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MAXIMAL DISSIPATIVITY OF A CLASS OF ELLIPTIC DEGENERATE OPERATORS IN WEIGHTED L^2 SPACES

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Abstract. We consider a degenerate elliptic Kolmogorov–type operator arising from second order stochastic differential equations in $\mathbb{R}^n$ perturbed by noise. We study a realization of such an operator in L^2 spaces with respect to an explicit invariant measure, and we prove that it is m-dissipative.

1. Introduction.

We are concerned with a Kolmogorov operator in $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$,

$$K\varphi(x,y) = \frac{1}{2}\,\Delta_x\varphi(x,y) - \langle My + x + D_yU(y), D_x\varphi(x,y)\rangle + \langle x, D_y\varphi(x,y)\rangle, \quad (1)$$

where M is a symmetric positive definite matrix, and $U \in C^1(\mathbb{R}^n,\mathbb{R})$ is a nonnegative function satisfying suitable assumptions. We stress that U and its derivatives may be unbounded, and even grow exponentially as $|y| \to +\infty$.

The operator K arises in the study of the second order stochastic initial value problem in $\mathbb{R}^n$,

$$\begin{cases} Y''(t) = -MY(t) - Y'(t) - DU(Y(t)) + W'(t), \\ Y(0) = y, \quad Y'(0) = x. \end{cases} \quad (2)$$

See [6] for a discussion and several developements. Setting $Y'(t) = X(t)$, problem (2) is equivalent to the system

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} -\mathbb{I} & -M \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} - \begin{pmatrix} DU(Y(t)) \\ 0 \end{pmatrix} + \begin{pmatrix} W'(t) \\ 0 \end{pmatrix}, \\ \\ \qquad\begin{pmatrix} X(0) \\ Y(0) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases} \quad (3)$$

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and K is precisely the Kolmogorov operator in $\mathbb{R}^{2n}$ associated to (3).

In the previous paper [4] we have studied a realization of K in $L^1(\mathbb{R}^{2n},\mu)$, where μ is the infinitesimally invariant measure defined by

$$\mu(dx,dy)=\rho(x,y)dxdy,$$

and

$$\rho(x,y)=\frac{e^{-(\langle My,y\rangle+|x|^2)}e^{-2U(y)}}{\int_{\mathbb{R}^{2n}}e^{-(\langle My',y'\rangle+|x'|^2)}e^{-2U(y')}dx'dy'}:=c\,e^{-(\langle My,y\rangle+|x|^2)}e^{-2U(y)}.$$

Here we study a realization of K in $L^2(\mathbb{R}^{2n},\mu)$. The main result of this paper is that $K: C^2_b(\mathbb{R}^{2n})\mapsto L^2(\mathbb{R}^{2n},\mu)$ is closable, and its closure is m-dissipative. $C^2_b(\mathbb{R}^{2n})$ denotes the space of all bounded and twice continuously differentiable functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ with bounded first and second order derivatives.

Maximal dissipativity of Kolmogorov operators was studied in the last few years by several authors, but most results concern nondegenerate elliptic operators. See for instance the papers [9, 10, 11, 3], the books [5, 2] and the references therein.

Let us explain our method. Our assumptions imply that $\int_{\mathbb{R}^{2n}}|DU|^2d\mu<+\infty$, so that $K\varphi\in L^2(\mathbb{R}^{2n},\mu)$ for every $\varphi\in C^2_b(\mathbb{R}^{2n})$, and integrating by parts we obtain

$$\int_{\mathbb{R}^{2n}}K\varphi\,\varphi\,d\mu=-\frac{1}{2}\int_{\mathbb{R}^{2n}}|D_x\varphi|^2d\mu,\quad \varphi\in C^2_b(\mathbb{R}^{2n}),\tag{4}$$

so that K is dissipative in $L^2(\mathbb{R}^{2n},\mu)$. Consequently it is closable, and we denote by $\overline{K}$ its closure. To prove that $\overline{K}$ is m-dissipative, we have to show that for each $\lambda>0$ the range of $\lambda I-K$ is dense in $L^2(\mathbb{R}^{2n},\mu)$, i.e. we have to solve the resolvent equation

