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On a class of self-adjoint elliptic operators in L^2 spaces with respect to invariant measures

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Abstract

We consider the operator $\mathcal{A}u = \Delta u/2 - \langle DU, Du\rangle$, where U is a convex real function defined in a convex open set $\Omega \subset \mathbb{R}^N$ and $\lim_{|x|\to\infty} U(x) = +\infty$. Setting $\mu(dx) = \exp(-2U(x))\,dx$, we prove that the realization of $\mathcal{A}$ in $L^2(\Omega,\mu)$ with domain $\{u \in H^2(\Omega,\mu)\colon\ \langle DU, Du\rangle \in L^2(\Omega,\mu),\ \partial u/\partial n = 0$ at $\Gamma_1\}$, is a self-adjoint dissipative operator. Here Γ_1 is the set of points y in the boundary of Ω such that $\limsup_{x\to y} U(x) < +\infty$. Then we discuss several properties of $\mathcal{A}$ and of the measure μ, including Poincaré and log-Sobolev inequalities in $H^1(\Omega,\mu)$.

MSC: 35J25; 37L40

Keywords: Elliptic operators; Unbounded coefficients; Invariant measures

1. Introduction

In this paper we study differential operators $\mathcal{A}$ of the type

$$(\mathcal{A}u)(x) = \frac{1}{2}\Delta u(x) - \big\langle DU(x), Du(x)\big\rangle, \quad x \in \Omega,$$

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where $\Omega \subset \mathbb{R}^N$ is a (possibly unbounded) convex open set, and $U : \Omega \mapsto \mathbb{R}$ is a convex function such that $\lim_{|x|\to\infty} U(x) = +\infty$.

The symbol D denotes the gradient; since U is convex then DU exists almost everywhere in Ω and it is locally bounded. If it is globally bounded, the operator $\mathcal{A}$ belongs to a class of operators that has been widely studied in the last fifty years, and several results of existence, uniqueness, and properties of the solutions to $\lambda u - \mathcal{A}u = f$ are available. On the contrary, if DU is unbounded there are not many results in the literature. In this case, it is clear that the realizations of $\mathcal{A}$ in the usual L^p spaces with respect to the Lebesgue measure dx do not enjoy nice properties. For instance, in the simplest situation $\Omega = \mathbb{R}^N$ and $p = 2$, it is possible to show that the domain of the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N, dx)$ is contained in $H^2(\mathbb{R}^N, dx)$ only under very restrictive assumptions, for example when DU is globally Lipschitz continuous, see e.g. [16,17].

If DU is unbounded, natural settings for the operator $\mathcal{A}$ are suitably weighted spaces. The best weight is $\rho(x) = e^{-2U(x)}$, for several reasons. First, as it is easy to see,

$$\int_\Omega \mathcal{A}uve^{-2U}\,dx = \int_\Omega \mathcal{A}vue^{-2U}\,dx = -\int_\Omega \langle Du, Dv\rangle e^{-2U}\,dx, \quad u, v \in C_0^\infty(\Omega),$$

so that $\mathcal{A}$ is associated to a nice Dirichlet form and it is formally self-adjoint in the space $L^2(\Omega, e^{-2U(x)}dx)$. Note that if an operator $\mathcal{B}$ of the type

$$(\mathcal{B}u)(x) = \frac{1}{2}\Delta u(x) - \big\langle F(x), Du(x)\big\rangle$$

with measurable F, and a weight $\rho \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfy

$$\int_\Omega \mathcal{B}uv\rho\,dx = \int_\Omega \mathcal{B}vu\rho\,dx, \quad u, v \in C_0^\infty(\Omega),$$

then necessarily F is the gradient of a function U and $\rho(x) = \mathrm{const.}\exp(-2U(x))$. So, we set

$$\mu(dx) = \bigg(\int_\Omega e^{-2U(y)}\,dy\bigg)^{-1} e^{-2U(x)}\,dx,$$

where the normalization constant $(\int_\Omega e^{-2U(y)}\,dy)^{-1}$ lets μ be a probability measure. In this paper we show that in fact a realization A of $\mathcal{A}$ in $L^2(\Omega, \mu)$, with suitable domain $D(A)$, is self-adjoint and dissipative. The domain $D(A)$ consists of the functions u in $H^2(\Omega, \mu)$ such that $\langle DU, Du\rangle$ is in $L^2(\Omega, \mu)$, and that satisfy suitable boundary conditions. The boundary conditions that let A to be self-adjoint are obvious—either Dirichlet or Neumann—if U has good behavior near the boundary Γ of Ω, in particular if it has a convex real valued (hence, continuous) extension to the whole $\mathbb{R}^N$. This is the situation considered in the papers [6] with Neumann boundary condition and [12] with Dirichlet boundary condition.

