ELLIPTIC OPERATORS WITH UNBOUNDED DIFFUSION COEFFICIENTS IN L^2 SPACES WITH RESPECT TO INVARIANT MEASURES

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ABSTRACT. We study the self-adjoint and dissipative realization A of a second order elliptic differential operator $\mathcal{A}$ with unbounded regular coefficients in $L^2(\mathbb{R}^N, \mu)$, where $\mu(dx) = \rho(x)dx$ is the associated invariant measure. We prove a maximal regularity result under suitable assumptions, that generalize the well known conditions in the case of constant diffusion part.

1. Introduction

In this paper we deal with an elliptic operator $\mathcal{A}$ in $\mathbb{R}^N$ defined by

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} q_{ij}(x)D_{ij}\varphi(x) + \sum_{j=1}^{N} b_j(x)D_j\varphi(x) = \mathrm{Tr}(Q(x)D^2\varphi(x)) + \langle B(x), D\varphi(x)\rangle,$$

with regular (continuously differentiable, with locally Hölder continuous derivatives) and possibly unbounded coefficients q_{ij}, b_j $(i, j = 1, \ldots, N)$.

It is well known that, if the coefficients of an elliptic operator are unbounded, its realizations in the Lebesgue spaces $L^p(\mathbb{R}^N, dx)$ do not enjoy good properties, unless we make very strong assumptions. So, here we consider a weighted Lebesgue measure

$$\mu(dx) = \rho(x)dx,$$

such that a realization A of $\mathcal{A}$ in $L^2(\mu) := L^2(\mathbb{R}^N, \mu)$ is self-adjoint. It is not hard to see that, given any differentiable weight $\rho(x) > 0$, we have

$$\int_{\mathbb{R}^N} u\,\mathcal{A}v\,d\mu = \int_{\mathbb{R}^N} v\,\mathcal{A}u\,d\mu,$$

for all u, $v \in C_0^\infty(\mathbb{R}^N)$ (the space of smooth functions with compact support in $\mathbb{R}^N$) if and only if

$$Q^{-1}(B - \mathrm{div}\,Q) = D(\log\rho),$$

where $\mathrm{div}\,Q$ is the vector with entries $\xi_j = \sum_{i=1}^N D_i q_{ij}$. Therefore, our first assumption is the existence of a function Φ such that

$$D\Phi = Q^{-1}(\mathrm{div}\,Q - B),$$

and we set $\rho(x) = e^{-\Phi(x)}$, i.e.

$$\mu(dx) = e^{-\Phi(x)}dx.$$

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Note that Φ is uniquely determined, up to a constant. Now, several properties follow. First, taking $v \equiv 1$, we get

$$\int_{\mathbb{R}^N} \mathcal{A}u \, d\mu = 0, \qquad u \in C_0^\infty(\mathbb{R}^N),$$

that is, μ is infinitesimally invariant for $\mathcal{A}$, at least on $C_0^\infty(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} \mathcal{A}u \, v \, d\mu = -\int_{\mathbb{R}^N} \langle QDu, Dv\rangle d\mu, \qquad u,\ v \in C_0^\infty(\mathbb{R}^N),$$

so that $\mathcal{A}$ is associated to a nice quadratic form in the gradient. Taking in particular $v = u$, we see that $\mathcal{A}$ is dissipative on $C_0^\infty(\mathbb{R}^N)$. In view of the above identity, it is natural to introduce the space $H^1_Q(\mu)$, consisting of the functions $u \in L^2(\mu)$ such that $|Q^{1/2}Du| \in L^2(\mu)$. Similarly, we denote by $H^2_Q(\mu)$ the subspace of $H^1_Q(\mu)$ consisting of the functions u such that $|Q^{1/2}D^2u\, Q^{1/2}| \in L^2(\mu)$. Here, first and second order derivatives are understood in the weak sense, and we use the symbol $|\cdot|$ for the euclidean norms both of vectors and matrices.

The main result of this paper is that the realization A of $\mathcal{A}$ with domain

$$D(A) = \{u \in H^2_Q(\mathbb{R}^N) : \langle B\cdot, Du\rangle \in L^2(\mu)\}$$

is self-adjoint and dissipative in $L^2(\mu)$, provided suitable growth and structural conditions on the coefficients hold. Moreover, the domain $D(A)$ is continuously embedded in $H^2_Q(\mu)$ and it coincides with the maximal domain $\{u \in L^2(\mu) \cap H^2_{\rm loc}(\mathbb{R}^N, dx) : \mathcal{A}u \in L^2(\mu)\}$. This can be seen as an optimal regularity result for the equation

$$\lambda u - \mathcal{A}u = f,$$

with $\lambda > 0$ and $f \in L^2(\mu)$. Indeed, existence and uniqueness of a weak solution $u \in H^1_Q(\mu)$ can be obtained by the Lax-Milgram lemma. What is not obvious is that u belongs to $H^2_Q(\mu)$.

Our growth condition is only on the coefficients q_{ij}: we assume that there exists $C > 0$ such that

$$|Q(x)| \le C(1 + |x|^2), \qquad x \in \mathbb{R}^N.$$

The structural condition is a generalization of the well known dissipativity assumption on B in the case of constant diffusion coefficients (e.g., [2, 3, 4, 6]). More precisely, we assume that there exist two constants $k_1 > 0$ and $k_2 \in (0,1)$ such that

$$\begin{aligned}&\langle Q(x)\xi, D(\mathrm{Tr}(Q(x)S))\rangle - \mathrm{Tr}((D(Q(x)\xi))Q(x)S) + \langle Q(x)(DB(x))^*\xi, \xi\rangle \\ \le\ & k_1|Q^{1/2}(x)\xi|^2 + k_2|Q^{1/2}(x)SQ^{1/2}(x)|^2,\end{aligned}$$

for any symmetric matrix S and any x, $\xi \in \mathbb{R}^N$. Examples such that these conditions are satisfied are given in Section 2.

Most of the papers about elliptic operators in L^p spaces with respect to invariant measures are devoted to show that such operators possess m-dissipative realizations, that generate contraction semigroups. The characterization of the domains of such realizations is a more difficult problem. It has been considered in [7, 10] in the case of constant diffusion coefficients. See also [8, 9] where $\mathbb{R}^N$ is replaced by an unbounded open set Ω with suitable boundary conditions.

2. Assumptions and function spaces

Our assumptions have been already mentioned in the introduction, for the reader's convenience we list them again.