$$\lambda\varphi-K\varphi=f\tag{5}$$

for every f in a dense set in $L^2(\mathbb{R}^{2n},\mu)$. This is not obvious, because K is a degenerate elliptic operator with unbounded coefficients. If the coefficients of K were smooth enough and had bounded derivatives, for smooth f (say, $f\in C^2_b(\mathbb{R}^{2n})$) a solution $u\in C^2_b(\mathbb{R}^{2n})$ would be easily obtained by the classical stochastic characteristics method. Therefore, we assume that U has good bounded approximations $U_\alpha\le U$, $\alpha>0$, and for every $\alpha>0$ we solve

$$\lambda\varphi_\alpha-K_\alpha\varphi_\alpha=f,\tag{6}$$

where $K_\alpha: C^2_b(\mathbb{R}^{2n})\mapsto L^2(\mathbb{R}^{2n},\mu)$ is the operator

$$K_\alpha\varphi(x,y)=\frac{1}{2}\,\Delta_x\varphi(x,y)-\langle My+x+D_yU_\alpha(y),D_x\varphi(x,y)\rangle+\langle x,D_y\varphi(x,y)\rangle.\tag{7}$$

Of course, (6) may be rewritten as

$$\lambda\varphi_\alpha-K\varphi_\alpha=f+\langle D_yU-D_yU_\alpha,D_x\varphi_\alpha\rangle,$$

and our aim is to prove that $\langle D_yU-D_yU_\alpha,D_x\varphi_\alpha\rangle$ goes to 0 in $L^2(\mathbb{R}^{2n},\mu)$ as $\alpha\to 0$. The simplest way to reach this goal is to assume that $|D_yU_\alpha-D_yU|$ goes to 0 in $L^4(\mathbb{R}^{2n},\mu)$, and to prove that the derivatives $D_{x_i}u_\alpha$ are bounded in $L^4(\mathbb{R}^{2n},\mu)$ by a constant independent of α. Since $U_\alpha\le U$, it is enough to prove that

$$\int_{\mathbb{R}^{2n}}(D_{x_i}\varphi_\alpha)^4d\mu_\alpha\le C,\quad i=1,\dots,n,\tag{8}$$

with a constant C independent of α. Here

$$\mu_\alpha(dx,dy)=\rho_\alpha(x,y)dxdy,\tag{9}$$

$$\rho_\alpha(x,y) = \frac{e^{-(\langle My,y\rangle+|x|^2)}e^{-2U_\alpha(y)}}{\int_{\mathbb{R}^{2n}} e^{-(\langle My',y'\rangle+|x'|^2)}e^{-2U_\alpha(y')}dx'dy'} := c_\alpha e^{-(\langle My,y\rangle+|x|^2)}e^{-2U_\alpha(y)}. \tag{10}$$

Note that, even for $n = 1$, estimate (8) cannot follow from L^2 estimates on the second order x-derivatives of φ_α through Sobolev embedding, because our measures μ_α are of Gaussian type, and no Sobolev embeddings are available.

The core of the paper is in fact the proof of estimate (8) for $f \in C_b^3(\mathbb{R}^{2n})$, which is dense in $L^2(\mathbb{R}^{2n},\mu)$. To this aim we need further assumptions on the approximations U_α, that turn out to be assumptions on U. Such assumptions are far from being optimal, however they are satisfied if U is any polynomial such that $\lim_{|y|\to\infty} U(y) = +\infty$, or any positive smooth function such that U and its derivatives up to the third order have polynomial (resp. exponential) growth as $|y| \to +\infty$. See next section.

We note that proving m-dissipativity in $L^2(\mathbb{R}^{2n},\mu)$ instead of in $L^1(\mathbb{R}^{2n},\mu)$, which is the setting considered in [4], is an important starting point for the study of further properties, such as asymptotic behavior as $t \to +\infty$ of the semigroup generated by $\overline{K}$. This study will be the object of a future paper.

2. **The approximating operators.** We denote by $C_b^k(\mathbb{R}^{2n})$ the space of all bounded and k times continuously differentiable functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ with bounded derivatives up to the order k. Moreover we shall use the following notation:

$$|D_x\varphi|^2 = \sum_{k=1}^n |D_{x_k}\varphi|^2, \quad |D_y\varphi|^2 = \sum_{k=1}^n |D_{y_k}\varphi|^2,$$

$$|D_x^2\varphi|^2 = \sum_{k,h=1}^n |D_{x_k x_k}\varphi|^2, \quad |D_y^2\varphi|^2 = \sum_{k,h=1}^n |D_{y_k y_h}\varphi|^2,$$

$$|D_x^3\varphi|^2 = \sum_{k,h,l=1}^n |D_{x_k x_k x_l}\varphi|^2,$$

for functions φ that have the derivatives appearing in the formulas.