If U is not regular near Γ, several problems may arise. If U is unbounded near a subset $\Gamma' \subset \Gamma$ with positive $(N-1)$-dimensional measure, the traces on Γ' of the functions in $H^1(\Omega, \mu)$ or in $H^2(\Omega, \mu)$ are not necessarily well defined, so that the Neumann and the Dirichlet boundary conditions may not be meaningful in Γ'.

We split the boundary Γ in three parts: Γ_1, consisting of the points $y \in \Gamma$ such that $\limsup_{x\to y} U(x) < +\infty$, Γ_2, consisting of the points $y \in \Gamma$ such that $\liminf_{x\to y} U(x) \in \mathbb{R}$ and $\limsup_{x\to y} U(x) = +\infty$, and Γ_∞, consisting of the points $y \in \Gamma$ such that $\lim_{x\to y} U(x) = +\infty$. We prove that if U is not too crazy, specifically if the $(N-1)$-dimensional measure of Γ_2 and of the relative boundary of Γ_∞ in Γ is zero, then A is self-adjoint and dissipative if we assign the Neumann boundary condition $\partial u/\partial n = 0$ at Γ_1 and no boundary condition at Γ_∞ to the functions of $D(A)$. In particular, if $\lim_{x\to y} U(x) = +\infty$ at each $y \in \Gamma$, we have $\Gamma = \Gamma_\infty$ and no boundary condition is needed. A partial result in this direction may be found in [7]. But here we allow that both Γ_1 and Γ_∞ have positive $(N-1)$-dimensional measure.

For instance, if Ω is the half ball $\{x = (x_1, \dots, x_N): |x| < 1,\ x_N > 0\}$, $\mathcal{A}u(x) = \frac{1}{2}(\Delta u(x) - \alpha(1-|x|^2)^{-1}\langle x, Du(x)\rangle)$ for some $\alpha > 0$, the associated measure is $\mu(dx) = c(\alpha)(1-|x|^2)^{\alpha/2}\,dx$, and

$$D(A) = \bigl\{u \in H^2(\Omega,\mu):\ x \mapsto \bigl\langle x, Du(x)\bigr\rangle\bigl(1-|x|^2\bigr)^{-1} \in L^2(\Omega,\mu),\ \partial u/\partial x_N(x',0) = 0\bigr\}.$$

Then the operator $A : D(A) \mapsto L^2(\Omega,\mu)$ is self-adjoint and dissipative. It is known that the traces at $|x| = 1$ of the functions in $H^1(\Omega,\mu)$ are well defined iff $\alpha < 1$, hence the normal derivative at $|x| = 1$ is well defined for all functions in $H^2(\Omega,\mu)$ iff $\alpha < 1$. In any case, we need to have no boundary condition at $|x| = 1$ if we want A to be self-adjoint.

Together with the boundary conditions, the other important feature of the functions in $D(A)$ is their regularity, and the degree of summability of their derivatives. A classical approach to the study of $\mathcal{A}$ is to consider the associated bilinear form,

$$a(u,v) = \frac{1}{2}\int_\Omega \bigl\langle Du(x), Dv(x)\bigr\rangle e^{-2U(x)}\,dx, \quad u,v \in H^1(\Omega,\mu).$$

a is continuous in $H^1(\Omega,\mu)$, and $a(u,u) + \lambda\|u\|^2_{L^2(\Omega,\mu)} \geqslant \text{const.}\|u\|^2_{H^1(\Omega,\mu)}$ for $\lambda > 0$, so that there exists a nonpositive self-adjoint operator $A_0 : D(A_0) \mapsto L^2(\Omega,\mu)$, such that $\langle A_0 u, v\rangle = -a(u,v)$ for each $v \in L^2(\Omega,\mu)$. The domain of A_0 consists of the functions $u \in H^1(\Omega,\mu)$ such that $\int_\Omega \langle Du(x), Dv(x)\rangle\mu(dx) \leqslant C\|v\|_{L^2(\Omega,\mu)}$ for some $C > 0$ and for every $v \in H^1(\Omega,\mu)$, and its characterization as a subset of $H^2(\Omega,\mu)$ needs further steps that necessarily go beyond the theory of the quadratic forms in Hilbert spaces.

So, our main theorem may be seen as an optimal L^2 regularity result for the solution to the elliptic equation

$$\lambda u - \mathcal{A}u = f, \quad x \in \Omega,$$

satisfying the specified boundary condition.

