = \lim_{\varepsilon\to 0}\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}(x)\psi(x)\, e^{-2V_\varepsilon(x)}dx$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} e^{-2V_\varepsilon(x)}dx \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}(x)\psi(x)\, \nu_\varepsilon(dx) = 0.$$

This implies that $\partial u/\partial n = 0$ at $\partial\Omega$, and $u \in D(A_N)$. So, u is the unique solution in $D(A_N)$ of problem (3.4). Therefore $\rho(A_N) \supset (0,+\infty)$. Estimates (3.3) follow letting ε to 0 in (3.7). □

Corollary 3.4 *The operator A_N is self-adjoint and dissipative in $L^2(\Omega,\mu)$. The measure μ is an infinitesimally invariant measure for A_N. The space $H^1(\Omega,\mu)$ is the domain of $(-A_N)^{1/2}$.*

Proof — The first statement is an immediate consequence of theorem 3.3, and of formula (3.2) which yields that A_N is symmetric. Taking $\psi \equiv 1$, formula (3.2) shows that $\int_\Omega \mathcal{A}_N u\, \mu(dx) = 0$ for each $u \in D(A_N)$, and therefore μ is an infinitesimally invariant measure for A_N. It proves also that

$$\int_\Omega |(-A_N)^{1/2}u|^2\mu(dx) = \frac{1}{2}\int_\Omega |Du|^2\mu(dx)$$

for each $u \in D(A_N)$, and this implies that $D((-A_N)^{1/2}) = H^1(\Omega,\mu)$. □

4 Poincaré and Log–Sobolev inequalities

In this section we prove the Poincaré and Log–Sobolev inequalities for the measure μ. In addition to (1.4) we assume that

$$\exists\omega > 0 \text{ such that } x \mapsto U(x) - \omega|x|^2/2 \text{ is convex.} \tag{4.1}$$

In the case of a twice continuously differentiable U this means just $D^2U(x) \geq \omega I$ for each x.

If (Λ, m) is any measure space and $u \in L^1(\Lambda, m)$ we set

$$\overline{u}_m = \int_\Lambda u(x)m(dx). \tag{4.2}$$

Poincaré and Log–Sobolev inequalities in the whole $\mathbb{R}^N$ were proved in [1] under the further assumptions that U is continuously differentiable. In our case such assumptions may be avoided.

Proposition 4.1 *Let U satisfy (1.4) and (4.1), and let ν be defined by (1.3). Then*

$$\int_{\mathbb{R}^N} |u(x) - \overline{u}_\nu|^2\nu(dx) \leq \frac{1}{2\omega}\int_{\mathbb{R}^N} |Du(x)|^2 d\nu(dx), \quad u \in H^1(\mathbb{R}^N,\nu), \tag{4.3}$$

and

$$\int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\nu(dx) \leq \frac{1}{\omega}\int_{\mathbb{R}^N} |Du(x)|^2\nu(dx) + \overline{u^2}_\nu \log(\overline{u^2}_\nu), \quad u \in H^1(\mathbb{R}^N,\nu). \tag{4.4}$$

Proof — Let us consider the Moreau-Yosida approximations U_α of U. For each α, U_α satisfies (1.4) with constant ω_α which goes to ω as α goes to 0, and DU_α is Lipschitz continuous. Therefore (see e.g. [1]) Poincaré and Log–Sobolev inequalities hold for the measures ν_α, in the form

$$\int_{\mathbb{R}^N} |u(x)-\overline{u}_\alpha|^2\nu_\alpha(dx) \leq \frac{1}{2\omega_\alpha}\int_{\mathbb{R}^N}|Du(x)|^2\nu_\alpha(dx), \quad u\in H^1(\mathbb{R}^N,\nu_\alpha),$$

(where $\overline{u}_\alpha$ stands for $\overline{u}_{\nu_\alpha}$) and

$$\int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\nu_\alpha(dx) \leq \frac{1}{\omega_\alpha}\int_{\mathbb{R}^N}|Du(x)|^2\nu_\alpha(dx) + \overline{u^2}_\alpha\log(\overline{u^2}_\alpha), \quad u\in H^1(\mathbb{R}^N,\nu_\alpha).$$

Taking $u\in C_0^\infty(\mathbb{R}^N)$ and letting $\alpha\to 0$ we see that $\overline{u}_\alpha$ goes to $\overline{u}_\nu$, $\overline{u^2}_\alpha$ goes to $\overline{u^2}_\nu$, and u satisfies (4.3) and (4.4). Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N,\nu)$ the statement follows. □

Having estimates (4.3) and (4.4) as a tool, it is not hard to prove Poincaré and Log–Sobolev inequalities in our situation.