Throughout the paper we shall assume that the following conditions are satisfied.

Hypothesis 2.1. *For each $\alpha > 0$ there exists a function $U_\alpha \in C^4(\mathbb{R}^n,\mathbb{R})$ such that*

(i) $0 \le U_\alpha(y) \le U(y)$ *for all* $y \in \mathbb{R}^n$,

(ii) DU_α *has bounded derivatives up to the order* 3,

(iii) $\lim_{\alpha\to 0}\int_{\mathbb{R}^{2n}} (U_\alpha(y) - U(y))^4 d\mu = 0$,

(iv) *there exists $\kappa > 0$ such that for each $\alpha > 0$*

$$\int_{\mathbb{R}^{2n}} \left(|D_y^3U_\alpha|^2 + |D_y^2U_\alpha|^2\,|D_yU_\alpha|^2 + |D_y^2U_\alpha|^2(1+|y|^2)\right) d\mu_\alpha \le \kappa. \tag{11}$$

Here μ_α is defined by (9) – (10).

We notice that Hypothesis 2.1 is fulfilled if U is a C^4 nonnegative function having polynomial or exponential growth together with his derivatives up to the order 3, in the sense that there are m_0, m_1, c_0, $c_1 > 0$, such that for large $|y|$ and for every i, j, $k = 1,\dots,n$ we have

$$\begin{cases} U(y) \ge c_0|y|^{m_0}, \\ \\ |D_iU(y)| + |D_{ij}U(y)| + |D_{ijk}U(y)| \le c_1|y|^{m_1}, \end{cases}$$

or else

$$\begin{cases} U(y) \geq c_0 \exp(m_0|y|), \\ \\ |D_iU(y)| + |D_{ij}U(y)| + |D_{ijk}U(y)| \leq c_1 \exp(m_1|y|). \end{cases}$$

Indeed, in these cases the functions

$$U_\alpha(y) := \frac{U(y)}{1 + \alpha U(y)^m}, \quad y \in \mathbb{R}^n,$$

satisfy Hypothesis 2.1 if m is large enough.

We fix here $\alpha > 0$ and consider the approximating equation (6), where $\lambda > 0$ and $f \in C_b^3(\mathbb{R}^{2n})$ are given.

Proposition 2.2. *Assume that Hypothesis* 2.1 *is fulfilled. Then for arbitrary* $\lambda > 0$ *and* $f \in C_b^3(\mathbb{R}^{2n})$ *there exists a unique solution* $\varphi_\alpha \in C_b^3(\mathbb{R}^{2n})$ *of equation* (6). *Moreover,*

$$\|\varphi_\alpha\|_\infty \leq \frac{1}{\lambda}\, \|f\|_\infty. \tag{12}$$

Proof. To solve equation (6) we shall use the classical stochastic characteristics method. It is based on the solution of the following system of stochastic differential equations in $\mathbb{R}^{2n}$,

$$\begin{cases} dX_\alpha(t) = -[MY_\alpha(t) + X_\alpha(t) + D_yU_\alpha(Y_\alpha(t))]dt + dW(t), \quad t > 0, \\ \\ dY_\alpha(t) = X_\alpha(t)dt, \quad t > 0, \\ \\ X_\alpha(0) = x, \; Y_\alpha(0) = y, \end{cases} \tag{13}$$

where $x,\, y \in \mathbb{R}^n$ and $W(t)$ is a standard Brownian motion in $\mathbb{R}^n$.

By Hypothesis 2.1–(ii), D_yU_α is Lipschitz continuous, so that problem (13) has a unique global solution $(X_\alpha(\cdot, x, y), Y_\alpha(\cdot, x, y))$. Moreover, since D_yU_α has bounded derivatives up to the order 3, in view of a classical result on the dependence of the solution of (13) upon initial data (see e.g. [7, Theorem 1, page 61], [1, Proposition 1.3.3]), it follows that $(X_\alpha(t, x, y), Y_\alpha(t, x, y))$ is thrice continuously differentiable with respect to (x, y), with bounded derivatives up to the third order.