Hypotheses 2.1. (i) *the functions* q_{ij} *and* b_i $(i,j=1,\dots,N)$ *are continuously differentiable, with locally Hölder continuous derivatives; for each* $x\in\mathbb{R}^N$ *there is* $\nu(x)>0$ *such that*

$$\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geq \nu(x)|\xi|^2, \qquad x,\xi\in\mathbb{R}^N; \tag{2.1}$$

(ii) *there exists a positive constant* C *such that*

$$|Q(x)|\leq C(1+|x|^2), \qquad x\in\mathbb{R}^N; \tag{2.2}$$

(iii) *there exists a function* $\Phi:\mathbb{R}^N\to\mathbb{R}$ *such that*

$$Q^{-1}(\operatorname{div} Q - B) = D\Phi, \tag{2.3}$$

where $(\operatorname{div} Q)_j := \sum_{i=1}^N D_i q_{ij}$ $(j=1,\dots,N)$*;*

(iv) *there exist two positive constants* $k_1>0$ *and* $k_2\in(0,1)$ *such that*

$$\langle Q(DB)^*\xi,\xi\rangle + \langle Q\xi, D(\mathrm{Tr}(QS))\rangle - \mathrm{Tr}((D(Q\xi))QS) \leq k_1|Q^{1/2}\xi|^2 + k_2|Q^{1/2}SQ^{1/2}|^2, \tag{2.4}$$

for any symmetric matrix S *and any* $\xi\in\mathbb{R}^N$*.*

Note that we do not assume that $\mathcal{A}$ is uniformly elliptic in the whole $\mathbb{R}^N$, i.e. we allow that the infimum of the ellipticity constant ν is zero.

We introduce the measure

$$\mu(dx) = e^{-\Phi(x)}dx$$

and we denote by $H^1_Q(\mu)$ the space of functions $u\in L^2(\mu)$ such that $|Q^{1/2}Du|\in L^2(\mu)$. It is a Hilbert space with the scalar product

$$\langle u,v\rangle_{H^1_Q(\mu)} := \int_{\mathbb{R}^N} u(x)v(x)\,d\mu + \int_{\mathbb{R}^N}\langle Q(x)Du(x), Dv(x)\rangle\,d\mu, \qquad u,\ v\in H^1_Q(\mu).$$

Similarly, we denote by $H^2_Q(\mu)$ the subspace of $H^1_Q(\mu)$ consisting of the functions u such that $|Q^{1/2}D^2u\,Q^{1/2}|\in L^2(\mu)$. We endow it with the norm

$$\|u\|_{H^2_Q(\mu)} = \|u\|_{H^1_Q(\mu)} + \|\,|Q^{1/2}D^2u\,Q^{1/2}|\,\|_{L^2(\mu)}, \qquad u\in H^2_Q(\mu).$$

Since q_{ij}, b_i $(i,j=1,\dots,N)$ are continuous, the function $e^{-\Phi}$ has positive minimum on each compact set, and the matrices $Q(x)$ are uniformly positive definite on each compact set. Therefore, the spaces $L^2(\mu)$, $H^1_Q(\mu)$, $H^2_Q(\mu)$ are locally equivalent to the usual L^2, H^1, H^2 spaces with respect to the Lebesgue measure.

Let us illustrate Hypothesis 2.1 by means of some examples.

Examples 2.2. **(a)** If $Q=I$, condition (2.3) means that $B=-D\Phi$, whereas condition (2.4) means that the symmetric matrix $DB=-D^2\Phi$ is upperly bounded, and more precisely the function $x\mapsto\Phi(x)+k_1|x|^2/2$ is convex. So, we recover the convexity hypotheses of [2, 7]. In the case $B=0$, μ is just a multiple of the Lebesgue measure.

(b) If $N=1$, condition (2.3) is obviously satisfied and

$$\mu(dx)=\frac{c}{q(x)}\exp\left(\int_0^x\frac{b(s)}{q(s)}\,ds\right)dx,$$

c being an arbitrary positive constant.

Condition (2.4) is simply reduced to $\sup_{x\in\mathbb{R}} b'(x)<+\infty$.

(c) Let $Q(x)=\nu(x)I$. Then,

$$Q^{-1}(\operatorname{div}Q-B)=D\log(\nu(x))-\nu(x)^{-1}B(x),\qquad x\in\mathbb{R}^N.$$

Condition (2.3) is satisfied provided that $B=\nu DF$, for some function $F:\mathbb{R}^N\to\mathbb{R}$. In this case we have

$$\mu(dx)=\frac{c}{\nu(x)}\exp(F(x))dx,$$

for some $c\in\mathbb{R}_+$. Moreover, (2.4) reads

$$\langle DB(x)\,\xi,\xi\rangle+\langle D\nu(x),(\mathrm{Tr}(S)I-S)\xi\rangle\le k_1|\xi|^2+k_2\nu(x)|S|^2,\ \ x,\ \xi\in\mathbb{R}^N.$$

Therefore, (2.2) and (2.4) are satisfied if

$$\nu(x)\le C(1+|x|^2),\quad |D\nu(x)|\le C\nu(x)^{1/2},\quad \langle DB\,\xi,\xi\rangle\le C|\xi|^2,\qquad x,\ \xi\in\mathbb{R}^N,$$

for some positive constant C.

(d) Suppose that

$$Q(x,y)=\begin{pmatrix} f(x) & 0\\ 0 & g(y)\end{pmatrix},\qquad B(x,y)=Q(x,y)DV(x,y)=\begin{pmatrix} f(x)V_x(x,y)\\ g(y)V_y(x,y)\end{pmatrix},$$

for any $(x,y)\in\mathbb{R}^2$ and some smooth functions $f,g:\mathbb{R}\to\mathbb{R}$ and $V:\mathbb{R}^2\to\mathbb{R}$, with f and g positive and such that $f(x)+g(x)\le C(1+x^2)$ for any $x\in\mathbb{R}$ and some positive constant C. Then, the condition (2.3) is satisfied with $\Phi(x,y)=-V(x,y)+\log(f(x)g(y))$ for any $(x,y)\in\mathbb{R}^2$. Therefore, the invariant measures are

$$\mu(dx,dy)=c\frac{e^{V(x,y)}}{f(x)g(y)}dxdy,$$

c being an arbitrary positive constant. Moreover, the condition (2.4) reduces to

$$\langle Q(x,y)(DB(x,y))^*\xi,\xi\rangle\le k_1|Q^{1/2}(x,y)\xi|^2,\qquad (x,y),\ \xi\in\mathbb{R}^2,$$

for some $k_1>0$, that is

$$\langle Q(x,y)D^2V(x,y)Q(x,y)\xi,\xi\rangle+f'(x)V_x(x,y)\xi_1^2+g'(y)V_y(x,y)\xi_2^2\le k_1|Q^{1/2}(x,y)\xi|^2,\ \ (2.5)$$

for any $(x,y),\xi\in\mathbb{R}^2$. Condition (2.5) is satisfied, for instance, in the case when $D^2V(x,y)\le 0$ for $|(x,y)|$ large and

$$f'(x)V_x(x,y)\le k_1f(x),\qquad g'(y)V_y(x,y)\le k_1g(y),\qquad (x,y)\in\mathbb{R}^2.$$

This is the case if we take

$$Q(x,y)=\begin{pmatrix} 1+x^2 & 0\\ 0 & 1\end{pmatrix},\qquad B(x,y)=\begin{pmatrix} 2x-(1+x^2)U_x(x,y)\\ -U_y(x,y)\end{pmatrix},\qquad (x,y)\in\mathbb{R}^2,\ \ (2.6)$$

and U is a smooth convex function such that

$$\inf_{(x,y)\in\mathbb{R}^2}\frac{xU_x(x,y)}{1+x^2}>-\infty.$$

The invariant measures are

$$\mu(dx,dy) = c\, e^{-U} dxdy, \tag{2.7}$$

c being an arbitrary positive constant.