Proposition 4.2 *Let U satisfy (1.4) and (4.1), and let μ be defined by (1.5). Then*

$$\int_\Omega |u(x)-\overline{u}_\mu|^2\mu(dx) \leq \frac{1}{2\omega}\int_\Omega |Du(x)|^2 d\mu(dx), \quad u\in H^1(\Omega,\mu), \tag{4.5}$$

and

$$\int_\Omega u^2(x)\log(u^2(x))\mu(dx) \leq \frac{1}{\omega}\int_\Omega |Du(x)|^2\mu(dx) + \overline{u^2}_\mu\log(\overline{u^2}_\mu), \;\; u\in H^1(\Omega,\mu). \tag{4.6}$$

Proof — Let $u\in H^1(\Omega,\mu)$ have compact support, and extend u to an $H^1(\mathbb{R}^N)$ function with compact support, still denoted by u. Let

$$\nu_\varepsilon(dx) = \left(\int_{\mathbb{R}^N} e^{-2V_\varepsilon(x)}dx\right)^{-1} e^{-2V_\varepsilon(x)}dx,$$

where V_ε is defined by (1.7). By proposition 4.2, for each $\varepsilon>0$ we have

$$\int_{\mathbb{R}^N} |u(x)-\overline{u}_\varepsilon|^2\nu_\varepsilon(dx) \leq \frac{1}{2\omega}\int_{\mathbb{R}^N}|Du(x)|^2\nu_\varepsilon(dx), \tag{4.7}$$

(where $\overline{u}_\varepsilon$ stands for $\overline{u}_{\nu_\varepsilon}$) and

$$\int_{\mathbb{R}^N} u^2(x)\log(u^2(x))\nu_\varepsilon(dx) \leq \frac{1}{\omega}\int_{\mathbb{R}^N}|Du(x)|^2\nu_\varepsilon(dx) + \overline{u^2}_\varepsilon\log(\overline{u^2}_\varepsilon). \tag{4.8}$$

Since

$$\lim_{\varepsilon\to 0} V_\varepsilon(x) = \begin{cases} U(x) & \text{if } x\in\Omega \\ +\infty & \text{if } x\notin\Omega, \end{cases}$$

then $\overline{u}_\varepsilon$ goes to $\overline{u}_\mu = \int_\Omega u(x)\mu(dx)$, $\overline{u^2}_\varepsilon$ goes to $\overline{u^2}_\mu$ as ε goes to 0, and letting ε go to 0 in (4.7), (4.8) we obtain that u satisfies (4.5) and (4.6). Since the compactly supported functions are dense in $H^1(\Omega,\mu)$ the statement follows. □

Proposition 4.2 yields important properties of the semigroup $T(t)$ generated by A_N.

Corollary 4.3 *Under assumptions (1.4) and (4.1), 0 is a simple isolated eigenvalue of A_N. The rest of the spectrum, $\sigma(A_N)\setminus\{0\}$ is contained in $(-\infty,-\omega]$, and*

$$\|T(t)u-\overline{u}_\mu\|_{L^2(\Omega,\mu)} \leq e^{-\omega t}\|u-\overline{u}_\mu\|_{L^2(\Omega,\mu)}, \;\; u\in L^2(\Omega,\mu),\; t>0. \tag{4.9}$$

Moreover we have

$$\|T(t)\varphi\|_{L^{q(t)}(\Omega,\mu)} \le \|\varphi\|_{L^p(\Omega,\mu)}, \quad p \ge 2,\ \varphi \in L^p(\Omega,\mu), \tag{4.10}$$

where

$$q(t) = 1 + (p-1)e^{2\omega t}, \quad t > 0. \tag{4.11}$$

Proof — 0 is obviously an eigenvalue of A_N, whose kernel contains the constant functions. In fact, the kernel of A_N consists only of constant functions: if $Au = 0$, then $T(t)u = u$ and hence $DT(t)u = Du$ for each $t > 0$, but since $\|DT(t)u\|_{L^2(\Omega,\mu)} \le Ct^{-1/2}\|u\|_{L^2(\Omega,\mu)}$ (this is a consequence of the equality $D((-A_N)^{1/2}) = H^1(\Omega,\mu)$) letting $t \to +\infty$ we get $Du = 0$ so that u is constant.

A standard argument shows now that 0 is isolated. Let $u \in H^1(\Omega,\mu)$ and set $g(t) = \|T(t)u - \overline{u}\|^2_{L^2(\Omega,\mu)}$. Using (1.8) and (4.5) we get easily $g'(t) \le -2\omega g(t)$, for each $t > 0$, so that u satisfies (4.9). Since $H^1(\Omega,\mu)$ is dense in $L^2(\Omega,\mu)$, then (4.9) holds for every $u \in L^2(\Omega,\mu)$. From the general theory of strongly continuous semigroups it follows that 0 is isolated in $\sigma(A)$, and, since $\sigma(A_N) \subset \mathbb{R}$, that $\sigma(A) \setminus \{0\} \subset (-\infty, -\omega]$.

Since A_N is self-adjoint, all its isolated eigenvalues are semisimple. The kernel of A_N is one dimensional, and hence 0 is a simple eigenvalue.

The hypercontractivity estimate (4.10) is a consequence of the Log–Sobolev inequality (4.6). See e.g. [1]. $\square$

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