By the Itô formula it follows that the parabolic problem $u_t = Ku$, $t > 0$, $u(0, \cdot) = f$, has $u_\alpha(t, x, y) := \mathbb{E}[f(X_\alpha(t, x, y), Y_\alpha(t, x, y))]$ as a solution, and consequently equation (6) has a solution $\varphi_\alpha \in C_b^3(\mathbb{R}^n)$, given by

$$\varphi_\alpha(x, y) = \int_0^\infty e^{-\lambda t}\mathbb{E}[f(X_\alpha(t, x, y), Y_\alpha(t, x, y))]dt, \quad (x, y) \in \mathbb{R}^{2n}. \tag{14}$$

The classical maximum principle may be easily adapted to elliptic operators with Lipschitz continuous coefficients; for a detailed proof see e.g. [4]. It implies estimate (12) as well as uniqueness of the solution. □

3. **Integral estimates for the solutions of the approximating problems.** Here we fix $\alpha > 0$ and we derive several estimates on the solutions of (6), which will be used in the next section to prove that $\overline{K}$ is m–dissipative in $L^2(\mathbb{R}^{2n}, \mu)$.

The starting point is that Hypothesis 2.1 implies that $K_\alpha\varphi \in L^2(\mathbb{R}^{2n}, \mu_\alpha)$ for each $\varphi \in C_b^2(\mathbb{R}^{2n})$, so that $K_\alpha\varphi \cdot \varphi \in L^2(\mathbb{R}^{2n}, \mu_\alpha)$ and integrating by parts we get

$$\int_{\mathbb{R}^{2n}} K_\alpha\varphi \cdot \varphi \, d\mu_\alpha = -\frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x\varphi|^2 d\mu_\alpha. \tag{15}$$

Proposition 3.1. *For each $\alpha > 0$ we have*

$$\int_{\mathbb{R}^{2n}} \varphi_\alpha^2 d\mu_\alpha \le \frac{1}{\lambda^2} \int_{\mathbb{R}^{2n}} f^2 d\mu_\alpha, \tag{16}$$

$$\int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^2 d\mu_\alpha \le \frac{2}{\lambda} \int_{\mathbb{R}^{2n}} f^2 d\mu_\alpha. \tag{17}$$

Proof. Multiplying both sides of (6) by φ_α, integrating with respect to μ_α over $\mathbb{R}^{2n}$ and taking into account (15) yields

$$\lambda \int_{\mathbb{R}^{2n}} \varphi_\alpha^2 d\mu_\alpha + \frac{1}{2} \int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^2 d\mu_\alpha = \int_{\mathbb{R}^{2n}} f\varphi_\alpha d\mu_\alpha. \tag{18}$$

Consequently,

$$\lambda \int_{\mathbb{R}^{2n}} \varphi_\alpha^2 d\mu_\alpha \le \int_{\mathbb{R}^{2n}} f\varphi_\alpha d\mu_\alpha \le \left(\int_{\mathbb{R}^{2n}} f^2 d\mu_\alpha\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} \varphi_\alpha^2 d\mu_\alpha\right)^{1/2},$$

and (16) follows. Similarly, (18) and (16) imply

$$\frac{1}{2} \int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^2 d\mu_\alpha \le \left(\int_{\mathbb{R}^{2n}} f^2 d\mu_\alpha\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} \varphi_\alpha^2 d\mu_\alpha\right)^{1/2} \le \frac{1}{\lambda} \int_{\mathbb{R}^{2n}} f^2 d\mu_\alpha,$$

which yields (17). □

Proposition 3.2. *There exists a constant $c_1 = c_1(\lambda, \|f\|_\infty, \| |Df| \|_\infty, \kappa)$ such that*

$$\int_{\mathbb{R}^{2n}} |D_y\varphi_\alpha|^2 d\mu_\alpha \le c_1. \tag{19}$$

Proof. Denote by $m_{ij} = \langle Me_j, e_i\rangle = \langle Me_i, e_j\rangle$ the entries of the matrix M, so that $\langle My, D_x\varphi_\alpha\rangle = \sum_{i,j=1}^n y_i m_{ij} D_{x_j}\varphi_\alpha$. Differentiating (6) with respect to y_i yields

$$\lambda D_{y_i}\varphi_\alpha - K_\alpha D_{y_i}\varphi_\alpha + \sum_{j=1}^n m_{ij} D_{x_j}\varphi_\alpha + \langle D_{y_i} D_y U(y), D_x\varphi_\alpha\rangle = D_{y_i} f.$$