The starting point of our analysis is the integration by parts formula

$$\int_\Omega (\mathcal{A}u)(x)v(x)e^{-2U(x)}\,dx = -\frac{1}{2}\int_\Omega \bigl\langle Du(x), Dv(x)\bigr\rangle e^{-2U(x)}\,dx + \frac{1}{2}\int_{\Gamma_1} \frac{\partial u}{\partial n}(x)v(x)e^{-2U(x)}\,d\sigma_x,$$

that holds for each $u \in H^2(\Omega,\mu)$ such that $\langle DU, Du\rangle \in L^2(\Omega,\mu)$, and for each $v \in H^1(\Omega,\mu)$. (For such functions, the right-hand side is shown to make sense, which is not a priori obvious.) This is the most technical part of the paper.

The integration formula implies immediately that A is symmetric, and that for each $\lambda > 0$ the unique solution $u \in D(A)$ of $\lambda u - Au = 0$ is $u \equiv 0$. It implies immediately that A is dissipative, too: if $\lambda > 0$ and $\lambda u - Au = f$ we multiply both sides by u, we integrate, and we get $\lambda \|u\|^2 \leqslant \|fu\|$, so that $\|u\| \leqslant \|f\|/\lambda$.

To prove that A is self-adjoint we show that for each $\lambda > 0$ and for each $f \in L^2(\Omega, \mu)$ the resolvent equation $\lambda u - Au = f$ has in fact a solution. Existence of a solution to $\lambda u - \mathcal{A}u = f$ may be shown in several ways if $\Omega = \mathbb{R}^N$, or if U has a continuous extension up to the boundary Γ. For instance, one may approximate an unbounded Ω by a sequence of bounded open sets Ω_n, solve the equation in Ω_n with Neumann boundary condition and then use the classical interior estimates and estimates up to the boundary for elliptic equations in bounded domains to find a solution of the original problem. This has been done in some papers about elliptic and parabolic operators with unbounded regular coefficients, such as [2,3]. However, if U has bad behavior near the boundary this method is of no help, because classical estimates near the boundary are missing, even if Ω is bounded and $\partial\Omega$ is smooth. In this paper we follow the approach of [6], that has the advantage to work without any regularity assumptions except convexity.

In any case, what is less obvious is the estimate of the second order derivatives of u in $L^2(\Omega, \mu)$. It is here that the convexity of U plays an essential role. Let us explain why, just by formal arguments.

If U and f are smooth, then any solution to $\lambda u - \mathcal{A}u = f$ is smooth in Ω, and every first order derivative $D_i u$ satisfies

$$\lambda D_i u - \mathcal{A}(D_i u) + \sum_{j=1}^{N} D_{ij} U \cdot D_j u = D_i f,$$

so that, multiplying both sides by $D_i u$ and summing up,

$$\lambda |Du|^2 - \sum_{i=1}^{N} \mathcal{A}(D_i u) \cdot D_i u + \langle D^2 U \cdot Du, Du \rangle = \langle Df, Du \rangle.$$

Now we integrate over Ω, using twice the integration formula: first, to integrate each product $\mathcal{A}(D_i u) \cdot D_i u$, and second, to integrate $\langle Df, Du \rangle$. We get

$$\lambda \| |Du| \|^2 + \frac{1}{2} \| |D^2 u| \|^2 + \int_{\Omega} \langle D^2 U \cdot Du, Du \rangle \mu(dx) = -2 \int_{\Omega} \mathcal{A}u \cdot f \mu(dx) + \text{boundary integrals}.$$

If the boundary integrals vanish (for instance, if Ω is the whole $\mathbb{R}^N$) or if they are negative we get

$$\| |D^2 u| \|^2 \leqslant 4 \|f\| \cdot \|\mathcal{A}u\| = 4 \|f\| \cdot \|\lambda u - f\|,$$

because $D^2 U$ is nonnegative definite at each x, and we are done: recalling that $\|u\| \leqslant \|f\|/\lambda$ we obtain $\| |D^2 u| \| \leqslant 2\sqrt{2} \|f\|$. Note that the constant is universal, i.e. it is independent of U.