If we take

$$Q(x,y) = \begin{pmatrix} 1+x^2 & 0 \\ 0 & \dfrac{1}{1+y^2} \end{pmatrix}, \qquad B(x,y) = \begin{pmatrix} 2x - (1+x^2)U_x(x,y) \\ -\dfrac{2y}{(1+y^2)^2} - \dfrac{U_y}{1+y^2} \end{pmatrix}, \tag{2.8}$$

for some convex function U, then the condition (2.5) is satisfied if

$$\inf_{(x,y)\in\mathbb{R}^2} \frac{xU_x(x,y)}{1+x^2} > -\infty, \qquad \sup_{(x,y)\in\mathbb{R}^2} \frac{yU_y(x,y)}{1+y^2} < +\infty,$$

and the invariant measures are still given by (2.7).

Note that in (2.8) the diffusion matrix Q degenerates at $+\infty$.

3. The self-adjoint realization of $\mathcal{A}$ in $L^2(\mu)$

We begin this section by proving two lemmas which will play a fundamental role in what follows.

Lemma 3.1. *Suppose that Hypotheses* 2.1*(i)–(iii) are satisfied. Then* $C_0^\infty(\mathbb{R}^N)$ *is dense in* $H^1_Q(\mu)$ *and in* $H^2_Q(\mu)$.

Proof. Let us prove that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2_Q(\mu)$. The same arguments show that $C_0^\infty(\mathbb{R}^N)$ is dense also in $H^1_Q(\mu)$.

We first assume that $u \in H^2_Q(\mu)$ has compact support. Then, $u \in H^2(\mathbb{R}^N, dx)$ and there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^N)$ compactly supported in $\mathrm{supp}(u) + B(0,1)$, which converges to u in $H^2(\mathbb{R}^N, dx)$. It follows that $\{u_n\}_{n\in\mathbb{N}}$ converges to u also in $H^2_Q(\mu)$.

Now we show that any function $u \in H^2_Q(\mu)$ can be approximated in the $H^2_Q(\mu)$-norm by a sequence of compactly supported functions $u_n \in H^2_Q(\mu)$. Let $\vartheta \in C_0^\infty(\mathbb{R}^N)$ be any smooth function with support contained in $B(0,1)$ and such that $0 \le \vartheta \le 1$ and $\vartheta \equiv 1$ in $B(0,1/2)$. Then, we set

$$u_n(x) = u(x)\vartheta\left(\frac{x}{n}\right), \qquad x \in \mathbb{R}^N,\ n \in \mathbb{N}. \tag{3.1}$$

Each u_n belongs to $H^2_Q(\mu)$, its support is contained in $B(0,n)$ and $u_n \equiv 1$ in $B(0,n/2)$. Moreover,

$$D_i u_n(x) = \vartheta\left(\frac{x}{n}\right) D_i u(x) + \frac{1}{n} u(x)(D_i\vartheta)\left(\frac{x}{n}\right)$$

and

$$\begin{aligned} D_{ij}u_n(x) \;=\; & \vartheta\left(\frac{x}{n}\right) D_{ij}u(x) + \frac{1}{n} D_j u(x)(D_i\vartheta)\left(\frac{x}{n}\right) \\ & + \frac{1}{n} D_i u(x)(D_j\vartheta)\left(\frac{x}{n}\right) + \frac{1}{n^2} u(x)(D_{ij}\vartheta)\left(\frac{x}{n}\right), \end{aligned}$$

for any $x \in \mathbb{R}^N$, any $n \in \mathbb{N}$ and any $i, j = 1, \ldots, N$. Therefore,

$$
\begin{aligned}
\|u_n - u\|^2_{H^2_Q(\mu)} \le & \int_{\mathbb{R}^N} |u(x)|^2 |1-\vartheta(x/n)|^2 d\mu + \int_{\mathbb{R}^N} |Q^{1/2}(x) Du(x)|^2 |1-\vartheta(x/n)|^2 d\mu \\
& + \int_{\mathbb{R}^N} |Q^{1/2}(x) D^2 u(x) Q^{1/2}(x)|^2 |1-\vartheta(x/n)|^2 d\mu \\
& + \frac{1}{n^2} \int_{\mathbb{R}^N} |Q^{1/2}(x) D\vartheta(x/n)|^2 |u(x)|^2 d\mu \\
& + \frac{2}{n^2} \int_{\mathbb{R}^N} |Q^{1/2}(x) Du(x)|^2 |Q^{1/2}(x) D\vartheta(x/n)|^2 d\mu \\
& + \frac{1}{n^4} \int_{\mathbb{R}^N} |Q^{1/2}(x) D^2\vartheta(x/n) Q^{1/2}(x)|^2 |u(x)|^2 d\mu. \qquad (3.2)
\end{aligned}
$$

The first three terms in the right hand side of (3.2) converge to 0 as n tends to $+\infty$ by dominated convergence. Taking (2.2) into account, we get

$$
\widetilde{C} := \sup_{x \in \mathbb{R}^N} \frac{|Q^{1/2}(x)|}{\sqrt{1+|x|^2}} < +\infty \qquad (3.3)
$$

and

$$
\begin{aligned}
\frac{1}{n^2} \int_{\mathbb{R}^N} |Q^{1/2}(x) D\vartheta(x/n)|^2 |u(x)|^2 d\mu = & \frac{1}{n^2} \int_{\frac{n}{2} \le |x| \le n} |Q^{1/2}(x) D\vartheta(x/n)|^2 |u(x)|^2 d\mu \\
\le & \frac{\widetilde{C}^2}{n^2} \| \, |D\vartheta| \, \|^2_\infty \int_{\frac{n}{2} \le |x| \le n} (1+|x|^2) |u(x)|^2 d\mu \\
\le & \widetilde{C}^2 \frac{1+n^2}{n^2} \| \, |D\vartheta| \, \|^2_\infty \int_{|x| \ge \frac{n}{2}} |u(x)|^2 d\mu,
\end{aligned}
$$