Multiplying both sides by $D_{y_i}\varphi_\alpha$, integrating with respect to μ_α over $\mathbb{R}^{2n}$, taking into account (15) and summing up yields

$$\begin{aligned} \int_{\mathbb{R}^{2n}} \langle D_y f, D_y\varphi_\alpha\rangle d\mu_\alpha = \; & \lambda \int_{\mathbb{R}^{2n}} |D_y\varphi_\alpha|^2 d\mu_\alpha + \frac{1}{2} \sum_{i,k=1}^n \int_{\mathbb{R}^{2n}} (D_{x_k y_i}\varphi_\alpha)^2 d\mu_\alpha \\ & + \int_{\mathbb{R}^{2n}} \langle MD_x\varphi_\alpha, D_y\varphi_\alpha\rangle d\mu_\alpha \\ & + \textstyle\sum_{i,k=1}^n \int_{\mathbb{R}^{2n}} (D_{y_i y_k} U_\alpha)(D_{x_k}\varphi_\alpha)(D_{y_i}\varphi_\alpha) d\mu_\alpha. \end{aligned} \tag{20}$$

Fixed any $i, k = 1, \dots n$, let us estimate the integral

$$I_{ik} := \int_{\mathbb{R}^{2n}} (D_{y_i y_k} U_\alpha)\,(D_{x_k}\varphi_\alpha)\,(D_{y_i}\varphi_\alpha) d\mu_\alpha = \int_{\mathbb{R}^{2n}} (D_{y_i y_k} U_\alpha)(D_{x_k}\varphi_\alpha)(D_{y_i}\varphi_\alpha)\rho_\alpha dx\, dy.$$

Integrating by parts with respect to y_i, we obtain

$$\begin{aligned}
I_{ik} \quad = \quad & -\int_{\mathbb{R}^{2n}} (D_{y_i y_i y_k} U_\alpha)\,(D_{x_k}\varphi_\alpha)\,\varphi_\alpha\ \rho_\alpha\ dx\,dy \\
& -\int_{\mathbb{R}^{2n}} (D_{y_i y_k} U_\alpha)\,(D_{y_i x_k}\varphi_\alpha)\,\varphi_\alpha\ \rho_\alpha\ dx\,dy \\
& +\int_{\mathbb{R}^{2n}} (D_{y_i y_k} U_\alpha)\,(D_{x_k}\varphi_\alpha)\,\varphi_\alpha \Big(2D_{y_i}U_\alpha + 2\sum_{i,j=1}^n m_{ij}y_j\Big)\rho_\alpha\ dx\,dy.
\end{aligned}$$

Taking into account (12) and using the Hölder inequality, we find

$$\begin{aligned}
& |I_{ik}| \\
= \quad & \frac{1}{\lambda}\|f\|_\infty \Big(\int_{\mathbb{R}^{2n}} (D_{y_i y_i y_k}U_\alpha)^2 d\mu_\alpha\Big)^{1/2} \Big(\int_{\mathbb{R}^{2n}} (D_{x_k}\varphi_\alpha)^2 d\mu_\alpha\Big)^{1/2} \\
& + \frac{1}{\lambda}\|f\|_\infty \Big(\int_{\mathbb{R}^{2n}} (D_{y_i y_k}U_\alpha)^2 d\mu_\alpha\Big)^{1/2} \Big(\int_{\mathbb{R}^{2n}} (D_{y_i x_k}\varphi_\alpha)^2 d\mu_\alpha\Big)^{1/2} \\
& + \frac{2}{\lambda}\|f\|_\infty \Big(\int_{\mathbb{R}^{2n}} (D_{y_i y_k}U_\alpha)^2\ (D_{y_i}U_\alpha)^2 d\mu_\alpha\Big)^{1/2} \Big(\int_{\mathbb{R}^{2n}} (D_{x_k}\varphi_\alpha)^2 d\mu_\alpha\Big)^{1/2} \\
& + \frac{2\|M\|_{L(\mathbb{R}^n)}}{\lambda}\|f\|_\infty \Big(\int_{\mathbb{R}^{2n}} (D_{y_i y_k}U_\alpha)^2\ |y|^2 d\mu_\alpha\Big)^{1/2} \Big(\int_{\mathbb{R}^{2n}} (D_{x_k}\varphi_\alpha)^2 d\mu_\alpha\Big)^{1/2}.
\end{aligned}$$