To make this procedure rigorous, even for less regular data, we introduce the Moreau–Yosida approximations of U,

$$U_\alpha(x) = \inf\left\{U(y) + \frac{1}{2\alpha}|x-y|^2\colon\ y \in \Omega\right\}, \quad x \in \mathbb{R}^N,\ \alpha > 0,$$

and the approximating problems

$$\lambda u_\alpha - A_\alpha u_\alpha = \tilde{f}, \quad x \in \mathbb{R}^N,$$

where $\tilde{f}$ is the extension of f to the whole $\mathbb{R}^N$ that vanishes outside Ω, and

$$A_\alpha : D(A_\alpha) := H^2\big(\mathbb{R}^N, \mu_\alpha\big) \mapsto L^2\big(\mathbb{R}^N, \mu_\alpha\big), \quad A_\alpha = \frac{1}{2}\Delta - \langle DU_\alpha, D\cdot\rangle,$$

$$\mu_\alpha(dx) = \left(\int_{\mathbb{R}^N} e^{-2U_\alpha(y)}dy\right)^{-1} e^{-2U_\alpha(x)}\,dx.$$

The approximating problems are much easier than our original problem, because we work in the whole $\mathbb{R}^N$, and because each DU_α is globally Lipschitz continuous. Indeed, each U_α is the Moreau–Yosida approximation of the function that coincides with U in Ω, with $\liminf_{y\to x} U(y)$ at each $x \in \partial\Omega$, and with $+\infty$ in $\mathbb{R}^N \setminus \Omega$, which is convex and lower semicontinuous in the whole $\mathbb{R}^N$: hence U_α is convex in the whole $\mathbb{R}^N$, it is differentiable at each point, and it has globally Lipschitz continuous gradient. Then it is not hard to solve uniquely the approximating problems, to justify the above procedure in order to get a bound independent of α for the $H^2(\mathbb{R}^N, \mu_\alpha)$-norm of the solutions, and to find a solution to the equation $\lambda u - \mathcal{A}u = f$ in Ω as the limit of a subsequence $u_{\alpha_n|\Omega}$. That u satisfy the boundary condition $\partial u/\partial n = 0$ at Γ_1 is less obvious, but still it is proved as a consequence of the integration formula. Again, this should not be surprising, because the gradients $DU_\alpha(x)$ behave like $n(x)/\alpha$ at any $x \in \Gamma_1$, so that our approach is a sort of penalization method for the Neumann boundary condition.

The main interest of this method is that it works under very general assumptions; however we remark that it seems to be new even for bounded smooth Ω and for regular U.

The basic integration formula is proved in Section 2. Section 3 contains the results that we need about operators with Lipschitz continuous coefficients defined in the whole $\mathbb{R}^N$. In Section 4 we prove that the resolvent set of A contains $(0, +\infty)$ and we estimate the norm of $R(\lambda, A)f$ and of its first and second order derivatives in $L^2(\Omega, \mu)$, for any $f \in L^2(\Omega, \mu)$. At the end of the paper, in Section 5, we describe several properties of A, of the semigroup $T(t)$ generated by A, and of the measure μ. In particular we prove that μ is an invariant measure for $T(t)$, i.e.

$$\int_\Omega T(t)f\,\mu(dx) = \int_\Omega f\,\mu(dx), \quad f \in L^2(\Omega, \mu),\ t > 0,$$

and we discuss Poincaré and log-Sobolev inequalities in $H^1(\Omega, \mu)$.

2. Notation and preliminaries. The integration formula

Let Ω be an open convex set in $\mathbb{R}^N$, with boundary Γ.

Let $U:\Omega \mapsto \mathbb{R}$ be a convex function, and extend U to a lower semicontinuous convex function (still denoted by U) with values in $\mathbb{R} \cup \{+\infty\}$, setting

$$U(x) = \begin{cases} \liminf_{y\to x} U(x), & x \in \Gamma, \\ +\infty, & x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{1}$$

Set moreover

$$\Gamma_\infty = \Big\{ y \in \Gamma \colon \lim_{x\to y} U(x) = +\infty \Big\}.$$

Since U is convex, $U(x) \neq -\infty$ for each $x \in \Gamma$. Therefore, $\Gamma \setminus \Gamma_\infty = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \Big\{ y \in \Gamma \colon \liminf_{x\to y} U(x),\ \limsup_{x\to y} U(x) \in \mathbb{R} \Big\},$$

$$\Gamma_2 = \Big\{ y \in \Gamma \colon \liminf_{x\to y} U(x) \in \mathbb{R},\ \limsup_{x\to y} U(x) = +\infty \Big\}.$$

Note that Γ_1 is relatively open in Γ. In what follows we shall need that Γ_2 and the relative boundary of Γ_∞ in Γ be negligible, i.e. with zero $(N-1)$-dimensional measure. This is not satisfied in general, as the following counterexample shows.