which goes to 0 as n tends to $+\infty$. Similarly,

$$
\begin{aligned}
& \frac{2}{n^2} \int_{\mathbb{R}^N} |Q^{1/2}(x) Du(x)|^2 |Q^{1/2}(x) D\vartheta(x/n)|^2 d\mu \\
\le & 2\widetilde{C}^2 \frac{1+n^2}{n^2} \| \, |D\vartheta| \, \|^2_\infty \int_{|x| \ge \frac{n}{2}} |Q^{1/2}(x) Du(x)|^2 d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{n^4} \int_{\mathbb{R}^N} |Q^{1/2}(x) D^2\vartheta(x/n) Q^{1/2}(x)|^2 |u(x)|^2 d\mu \\
\le & \widetilde{C}^4 \frac{(1+n^2)^2}{n^4} \| \, |D^2\vartheta| \, \|^2_\infty \int_{|x| \ge \frac{n}{2}} |u(x)|^2 d\mu,
\end{aligned}
$$

and the right hand sides go to 0 as n tends to $+\infty$. □

The starting point of our estimates is the following lemma.

Lemma 3.2. *Under Hypotheses* 2.1*, for any* $u \in H^2_{\rm loc}(\mathbb{R}^N, dx)$ *and* $v \in H^1_{\rm loc}(\mathbb{R}^N, dx)$*, such that* u *or* v *has compact support, we have*

$$
\int_{\mathbb{R}^N} \mathcal{A}u \, v \, d\mu = -\int_{\mathbb{R}^N} \langle QDu, Dv \rangle d\mu. \qquad (3.4)
$$

Proof. Integrating by parts we get

$$\begin{aligned}\int_{\mathbb{R}^N}\sum_{i,j=1}^N q_{ij}D_{ij}u\,v\,d\mu &= -\int_{\mathbb{R}^N}\sum_{i,j=1}^N D_i(q_{ij}ve^{-\Phi})D_ju\,dx\\ &= -\int_{\mathbb{R}^N}\langle QDu,Dv\rangle\,d\mu-\int_{\mathbb{R}^N}\sum_{i,j=1}^N D_iq_{ij}vD_ju\,d\mu\\ &\quad+\int_{\mathbb{R}^N}\langle QD\Phi,Du\rangle v\,d\mu.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^N}\mathcal{A}u\,vd\mu &= -\int_{\mathbb{R}^N}\langle QDu,Dv\rangle\,d\mu-\int_{\mathbb{R}^N}\sum_{i,j=1}^N D_iq_{ij}vD_ju\,d\mu\\ &\quad+\int_{\mathbb{R}^N}\langle QD\Phi,Du\rangle v\,d\mu+\int_{\mathbb{R}^N}\sum_{j=1}^N b_jD_ju\,v\,d\mu. \qquad (3.5)\end{aligned}$$

By assumption (2.3), the last three terms in the right hand side of (3.5) vanish, and formula (3.4) follows. □

The main result of the paper is the next theorem.

Theorem 3.3. *Under the Hypotheses* 2.1*, the realization A of the operator $\mathcal{A}$ in $L^2(\mu)$ with domain*

$$D(A)=\{u\in H^2_Q(\mu):\ \langle B\cdot,Du\rangle\in L^2(\mu)\},$$

is a dissipative self-adjoint operator in $L^2(\mu)$. For each $u\in D(A)$ and $\lambda>0$, setting $\lambda u-Au=f$, we have

$$\begin{aligned}&(a)\quad \|u\|_{L^2(\mu)}\le\frac{1}{\lambda}\|f\|_{L^2(\mu)},\\ &(b)\quad \|\,|Q^{1/2}Du|\,\|_{L^2(\mu)}\le\frac{1}{\sqrt{\lambda}}\|f\|_{L^2(\mu)},\\ &(c)\quad \|\,|Q^{1/2}D^2u\,Q^{1/2}|\,\|_{L^2(\mu)}\le C(\lambda)\|f\|_{L^2(\mu)},\end{aligned}$$

with $C(\lambda)>0$ independent of u.

Proof. As a first step, we remark that since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1_Q(\mu)$ by Lemma 3.1, then formula (3.4) holds for any u, $v\in D(A)$. It implies immediately that A is symmetric. It also implies that A is dissipative: indeed, if $u\in D(A)$ and $\lambda>0$, then

$$\int_{\mathbb{R}^N}(\lambda u-Au)u\,d\mu=\lambda\int_{\mathbb{R}^N}u^2d\mu-\int_{\mathbb{R}^N}Au\,u\,d\mu=\lambda\int_{\mathbb{R}^N}u^2d\mu+\int_{\mathbb{R}^N}\langle QDu,Du\rangle\,d\mu.$$

Therefore,

$$\lambda\int_{\mathbb{R}^N}u^2d\mu+\int_{\mathbb{R}^N}|Q^{1/2}Du|^2d\mu\le\|\lambda u-Au\|_{L^2(\mu)}\|u\|_{L^2(\mu)},$$

which yields

$$\lambda\int_{\mathbb{R}^N}u^2d\mu\le\|\lambda u-Au\|_{L^2(\mu)}\|u\|_{L^2(\mu)}=\|f\|_{L^2(\mu)}\|u\|_{L^2(\mu)},$$

so that A is dissipative.

The main part of the proof consists in showing that, for any $\lambda > 0$ and any $f \in L^2(\mu)$, the equation

$$\lambda u - \mathcal{A}u = f, \tag{3.6}$$

has a (unique) solution in $D(A)$. This will imply that the resolvent of A is not empty, so that A is self-adjoint.