By Hypothesis 2.1 and Proposition 3.1 we get

$$\begin{aligned}
|I_{ik}| \quad \le \quad & \frac{3\sqrt{2} + 2\sqrt{2}\|M\|_{L(\mathbb{R}^n)}}{\lambda^{3/2}}\,\kappa^{1/2}\|f\|_\infty^2 \\
& + \frac{1}{\lambda}\|f\|_\infty \Big(\frac{1}{2\varepsilon}\kappa + \frac{\varepsilon}{2}\int_{\mathbb{R}^{2n}} (D_{y_i x_k}\varphi_\alpha)^2 d\mu_\alpha\Big),
\end{aligned} \tag{21}$$

for any $\varepsilon > 0$. Coming back to (20) we get

$$\begin{aligned}
& \lambda \|\,|D_y\varphi_\alpha|\,\|^2_{L^2(\mu_\alpha)} + \frac{1}{2}\sum_{i,k=1}^n \int_{\mathbb{R}^{2n}} (D_{y_i x_k}\varphi_\alpha)^2 d\mu_\alpha \\
\le \quad & (\|\,|D_y f|\,\|_{L^2(\mu_\alpha)} + \|\,|MD_x\varphi_\alpha|\,\|_{L^2(\mu_\alpha)})\|\,|D_y\varphi_\alpha|\,\|_{L^2(\mu_\alpha)} + \textstyle\sum_{i,k=1}^n |I_{ik}| \\
\le \quad & \frac{1}{2\lambda}(\|\,|D_y f|\,\|_{L^2(\mu_\alpha)} + \|\,|MD_x\varphi_\alpha|\,\|_{L^2(\mu_\alpha)})^2 \\
& + \tfrac{\lambda}{2}\|\,|D_y\varphi_\alpha|\,\|^2_{L^2(\mu_\alpha)} + \textstyle\sum_{i,k=1}^n |I_{ik}|,
\end{aligned}$$

and the statement follows using estimate (21) with $\varepsilon = \lambda/(2\|f\|_\infty)$, if $f \not\equiv 0$. If $f \equiv 0$ then $\varphi_\alpha \equiv 0$, and the statement is obvious. □

Corollary 3.3. *There exists a constant* $c_2 = c_2(\lambda, \|f\|_\infty, \|\,|Df|\,\|_\infty, \kappa)$ *such that*

$$\int_{\mathbb{R}^{2n}} |D_x^2\varphi_\alpha|^2 d\mu_\alpha \le c_2. \tag{22}$$

Proof. Differentiating (6) with respect to x_i yields

$$\lambda D_{x_i}\varphi_\alpha - K_\alpha D_{x_i}\varphi_\alpha - D_{y_i}\varphi_\alpha + D_{x_i}\varphi_\alpha = D_{x_i}f.$$

Multiplying both sides by $D_{y_i}\varphi_\alpha$, integrating with respect to μ_α over $\mathbb{R}^{2n}$, taking into account (15) and summing up yields

$$(\lambda+1)\int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^2 d\mu_\alpha + \frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x^2\varphi_\alpha|^2 d\mu_\alpha$$

$$= \int_{\mathbb{R}^{2n}} \langle D_y\varphi_\alpha + D_x f, D_x\varphi_\alpha\rangle d\mu_\alpha. \tag{23}$$

The conclusion follows using (19). □

Proposition 3.4. *There exists a constant* $c_3 = c_3(\lambda, \|f\|_\infty, \| |Df| \|_\infty, \kappa)$ *such that*

$$\int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^4 d\mu_\alpha \le c_3. \tag{24}$$

Proof. For $i = 1, \ldots, n$ let us estimate

$$\int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^4 d\mu_\alpha = \int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^3 D_{x_i}\varphi_\alpha d\mu_\alpha.$$

Integrating by parts with respect to x_i yields

$$\int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^4 d\mu_\alpha = -3\int_{\mathbb{R}^{2n}} \varphi_\alpha (D_{x_i}\varphi_\alpha)^2 D_{x_ix_i}\varphi_\alpha d\mu_\alpha + 2\int_{\mathbb{R}^{2n}} \varphi_\alpha (D_{x_i}\varphi_\alpha)^3 x_i d\mu_\alpha$$

$$:= I_1 + I_2.$$

Now, using (12) and the Hölder inequality, we obtain

$$|I_1| \le \frac{3}{\lambda}\|f\|_\infty \left(\int_{\mathbb{R}^{2n}} (D^2_{x_i}\varphi_\alpha)^4 d\mu_\alpha\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} (D_{x_ix_i}\varphi_\alpha)^2 d\mu_\alpha\right)^{1/2}$$

and

$$|I_2| \le \frac{2}{\lambda}\|f\|_\infty \left(\int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^4 d\mu_\alpha\right)^{3/4} \left(\int_{\mathbb{R}^{2n}} x_i^4 d\mu_\alpha\right)^{1/4}.$$