Example 2.1. *Let Ω be the open unit disk in $\mathbb{R}^2$. For each $\alpha < 2\pi$ there exists a convex function $f:\Omega \mapsto \mathbb{R}$ such that the measure of Γ_2 is bigger than α.*

Proof. For $i \in \mathbb{N}$, let $a_i, b_i > 0$ and let e_i be unit vectors in $\mathbb{R}^2$. Define the halfplanes $H_i = \{x \in \mathbb{R}^2 \colon \langle x, e_i\rangle > a_i\}$ and the open sets $\Omega_i = H_i \cap \Omega$.

We choose the numbers a_i and the vectors e_i in such a way that $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$ (consequently, the diameter of Ω_i goes to 0 as $i \to \infty$), and that $\bigcup_{i\in\mathbb{N}} H_i \cap \partial\Omega$ is a dense open set in $\partial\Omega$, and the 1d measure of its complement, a Cantor-like set, is larger than α. Moreover, we choose b_i in such a way that $\sup_{x\in\Omega_i} b_i(\langle x, e_i\rangle - a_i) = +\infty$.

Define the functions

$$f_0(x) = 0, \qquad f_i(x) = b_i\big(\langle x, e_i\rangle - a_i\big), \quad i \in \mathbb{N},\ x \in \Omega,$$

and

$$f(x) = \sup_{i\in\mathbb{N}\cup\{0\}} f_i(x), \quad x \in \Omega.$$

Being the supremum of a family of affine functions, f is convex. Moreover, f coincides with f_i on Ω_i for each $i \in \mathbb{N}$. Let $C = \partial\Omega \setminus (\bigcup_{i\in\mathbb{N}} H_i)$. Then we have

$$C = \Big\{ x \in \partial\Omega \colon x = \lim_{i\to+\infty} x_i,\ x_i \in \Omega_i \Big\},$$

because $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$, so that $f_j(x) \leqslant 0$ in Ω_i. For each $y \in C$ we have $\liminf_{x\to y} f(x) = 0$ and $\limsup_{x\to y} f(x) = +\infty$, because $\inf_{x\in\Omega_i} f_i(x) = 0$ for each i, $\sup_{x\in\Omega_i} f_i(x)$ goes to $+\infty$ and the diameter of Ω_i goes to 0 as $i \to \infty$. Therefore, $C \subset \Gamma_2$ (in fact, $C = \Gamma_2$), and the measure of Γ_2 is larger than α. □

Example 2.1 shows that the relative boundary of Γ_∞ may have positive $(N-1)$-dimensional measure. Take for instance $\Omega = B(0,1) \setminus \bigcup_{i\in\mathbb{N}} \overline{\Omega_i}$, and $U(x) = (1-|x|^2)^{-1}$. Then $\Gamma_\infty = C$ coincides with its relative boundary in Γ, and it has positive 1d measure. (We note however that Γ is not piecewise C^2.)

However, if the boundary of Ω is flat or piecewise flat, the above situations cannot occur, as the next lemma shows.

Lemma 2.2. *If Ω is either a halfspace, or a polyhedral set, the $(N-1)$-dimensional measure of Γ_2 is zero. The $(N-1)$-dimensional measure of the relative boundary of Γ_∞ in Γ is zero.*

Proof. Let us point out a general property of convex hulls: if $Co(I)$ is the convex hull of a set I, for each x in the interior part of $Co(I)$ there is a finite number of points in I such that x belongs to the interior part of the convex hull of such points. Indeed, each x in the interior part of $Co(I)$ is the center of a closed hypercube centered at x and contained in $Co(I)$, such a hypercube is the convex envelope of its edges. In their turn, the edges belong to the convex hull of a finite number of points of I.

Without loss of generality, we may assume that Ω is contained in the halfspace $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N\colon x_N > 0\}$ and that Γ_2 is contained in $\mathbb{R}^{N-1} \times \{0\}$.

Let $(x', 0)$, with $x' \in \mathbb{R}^{N-1}$, be in the interior part (in $\mathbb{R}^{N-1}$) of Γ_2. Then

$$\lim_{y\to 0^+} U(x', y) = \liminf_{x\to(x',0)} U(x)$$

so that there exists $\delta > 0$ such that the restriction of U to the segment $x' \times (0,\delta)$ is bounded from above. Since U is convex, the restriction of U to the convex hull of a finite number of such segments is bounded from above. In particular, if $P_1, \dots, P_k$ are in Γ_2, the restriction of U to the convex hull of the segments $P_i \times (0,\delta)$, with $\delta = \min\{\delta_i\colon i = 1,\dots,k\}$, i.e., the restriction of U to $Co(\{P_i\colon i = 1,\dots,k\}) \times (0,\delta)$, is bounded from above. Therefore the interior part (in $\mathbb{R}^{N-1}$) of $Co(\{P_i\colon i = 1,\dots,k\})$ does not intersect Γ_2.