Let us assume that $f \in C_0^\infty(\mathbb{R}^N)$. We solve the equation

$$\lambda \int_{\mathbb{R}^N} u\, v\, d\mu + \int_{\mathbb{R}^N} \langle QDu, Dv\rangle\, d\mu = \int_{\mathbb{R}^N} f\, v\, d\mu, \qquad v \in H^1_Q(\mu), \tag{3.7}$$

using Lax-Milgram theorem, that gives a unique solution $u \in H^1_Q(\mu)$. Then, u is a distributional solution of $\lambda u - \mathcal{A}u = f$ and, by elliptic regularity, u belongs to $C^3(\mathbb{R}^N)$ and it satisfies $\lambda u - \mathcal{A}u = f$ pointwise. Moreover, choosing $v = u$ in (3.7) gives

$$(i)\ \lambda \|u\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)}, \qquad (ii)\ \int_{\mathbb{R}^N} |Q^{1/2}Du|^2 d\mu \le \|f\|_{L^2(\mu)}\|u\|_{L^2(\mu)} \le \frac{1}{\lambda}\|f\|^2_{L^2(\mu)}. \tag{3.8}$$

To prove that u belongs to $D(A)$, we still have to show that $u \in H^2_Q(\mu)$. To this aim we differentiate (3.6) with respect to any variable x_h, obtaining

$$\lambda D_h u - \mathcal{A}D_h u - \sum_{i,j=1}^N D_h q_{ij} D_{ij} u - \sum_{j=1}^N D_h b_j D_j u = D_h f. \tag{3.9}$$

Next, we fix $n_0 \in \mathbb{N}$ such that the support of f is contained in the ball centered at 0 with radius $n_0/2$. For any $n \ge n_0$ we multiply both sides of (3.9) by $\vartheta_n^2 \sum_{k=1}^N q_{hk} D_k u$, where

$$\vartheta_n(x) = \vartheta(x/n), \;\; x \in \mathbb{R}^N, \; n \in \mathbb{N},$$

and ϑ is as in the proof of Lemma 3.1. Then, we sum with respect to h and integrate in $\mathbb{R}^N$, obtaining

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2}Du|^2 d\mu - \int_{\mathbb{R}^N} \vartheta_n^2 \sum_{h,k=1}^N q_{hk} D_k u\, \mathcal{A}(D_h u) d\mu \\
&- \int_{\mathbb{R}^N} \vartheta_n^2 \sum_{i,j,h,k=1}^N q_{hk} D_h q_{ij} D_k u D_{ij} u\, d\mu - \int_{\mathbb{R}^N} \vartheta_n^2 \sum_{j,h,k=1}^N q_{hk} D_k u D_h b_j D_j u\, d\mu \\
= &\int_{\mathbb{R}^N} \vartheta_n^2 \sum_{h,k=1}^N q_{hk} D_h f D_k u\, d\mu \\
= &\int_{\mathbb{R}^N} \langle QDu, Df\rangle d\mu \\
= &- \int_{\mathbb{R}^N} \mathcal{A}u\, f d\mu,
\end{aligned} \tag{3.10}$$

where we have used formula (3.4) in the last equality. In the left hand side of (3.10) we still have third order derivatives of u, that we eliminate using again formula (3.4) in each integral $\int_{\mathbb{R}^N} \vartheta_n^2\, q_{hk} D_k u \mathcal{A}(D_h u) d\mu$, obtaining

$$\begin{aligned}
&\int_{\mathbb{R}^N} \vartheta_n^2 \sum_{h,k=1}^N q_{hk} D_k u\, \mathcal{A}(D_h u)\, d\mu \\
= &-\sum_{h,k=1}^N \int_{\mathbb{R}^N} \langle QD(\vartheta_n^2 q_{hk} D_k u), D(D_h u)\rangle d\mu \\
= &-\int_{\mathbb{R}^N} \vartheta_n^2 \sum_{i,j,h,k=1}^N q_{ij} D_i q_{hk} D_k u D_{jh} u\, d\mu - \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2} D^2 u Q^{1/2}|^2 d\mu \\
&-2\int_{\mathbb{R}^N} \vartheta_n \sum_{i,j,h,k=1}^N q_{ij} D_i \vartheta_n\, q_{hk} D_k u D_{jh} u\, d\mu,
\end{aligned} \tag{3.11}$$

where the last equality follows from the formula

$$\mathrm{Tr}(QD^2uQD^2u) = \mathrm{Tr}(Q^{1/2}D^2uQ^{1/2}Q^{1/2}D^2uQ^{1/2}) = |Q^{1/2}D^2uQ^{1/2}|^2. \tag{3.12}$$

Combining (3.10) and (3.11) we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2} Du|^2 d\mu + \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2} D^2 u Q^{1/2}|^2 d\mu \\
= &-\int_{\mathbb{R}^N} \mathcal{A}u f d\mu + \int_{\mathbb{R}^N} \vartheta_n^2 \sum_{i,j,h=1}^N (QDu)_h D_h q_{ij} D_{ij} u\, d\mu \\
&-\int_{\mathbb{R}^N} \vartheta_n^2 \sum_{i,h,k=1}^N D_i q_{hk} D_k u (QD^2 u)_{ih} d\mu \\
&+\int_{\mathbb{R}^N} \vartheta_n^2 \sum_{j,h=1}^N (QDu)_h D_h b_j D_j u\, d\mu - 2\int_{\mathbb{R}^N} \vartheta_n \sum_{i,j,h,k=1}^N q_{ij} D_i \vartheta_n\, q_{hk} D_k u D_{jh} u\, d\mu.
\end{aligned} \tag{3.13}$$

Using Hölder inequality and estimate (3.8)(i), we get

$$\int_{\mathbb{R}^N} \mathcal{A}u f d\mu \le \|\mathcal{A}u\|_{L^2(\mu)} \|f\|_{L^2(\mu)} = \|\lambda u - f\|_{L^2(\mu)} \|f\|_{L^2(\mu)} \le 2\|f\|^2_{L^2(\mu)}. \tag{3.14}$$

The second, the third, and the fourth integral in the right hand side of (3.13) are estimated using Hypothesis 2.1(iv). Indeed, assumption (2.4), with $\xi = Du$ and $S = D^2u$, implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \vartheta_n^2 \Big(\sum_{i,j,h=1}^N (QDu)_h D_h q_{ij} D_{ij} u - \sum_{i,h,k=1}^N D_i q_{hk} D_k u (QD^2 u)_{ih} + \sum_{j,h=1}^N (QDu)_h D_h b_j D_j u \Big) d\mu \\
&\le k_1 \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2} Du|^2 d\mu + k_2 \int_{\mathbb{R}^N} \vartheta_n^2 |Q^{1/2} D^2 u Q^{1/2}|^2 d\mu.
\end{aligned} \tag{3.15}$$

In the last integral we have

$$\begin{aligned}
&\Big|\sum_{i,j,h,k=1}^N q_{ij}D_i\vartheta_n\, q_{hk}D_ku D_{jh}u\Big|\\
&= |\langle D^2u(x)\,Q(x)\,D\vartheta_n(x), Q(x)Du(x)\rangle|\\
&\le |Q^{1/2}(x)D\vartheta_n(x)||Q^{1/2}(x)\,D^2u(x)\,Q^{1/2}(x)||Q^{1/2}(x)Du(x)|,
\end{aligned}$$

for any $x\in\mathbb{R}^N$. Note that

$$\sup_{x\in\mathbb{R}^N}|Q^{1/2}(x)D\vartheta_n(x)| = \frac{1}{n}\sup_{\frac{n}{2}\le|x|\le n}|Q^{1/2}(x)D\vartheta(x/n)| \le \widetilde{C}\frac{(1+n^2)^{1/2}}{n}\|\,|D\vartheta|\,\|_\infty,$$

where $\widetilde{C}$ is given by (3.3), so that for every $n\in\mathbb{N}$

$$\sup_{x\in\mathbb{R}^N}|Q^{1/2}(x)D\vartheta_n(x)| \le \sqrt{2}\widetilde{C}\|\,|D\vartheta|\,\|_\infty := C_1. \tag{3.16}$$