Consequently, setting

$$p: = \int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^4 d\mu_\alpha$$

and using (22) we obtain

$$p \le \frac{1}{\lambda}\|f\|_\infty \left(3p^{1/2}c_2^{1/2} + 2p^{3/4}\left(\int_{\mathbb{R}^{2n}} x_i^4 d\mu_\alpha\right)^{1/4}\right),$$

which yields

$$\int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^4 d\mu_\alpha \le C \tag{25}$$

where $C = C(\lambda, \|f\|_\infty, \| |Df| \|_\infty, \kappa)$.

Now using the Hölder inequality we obtain, for all $i \ne j$, $i, j = 1, ..., n$,

$$\int_{\mathbb{R}^{2n}} (D_{x_i}\varphi_\alpha)^2 (D_{x_j}\varphi_\alpha)^2 d\mu_\alpha \le C, \tag{26}$$

and the statement follows. □

4. m–**dissipativity.** We prove here the main result of the paper.

Theorem 4.1. *Assume that Hypothesis* (2.1) *is fulfilled. Then the closure* $\overline{K}$ *of the operator* $K : C_b^2(\mathbb{R}^{2n}) \mapsto L^2(\mathbb{R}^{2n}, \mu)$ *is* m*–dissipative.*

Proof. Let $\lambda > 0$, $f \in C_b^3(\mathbb{R}^{2n})$ and let φ_α be the solution of (6). Then

$$\lambda\varphi_\alpha - K\varphi_\alpha = f + \langle D_yU(y) - D_yU_\alpha(y), D_x\varphi_\alpha\rangle.$$

We claim that

$$\lim_{\alpha\to 0}\langle D_yU(y) - D_yU_\alpha(y), D_x\varphi_\alpha\rangle = 0 \quad \text{in } L^2(\mathbb{R}^{2n}, \mu). \tag{27}$$

In fact, taking into account Proposition 3.4 and using the Hölder inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} \langle D_yU(y) - D_yU_\alpha(y), D_x\varphi_\alpha\rangle^2 d\mu \\
\le\ & \left(\int_{\mathbb{R}^{2n}} |D_yU(y) - D_yU_\alpha(y)|^4 d\mu\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^4 d\mu\right)^{1/2} \\
\le\ & \left(\int_{\mathbb{R}^{2n}} |D_yU(y) - D_yU_\alpha(y)|^4 d\mu\right)^{1/2} \left(\frac{c}{c_\alpha}\int_{\mathbb{R}^{2n}} |D_x\varphi_\alpha|^4 d\mu_\alpha\right)^{1/2} \\
\le\ & \left(\frac{c\,c_3}{c_\alpha}\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} |D_yU(y) - D_yU_\alpha(y)|^4 d\mu\right)^{1/2}.
\end{aligned}$$

Now the claim follows from Hypothesis 2.1–(iii), recalling that $1/c_\alpha$ is bounded by a constant independent of α.

Since $C_b^3(\mathbb{R}^{2n})$ is dense in $L^2(\mathbb{R}^{2n}, \mu)$, (27) implies that the range of $\lambda I - K$ is dense in $L^2(\mathbb{R}^{2n}, \mu)$ and the statement of the theorem follows from the Lumer–Phillips Theorem. □

5. **Concluding remarks.** First of all, formula (4) is easily extendable to the whole domain of $\overline{K}$.

Proposition 5.1. *For every* $\varphi \in D(\overline{K})$ *and* $i = 1, \ldots, n$ *there exist the weak derivatives* $D_{x_i}\varphi \in L^2(\mathbb{R}^{2n}, \mu)$, *and*

$$\int_{\mathbb{R}^{2n}} \overline{K}\varphi\, \varphi\, d\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x\varphi|^2 d\mu. \tag{28}$$

Proof. If $\varphi \in C_b^2(\mathbb{R}^{2n})$, formula (28) coincides with (4). Now, let $\varphi \in D(\overline{K})$. This means that there is a sequence $\{\varphi_k\} \subset C_b^2(\mathbb{R}^{2n})$ such that

$$\varphi_k \to \varphi, \quad K\varphi_k \to \overline{K}\varphi \quad \text{in } L^2(\mathbb{R}^{2n}, \mu).$$