Let us consider now the convex hull of Γ_2. If its interior part (in $\mathbb{R}^{N-1}$) is empty, since it is convex then it is contained in a subspace with dimension $\leqslant N-2$ so that it has null $(N-1)$-dimensional measure, and Γ_2 too has null $(N-1)$-dimensional measure. If its interior is not empty, each point in the interior belongs also to the interior of the convex hull of a finite number of points in Γ_2, and therefore its interior does not intersect Γ_2. Consequently, Γ_2 is contained in the relative boundary of $Co(\Gamma_2)$ in $\mathbb{R}^{N-1}$. Since $Co(\Gamma_2)$ is convex, its boundary is a locally Lipschitz continuous surface, and it has null $(N-1)$-dimensional measure. This shows the first part of the statement.

To prove the second claim, we still may assume that Ω is contained in $\mathbb{R}^N_+$ and that Γ_∞ is contained in $\mathbb{R}^{N-1} \times \{0\}$. We remark that the set $\{x_0 \in \overline{\Omega}\colon \liminf_{x\to x_0} U(x) < \infty\}$ is convex, so that its intersection with $\mathbb{R}^{N-1} \times \{0\}$ is convex. The relative boundary of $\Gamma_1 \cup \Gamma_2$ (which coincides with the relative boundary of Γ_∞) in $\mathbb{R}^{N-1} \times \{0\}$ is a locally Lipschitz continuous surface and it has null measure. □

From now on we shall assume that

$$\lim_{|x|\to\infty} U(x) = +\infty. \tag{2}$$

The following lemma is easily proved.

Lemma 2.3. *There are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geqslant a + b|x|$ for each $x \in \mathbb{R}^N$.*

We set as usual $e^{-\infty} = 0$. The function

$$x \mapsto e^{-2U(x)}, \quad x \in \mathbb{R}^N,$$

is upper semicontinuous on the whole $\mathbb{R}^N$, it is continuous and positive in Ω, and it vanishes outside $\overline{\Omega}$. Lemma 2.3 implies that its restriction to Ω is in $L^1(\Omega)$. Therefore, the probability measure

$$\mu(dx) = \left(\int_\Omega e^{-2U(x)}\, dx \right)^{-1} e^{-2U(x)}\, dx \tag{3}$$

is well defined in $\mathbb{R}^N$, and it has Ω as support. Thus, we can identify $L^2(\mathbb{R}^N, \mu)$ and $L^2(\Omega, \mu)$.

The spaces $H^1(\Omega, \mu)$ and $H^2(\Omega, \mu)$, consist of the functions $u \in H^1_{\mathrm{loc}}(\Omega)$ (respectively, $u \in H^2_{\mathrm{loc}}(\Omega)$) such that u and its first (respectively, first and second) order derivatives are in $L^2(\Omega, \mu)$. They are endowed with their natural norms.

Note that, although $L^2(\mathbb{R}^N, \mu)$ and $L^2(\Omega, \mu)$ are equivalent spaces, the same is not true in general for $H^1(\mathbb{R}^N, \mu)$ and $H^1(\Omega, \mu)$.

Lemma 2.4. *The following statements hold true.*

(i) *$C_0^\infty(\Omega)$ is dense in $L^2(\Omega, \mu)$.*
(ii) *The bounded functions in $H^1(\Omega, \mu)$ with compact support are dense in $H^1(\Omega, \mu)$.*
(iii) *If in addition $|DU| \in L^2(\Omega, \mu)$, the bounded functions in $H^1(\Omega, \mu)$ that have compact support with positive distance from $\Gamma_\infty \cup \Gamma_2$ are dense in $H^1(\Omega, \mu)$.*

Proof. Let $\theta_n : \mathbb{R} \mapsto \mathbb{R}$ be a sequence of smooth functions such that $0 \leqslant \theta_n(y) \leqslant 1$ for each y, $\theta_n \equiv 1$ for $y \leqslant n$, $\theta_n \equiv 0$ for $y \geqslant 2n$, and such that

$$\left|\theta_n'(y)\right| \leqslant \frac{C}{n}, \quad y \in \mathbb{R}.$$

For each $u \in L^2(\Omega, \mu)$ set

$$u_n(x) = u(x)\theta_n\big(|x|\big)\theta_n\big(U(x)\big), \quad x \in \Omega,\ u_n(x) = 0,\ x \notin \Omega. \tag{4}$$