Therefore, for any $\varepsilon>0$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\vartheta_n\sum_{i,j,h,k=1}^N q_{ij}D_i\vartheta_n\, q_{hk}D_ku D_{jh}u\,d\mu\Big|\\
&\le \varepsilon\int_{\mathbb{R}^N}\vartheta_n^2|Q^{1/2}D^2u\,Q^{1/2}|^2d\mu + \frac{C_1^2}{4\varepsilon}\int_{\mathbb{R}^N}|Q^{1/2}Du|^2d\mu.
\end{aligned}\tag{3.17}$$

Hence, from (3.13)–(3.17) we get

$$\begin{aligned}
&(1-k_2-2\varepsilon)\int_{\mathbb{R}^N}\vartheta_n^2|Q^{1/2}D^2u\,Q^{1/2}|^2d\mu\\
&\le 2\|f\|^2_{L^2(\mu)} + \frac{C_1^2}{2\varepsilon}\int_{\mathbb{R}^N}|Q^{1/2}Du|^2d\mu + (k_1-\lambda)\int_{\mathbb{R}^N}\vartheta_n^2|Q^{1/2}Du|^2d\mu,
\end{aligned}$$

and, taking (3.8)(ii) into account,

$$\begin{aligned}
&(1-k_2-2\varepsilon)\int_{\mathbb{R}^N}\vartheta_n^2|Q^{1/2}D^2u\,Q^{1/2}|^2d\mu\\
&\le \left(2+\frac{C_1^2}{2\varepsilon\lambda}+\frac{\max\{0,k_1-\lambda\}}{\lambda}\right)\|f\|^2_{L^2(\mu)}.
\end{aligned}$$

Choosing $\varepsilon=(1-k_2)/4$ we get

$$\frac{1-k_2}{2}\int_{\mathbb{R}^N}\vartheta_n^2|Q^{1/2}D^2uQ^{1/2}|^2d\mu \le \left(2+\frac{2C_1^2}{(1-k_2)\lambda}+\frac{\max\{0,k_1-\lambda\}}{\lambda}\right)\|f\|^2_{L^2(\mu)},$$

so that, letting n go to $+\infty$, we see that $u\in H^2_Q(\mu)$, and

$$\|\,|Q^{1/2}D^2u\,Q^{1/2}|\,\|_{L^2(\mu)} \le C(\lambda)\|f\|_{L^2(\mu)}. \tag{3.18}$$

Once we have solved (3.6) for $f\in C_0^\infty(\mathbb{R}^N)$ we solve it for any $f\in L^2(\mu)$ by standard arguments. Fix $f\in L^2(\mu)$ and let $\{f_n\}_{n\in\mathbb{N}}\subset C_0^\infty(\mathbb{R}^N)$ be a sequence converging to f in $L^2(\mu)$. For any $n\in\mathbb{N}$, let $u_n\in D(A)$ be the solution of (3.6) with f_n instead of f. From estimates (3.8) and (3.18) with (u,f) replaced by (u_n-u_m, f_n-f_m), it follows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^2_Q(\mu)$. Hence, u_n converges in $H^2_Q(\mu)$ to some function

$u \in H^2_Q(\mu)$. Then, $u \in D(A)$, it satisfies estimates (3.8) and (3.18), and $\lambda u - Au = f$. Therefore $\lambda \in \rho(A)$ and all the statements are proved. □

The next corollary shows that the domain of A is in fact the maximal domain of $\mathcal{A}$ in $L^2(\mu)$.

Corollary 3.4. *Under Hypotheses* 2.1*, we have*

$$D(A) = \{u \in L^2(\mu) \cap H^2_{\rm loc}(\mathbb{R}^N, dx) : \ \mathcal{A}u \in L^2(\mu)\}.$$

Proof. The inclusion "⊂" is obvious, we have to prove "⊃". Fix $u \in L^2(\mu) \cap H^2_{\rm loc}(\mathbb{R}^N, dx)$, such that $\mathcal{A}u \in L^2(\mu)$, and $\lambda > 0$. Moreover, set $\lambda u - \mathcal{A}u = f$. Then, the difference $v := u - R(\lambda, A)f$ satisfies $\lambda v - \mathcal{A}v = 0$. We shall show that $v \equiv 0$, provided λ is large enough.

Let ϑ_n be the cutoff functions used in the proof of Lemma 3.1 and Theorem 3.3. Integrating the identity $(\lambda v - \mathcal{A}v)v\vartheta_n^2 = 0$ on $\mathbb{R}^N$ we get, through formula (3.4),

$$0 = \lambda \int_{\mathbb{R}^N} v^2\vartheta_n^2 d\mu + \int_{\mathbb{R}^N} |Q^{1/2}Dv|^2\vartheta_n^2 d\mu + 2\int_{\mathbb{R}^N} v\langle Q^{1/2}Dv, Q^{1/2}D\vartheta_n\rangle\vartheta_n d\mu.$$

Recalling (3.16), the modulus of the last integral $\int_{\mathbb{R}^N} v\langle Q^{1/2}Dv, Q^{1/2}D\vartheta_n\rangle\vartheta_n d\mu$ does not exceed

$$C_1\int_{\mathbb{R}^N} |v|\,|Q^{1/2}Dv|\,\vartheta_n\,d\mu \le \frac{C_1}{2\varepsilon}\int_{\mathbb{R}^N} v^2 d\mu + \frac{C_1\varepsilon}{2}\int_{\mathbb{R}^N} |Q^{1/2}Dv|^2\vartheta_n^2 d\mu,$$

for each $\varepsilon > 0$. Choosing $\varepsilon = 1/C_1$ we get

$$0 \ge \lambda\int_{\mathbb{R}^N} v^2\vartheta_n^2 d\mu + \frac{1}{2}\int_{\mathbb{R}^N} |Q^{1/2}Dv|^2\vartheta_n^2 d\mu - C_1^2\int_{\mathbb{R}^N} v^2 d\mu,$$

so that, letting n go to $+\infty$,

$$0 \ge \left(\lambda - C_1^2\right)\int_{\mathbb{R}^N} v^2 d\mu,$$

which implies $v \equiv 0$ if λ is large enough. □

Theorem 3.3 has some immediate consequences.