(4) implies that

$$\int_{\mathbb{R}^{2n}} |D_x(\varphi_k - \varphi_h)|^2 d\mu \le 2\int_{\mathbb{R}^{2n}} |K(\varphi_k - \varphi_h)|\,|\varphi_k - \varphi_h|\, d\mu,$$

so that $\{D_{x_i}\varphi_k\}$ is a Cauchy sequence in $L^2(\mathbb{R}^{2n}, \mu)$, and the conclusion follows. □

Note that K is not symmetric: for any φ, $\psi \in C_b^2(\mathbb{R}^{2n})$ we have

$$\int_{\mathbb{R}^{2n}} K\varphi\,\psi\,d\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} \langle D_x\varphi, D_x\psi\rangle d\mu + \frac{1}{2}\int_{\mathbb{R}^{2n}} (\langle D_y\varphi, D_x\psi\rangle - \langle D_x\varphi, D_y\psi\rangle) d\mu$$

In particular, taking $\psi \equiv 1$ we obtain

$$\int_{\mathbb{R}^{2n}} K\varphi\,d\mu = 0$$

for each $\varphi \in C_b^2(\mathbb{R}^{2n})$, and since $C_b^2(\mathbb{R}^{2n})$ is dense in the domain of $\overline{K}$,

$$\int_{\mathbb{R}^{2n}} \overline{K}\varphi\,d\mu = 0, \quad \varphi \in D(\overline{K}).$$

In its turn, this implies that, denoting by $T(t)$ the semigroup generated by $\overline{K}$,

$$\int_{\mathbb{R}^{2n}} T(t)\varphi\,d\mu = \int_{\mathbb{R}^{2n}} \varphi\,d\mu, \quad \varphi \in L^2(\mathbb{R}^{2n}, \mu),\ t > 0,$$

i.e. μ is an invariant measure for $T(t)$.

Open problems. Several natural questions about $\overline{K}$, $T(t)$ and μ arise now. A first one is about the regularity properties of the functions in $D(\overline{K})$. Do their second order derivatives $D_{x_ix_j}\varphi$ exist and belong to $L^2(\mathbb{R}^{2n}, \mu)$? A related question is about the smoothing properties of $T(t)$. In the case $U \equiv 0$ we have the nice representation formula

$$(T(t)\varphi)(z) = \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^{2n}} e^{-\langle Q_t^{-1}\xi, \xi\rangle/2}\varphi(e^{tB}z - \xi)d\xi, \quad t > 0,\ z \in \mathbb{R}^{2n}, \tag{29}$$

where Q_t is the matrix

$$Q_t = \int_0^t e^{sB}Qe^{sB^*}ds, \quad 0 < t < +\infty, \tag{30}$$

and

$$B = \begin{pmatrix} -\mathbb{I} & -M \\ \mathbb{I} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}.$$

Using (29) it is not hard to see that $T(t)$ maps $L^2(\mathbb{R}^{2n}, \mu)$ into $C^\infty(\mathbb{R}^{2n})$ for $t > 0$. But a similar formula is not available for general U.

A third interesting question is whether the kernel of $\overline{K}$ consist of constant functions. As well known, this is equivalent to ergodicity of μ with respect to $T(t)$. Formula (28) implies immediately that every φ in the kernel of $\overline{K}$ depends only on the variables y. If φ were weakly differentiable, we would obtain that $\langle x, D_y\varphi(y)\rangle = 0$ for almost all x, $y \in \mathbb{R}^n$, so that $\varphi \equiv$ constant. But regularity of φ is not obvious, and we cannot conclude.

Last, but not least: under which conditions the domain of $\overline{K}$ is compactly embedded in $L^2(\mathbb{R}^{2n}, \mu)$? For nondegenerate Kolmogorov–type elliptic operators, under reasonable assumptions the domain is continuously embedded in $H^1(\mathbb{R}^{2n}, \mu)$, the space of the weakly differentiable functions with derivatives in $L^2(\mathbb{R}^{2n}, \mu)$, which is compactly embedded in $L^2(\mathbb{R}^{2n}, \mu)$ because logarithmic Sobolev inequalities hold. See e.g. [3]. But in our case the embedding $D(\overline{K}) \subset H^1(\mathbb{R}^{2n}, \mu)$ is out of reach.

We hope to be able to answer (a part of) these questions in a future paper.

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