Then u_n has compact support, and $u_n \to u$ in $L^2(\Omega, \mu)$. Indeed,

$$\int_\Omega |u_n - u|^2 \mu(dx) \leqslant \int_{\{x\in\Omega:\ U(x)\geqslant n\}} |u|^2 \mu(dx) + \int_{\{x\in\Omega:\ |x|\geqslant n\}} |u|^2 \mu(dx)$$

which goes to 0 as $n \to \infty$. In its turn, since the support of u_n is contained in the closure of the bounded open set $\Omega_n := \{x \in \Omega:\ U(x) < 2n\}$, u_n may be approximated in $L^2(\Omega_n, dx)$ by a sequence of $C_0^\infty(\Omega)$ functions obtained by convolution with smooth mollifiers. Since $\exp(-2n) \leqslant \exp(-2U(x)) \leqslant \exp(-2\inf U)$ on Ω_n, the Lebesgue measure is equivalent to μ on Ω_n, and such a sequence approximates u_n also in $L^2(\Omega, \mu)$. Statement (i) follows.

Let now $u \in H^1(\Omega, \mu)$ and for $\varepsilon > 0$ set

$$u_\varepsilon(x) = \frac{u(x)}{1 + \varepsilon u(x)^2}.$$

Then

$$\int_\Omega |u - u_\varepsilon|^2 \mu(dx) = \int_\Omega u^2 \left(1 - \frac{1}{1 + \varepsilon u^2}\right)^2 \mu(dx)$$

goes to 0 as $\varepsilon \to 0$, and

$$Du_\varepsilon = \frac{Du}{1 + \varepsilon u^2} - \frac{2\varepsilon u^2 Du}{(1 + \varepsilon u^2)^2}$$

so that $|Du - Du_\varepsilon|$ goes to 0 in $L^2(\Omega, \mu)$ as well. So, u is approximated by a sequence of bounded H^1 functions. Each bounded H^1 function v is approximated in its turn by the sequence

$$v_n(x) = v(x)\theta_n\big(|x|\big)$$

where θ_n are as above, and statement (ii) is proved.

Assume now that $|DU| \in L^2(\Omega, \mu)$. Let $u \in H^1(\Omega, \mu) \cap L^\infty(\Omega)$ have compact support, and set

$$u_n(x) = u(x)\theta_n\big(U(x)\big),$$

with θ_n as above. The functions u_n belong to $H^1(\Omega, \mu) \cap L^\infty(\Omega)$, their supports are compact and have positive distance from $\Gamma_\infty \cup \Gamma_2$, because U is bounded there. We already know that u_n goes to u in $L^2(\Omega, \mu)$. Concerning the first order derivatives, for almost each x in Ω we have

$$Du_n(x) = Du(x)\theta_n\big(U(x)\big) + u(x)\theta_n'\big(U(x)\big)DU(x),$$

where $|Du|\theta_n(U)$ goes to $|Du|$ in $L^2(\Omega, \mu)$, and $u\theta_n'(U)DU$ goes to 0 in $L^2(\Omega, \mu)$ as $n \to \infty$ because $u \in L^\infty$, $|DU| \in L^2(\Omega, \mu)$ and $\|\theta_n'\|_\infty \leqslant C/n$. The last statement follows. □

We remark that in general $C_0^\infty(\Omega)$ is not dense in $H^1(\Omega, \mu)$, even if $\Gamma = \Gamma_\infty$. For instance, if $\Omega = (-1, 1)$ and $U(x) = \alpha \log(1 - |x|)/2$, $\mu(dx) = (1 - |x|)^\alpha\, dx$, then $C_0^\infty(-1, 1)$ is dense in $H^1((-1, 1), \mu)$ iff $\alpha \geqslant 1$, see [18, Theorem 3.6.1]. (In the case $\alpha > 1$ the first order derivatives of U are in $L^2((-1, 1), \mu)$, and the density of $C_0^\infty(-1, 1)$ in $H^1((-1, 1), \mu)$ may be seen also as a consequence of Lemma 2.4.)