Corollary 3.5. *Let the Hypotheses* 2.1 *hold. Then:*

(i) *A generates a strongly continuous analytic semigroup of contractions in $L^2(\mu)$;*
(ii) *in the case when $\mu(\mathbb{R}^N) < +\infty$, the constant functions belong to $D(A)$. Then, taking $v \equiv 1$ in* (3.4) *implies that*

$$\int_{\mathbb{R}^N} Af\,d\mu = 0, \qquad f \in D(A).$$

It follows that μ is an invariant measure for $\{T(t)\}$, that is

$$\int_{\mathbb{R}^N} T(t)f\,d\mu = \int_{\mathbb{R}^N} f\,d\mu, \qquad f \in L^2(\mu).$$

4. Consequences and further properties of A

In this section we prove further properties of the semigroup $\{T(t)\}$ and its generator A. In the next proposition we list some straightforward consequences of the results in Section 3.

Proposition 4.1. *The following properties hold.*

(i) *the domain of* $(-A)^{1/2}$ *is* $H^1_Q(\mathbb{R}^N)$. *Therefore, the restriction of* $\{T(t)\}$ *to* $H^1_Q(\mathbb{R}^N)$ *is an analytic semigroup;*

(ii) $\{T(t)\}$ *is a positivity preserving semigroup in* $L^2(\mu)$, *i.e.* $T(t)f \geq 0$ *if* $f \geq 0$ *a.e. Moreover,*
$$\|T(t)f\|_\infty \leq \|f\|_\infty, \qquad f \in L^2(\mu) \cap L^\infty(\mu); \tag{4.1}$$

(iii) $\{T(t)\}$ *is a symmetric Markov semigroup that preserves* $L^1(\mu) \cap L^\infty(\mu)$ *and may be extended from* $L^1(\mu) \cap L^\infty(\mu)$ *to a contraction semigroup* $\{T_p(t)\}$ *on* $L^p(\mu)$ *for all* $p \in [1, +\infty]$, *in such a way that* $T_p(t)f = T_q(t)f$ *if* $f \in L^p(\mu) \cap L^q(\mu)$, *and* $T_2(t) = T(t)$. *Finally,* $\{T_p(t)\}$ *is analytic for any* $p \in (1, +\infty)$.

Proof. (i). $D(-A^{1/2})$ is the closure of $D(A)$ with respect to the norm induced by the inner product
$$\langle u, v\rangle := \int_{\mathbb{R}^N} uv\, d\mu - \int_{\mathbb{R}^N} Au\, v\, d\mu.$$
According to formula (3.4), it coincides with the inner product of $H^1_Q(\mu)$. Since $C_0^\infty(\mathbb{R}^N)$ is contained in $D(A)$ and dense in $H^1_Q(\mu)$, then $D(A)$ is dense in $H^1_Q(\mu)$. Therefore, $H^1_Q(\mu) = D((-A)^{1/2})$.

(ii). We use the Beurling-Deny criteria (see e.g. [5, Theorems 1.3.2, 1.3.3]). To prove that $T(t)$ preserves positivity, it is sufficient to check that if $u \in D((-A)^{1/2})$ then
$$|u| \in D((-A)^{1/2}), \qquad \|(-A)^{1/2}|u|\|_{L^2(\mu)} \leq \|(-A)^{1/2}u\|_{L^2(\mu)}. \tag{4.2}$$
Since the domain of $(-A)^{1/2}$ is contained in $H^1_{\rm loc}(\mathbb{R}^N, dx)$ by (i), then the gradient of $|u|$ is equal to $D(|u|) = \text{sign}(u)\, Du$ for each $u \in D((-A)^{1/2})$. This implies that $|u| \in D((-A)^{1/2})$ and estimate (4.2) follows.

To prove (4.1), it is sufficient to check that, for any nonnegative $u \in D((-A)^{1/2})$, the function $u \wedge 1$ is in $D((-A)^{1/2})$ and
$$\|(-A)^{1/2}(u \wedge 1)\|_{L^2(\mu)} \leq \|(-A)^{1/2}u\|_{L^2(\mu)}. \tag{4.3}$$
Again, since $(-A)^{1/2}$ is contained in $H^1_{\rm loc}(\mathbb{R}^N, dx)$, then the gradient of $u \wedge 1$ is equal to $\chi_{\{u \leq 1\}}\, Du$ for each $u \in D((-A)^{1/2})$. Therefore, $u \wedge 1 \in D((-A)^{1/2})$ and estimate (4.3) is satisfied.

(iii). Statement (ii) implies that $\{T(t)\}$ is a symmetric Markov semigroup, that preserves $L^1(\mu) \cap L^\infty(\mu)$. Then (iii) follows from e.g. [5, Thms. 1.4.1, 1.4.2]. □

Another consequence of the integration by parts formula (3.4) is the following Liouville type theorem.

Proposition 4.2. *Suppose that* $u \in D(A)$ *is such that* $Au = 0$. *Then:*

(i) u *is constant, if* $\mu(\mathbb{R}^N) < +\infty$;

(ii) u *is zero, if* $\mu(\mathbb{R}^N) = +\infty$.

Proof. If $u \in \mathrm{Ker}\,(A)$ then Du vanishes almost everywhere by (3.4). Therefore, u is constant and the statement follows. □

Note that even if $\mu(\mathbb{R}^N) = +\infty$ then 0 may belong to the spectrum of A, as in the case of the Laplacian with the Lebesgue measure. However, if $D(A)$ is compactly embedded in $L^2(\mu)$, then 0 is in the resolvent of A if $\mu(\mathbb{R}^N) = +\infty$, and it is a simple isolated eigenvalue if $\mu(\mathbb{R}^N) < +\infty$.

The compactness of the embedding $D(A) \subset L^2(\mu)$ is a nontrivial question. As the following example (adapted from [9]) shows, in general the embedding is not compact, even when $\mu(\mathbb{R}^N) < +\infty$ and $Q = I$.

Example 4.3. Let $\mathcal{A}$ be defined by

$$(\mathcal{A}u)(x,y) = (\Delta u)(x,y) - \varphi'(x)u_x(x,y) - 2yu_y(x,y), \quad (x,y) \in \mathbb{R}^2,$$

where φ is a smooth convex function such that $\varphi(x) = x$ for $x \geq 0$ and $\varphi(x) = -x$ for $x \leq -1$. The invariant measures associated with the operator $\mathcal{A}$ are given by $\mu(dx,dy) = ce^{-(\varphi(x)+y^2)}dxdy$, c being any positive constant.

Let $\vartheta \in C_0^\infty(\mathbb{R})$ be such that

$$\int_{\mathbb{R}} (\vartheta(y))^2 e^{-y^2}dy = 1$$

and consider the sequence $\{u_n\}_{n\in\mathbb{N}} \in L^2(\mu)$ defined by

$$u_n(x,y) = \frac{x^{n+2}}{\sqrt{(2n+4)!}}\vartheta(y)\chi_{(0,+\infty)}(x), \qquad (x,y) \in \mathbb{R}^2, \quad n \in \mathbb{N}.$$

Then, $u_n \in L^2(\mu) \cap C^2(\mathbb{R})$ and $\|u_n\|_{L^2(\mu)} = 1$, for any $n \in \mathbb{N}$ (if we choose $c = 1$). Moreover, the first and second order derivatives of u_n belong to $L^2(\mu)$ because they are polynomially bounded. As it is easy to check, $\mathcal{A}u_n \in L^2(\mu)$ and its norm is bounded by a positive constant, independent of n. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $D(A)$. Moreover, u_n converges pointwise to 0 as n tends to $+\infty$. Since $\|u_n\|_{L^2(\mu)} = 1$ for any $n \in \mathbb{N}$, it follows that no subsequence of $\{u_n\}_{n\in\mathbb{N}}$ may converge in $L^2(\mu)$.

The following proposition gives a sufficient condition for the embedding of $D(A)$ in $L^2(\mu)$ to be compact.

Proposition 4.4. *Under the Hypotheses* 2.1*, assume that* $q_{ij} \in C^2(\mathbb{R}^N)$ $(i,j = 1,\dots,N)$*,* $\mu(\mathbb{R}^N) < +\infty$ *and*

$$\sum_{i,j=1}^N D_{ij}q_{ij} - \sum_{j=1}^N D_j b_j \leq \alpha|Q^{-1/2}(\mathrm{div}Q - B)|^2 + \beta, \tag{4.4}$$

for some constants $\alpha \in (0,1)$ *and* $\beta > 0$*. Further, suppose that*

$$\lim_{|x|\to+\infty} |Q^{-1/2}(\mathrm{div}Q - B)| = \lim_{|x|\to+\infty} |Q^{1/2}D\Phi| = +\infty. \tag{4.5}$$

Then, $H^1_Q(\mu)$ *is compactly embedded in* $L^2(\mu)$ *and, hence,* $D(A)$ *is compactly embedded in* $L^2(\mu)$*.*

Proof. Let us fix $u \in C_0^\infty(\mathbb{R}^N)$. An integration by parts shows that

$$\int_{\mathbb{R}^N} u^2|Q^{1/2}D\Phi|^2 d\mu = -\int_{\mathbb{R}^N} u^2\langle QD\Phi, De^{-\Phi}\rangle dx$$
$$= 2\int_{\mathbb{R}^N} u\langle QD\Phi, Du\rangle d\mu + \int_{\mathbb{R}^N} u^2 \mathrm{div}(QD\Phi)d\mu.$$

Note that (4.4) is equivalent to

$$\mathrm{div}(QD\Phi) \le \alpha|Q^{1/2}D\Phi|^2 + \beta.$$

Then, we get

$$\int_{\mathbb{R}^N} u^2|Q^{1/2}D\Phi|^2 d\mu \le \alpha\int_{\mathbb{R}^N} u^2|Q^{1/2}D\Phi|^2 d\mu + \beta\int_{\mathbb{R}^N} u^2 d\mu$$
$$+2\left(\int_{\mathbb{R}^N}|Q^{1/2}Du|^2 d\mu\right)^{1/2}\left(\int_{\mathbb{R}^N} u^2|Q^{1/2}D\Phi|^2 d\mu\right)^{1/2}$$
$$\le (\alpha+\varepsilon)\int_{\mathbb{R}^N} u^2|Q^{1/2}D\Phi|^2 d\mu + \max\left\{\beta, \frac{1}{2\varepsilon}\right\}\|u\|^2_{H^1_Q(\mu)}, \quad (4.6)$$

for any $\varepsilon > 0$. Choosing ε such that $1-\alpha-\varepsilon > 0$, from (4.6) we deduce

$$\|\, u|Q^{1/2}D\Phi|\,\|_{L^2(\mu)} \le C\|u\|_{H^1_Q(\mu)}, \quad (4.7)$$

for any $u \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1_Q(\mu)$ by Lemma 3.1, (4.7) holds for any $u \in H^1_Q(\mu)$.

If $\mu(\mathbb{R}^N) < +\infty$, any estimate of the type $\|u\,\varphi\|_{L^2(\mu)} \le C\|u\|_{H^1_Q(\mu)}$ for all $u \in H^1_Q(\mu)$, with a function φ such that $\lim_{|x|\to+\infty}\varphi(x) = +\infty$, yields compactness of the embedding $H^1_Q(\mu) \subset L^2(\mu)$ in a standard way. See for instance the proof of [9, Proposition 3.4]. In our case, we can take $\varphi = |Q^{1/2}D\Phi|$ because of assumption (4.5). □

Examples 4.5.

(i) *Setting for each* $x \in \mathbb{R}^N$

$$Q(x) = (|x|^2+1)I, \qquad B(x) = -\gamma|x|^{\gamma-2}(|x|^2+1)x,$$

with $\gamma > 1$, *all the assumptions of Proposition* 4.4 *are satisfied, and consequently the domain of* A *is compactly embedded in* $L^2(\mu)$. *In this case,*

$$\mu(dx) = c\frac{e^{-|x|^\gamma}}{|x|^2+1}dx,$$

c being an arbitrary positive constant.

(ii) *Taking* Q *and* B *as in* (2.6) *with*

$$U(x,y) = (ax^2+by^2)^\gamma, \qquad (x,y) \in \mathbb{R}^2,$$

where a, b *and* γ *are positive constants with* $\gamma \ge 1$, *it is easy to see that all the assumptions of Proposition* 4.4 *are satisfied. Hence, the domain of* A *is compactly embedded in* $L^2(\mu)$. *The invariant measures are*

$$\mu(dx,dy) = ce^{-(ax^2+by^2)^\gamma}dxdy,$$

c being an arbitrary positive constant.

Remark 4.6.

(i) In Lemma 3.1 we have shown that, under our assumptions, $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1_Q(\mu)$ and in $H^2_Q(\mu)$. However, this is not enough for $C_0^\infty(\mathbb{R}^N)$ to be dense in $D(A)$ with respect to the graph norm. Sufficient conditions for $C_0^\infty(\mathbb{R}^N)$ to be a core for A may be found in the paper [1].

(ii) Our technique works in $L^2(\mu)$ and not in $L^p(\mu)$ with $p \neq 2$. In fact, even in the case $Q = I$ the L^p approach with general $p \in (1, +\infty)$ is different and much heavier than for $p = 2$, see [10].

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