We remark that the trace on Γ_1 of any function in $H^1(\Omega, \mu)$ is well defined. Indeed, if $x_0 \in \Gamma_1$, U is bounded from above in $\Omega \cap B(x_0, r)$ for some $r > 0$, e^{-2U} is bounded and far away from

0 in $\Omega \cap B(x_0, r)$, so that $H^1(\Omega \cap B(x_0, r), \mu) = H^1(\Omega \cap B(x_0, r), dx)$ with equivalence of the respective norms. Since Γ_1 is a locally Lipschitz continuous surface (see e.g. [8, Corollary 1.1.2.3]), the traces are well defined and they belong to $H^{1/2}_{\text{loc}}(\Gamma_1, d\sigma)$, where $d\sigma$ is the Lebesgue surface measure on Γ. See e.g. [8, Theorem 1.5.1.3]. Consequently, if $u \in H^2(\Omega, \mu)$ the traces of its first order derivatives belong to $H^{1/2}_{\text{loc}}(\Gamma_1, d\sigma)$; since the exterior normal vector field $n(x)$ is measurable and bounded then the normal derivative $\partial u/\partial n$ belongs to $L^2_{\text{loc}}(\Gamma_1, d\sigma)$. So, we may define

$$\begin{cases} D(A) = \{u \in H^2(\Omega, \mu)\colon \langle DU, Du\rangle \in L^2(\Omega, \mu),\, \partial u/\partial n = 0 \text{ at } \Gamma_1\}, \\ (Au)(x) = \mathcal{A}u(x), \quad x \in \Omega. \end{cases} \tag{5}$$

From now on we shall assume that

$$\Gamma_2 \textit{ and the relative boundary of } \Gamma_\infty \textit{ are negligible in } \Gamma. \tag{6}$$

The first important step in our analysis are the following integration formulas.

Theorem 2.5. *Let U be a convex function satisfying assumptions* (2) *and* (6). *For each $u \in H^2(\Omega, \mu)$ such that $\langle DU, Du\rangle \in L^2(\Omega, \mu)$, the function z defined by*

$$z(x) = \frac{\partial u}{\partial n}(x)e^{-U(x)}, \quad \text{if } x \in \Gamma_1, \qquad z(x) = 0, \quad \text{if } x \in \Gamma \setminus \Gamma_1,$$

belongs to $L^2(\Gamma, d\sigma)$, and $\|z\|_{L^2(\Gamma,d\sigma)} \leqslant C(\|u\|_{H^2(\Omega,\mu)} + \|\mathcal{A}u\|_{L^2(\Omega,\mu)})$. Moreover, for each $\psi \in H^1(\Omega, \mu) \cap L^\infty(\Omega)$ with compact support we have

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \big\langle Du(x), D\psi(x)\big\rangle \mu(dx) + \frac{1}{2}\bigg(\int_\Omega e^{-2U(x)}\,dx\bigg)^{-1} \int_{\Gamma_1} \frac{\partial u}{\partial n}(x)\psi(x)e^{-2U(x)}\,d\sigma_x. \tag{7}$$

Proof. The main point is the proof of (7). The first claim is in fact a part of the proof of (7). If the support of ψ is contained in Ω, or if its intersection with Γ is contained in Γ_1, formula (7) is obvious. If the support has nonempty intersection with $\Gamma \setminus \Gamma_1$, things are more difficult.

Let $r > 0$ be such that for each $x_0 \in \Gamma$, $\Gamma \cap B(x_0, r)$ is the graph of a Lipschitz continuous convex function g, defined in a convex open set in $\mathbb{R}^{N-1}$ and with values in $\mathbb{R}$. Without loss of generality we may assume that g is a function of the first $N-1$ variables, defined in $\Omega'_r := \{x' \in \mathbb{R}^{N-1}\colon \exists x_N \in \mathbb{R},\ (x', x_N) \in \Omega \cap B(x_0, r)\}$. Still without loss of generality we may assume that the translated graphs $(\textit{graph } g) + \delta e_N$ $(e_N = (0, \ldots, 0, 1))$ are contained in Ω for δ small, say $0 < \delta < r$.

For $\delta \in (0, r)$ set $\Omega_\delta = \{x = (x', x_N) \in \Omega \cap B(x_0, r)\colon x_N > g(x') + \delta\}$.

If $u \in H^2(\Omega, \mu)$ is such that $\langle DU, Du\rangle \in L^2(\Omega, \mu)$, and $\psi \in H^1(\Omega, \mu)$ has support contained in $B(x_0, r)$, we have

$$\int_{\Omega} (\mathcal{A}u)(x)\psi(x)\mu(dx) = \lim_{\delta\to 0}\int_{\Omega_\delta} (\mathcal{A}u)(x)\psi(x)\mu(dx),$$

$$\int_{\Omega} \langle Du(x), D\psi(x)\rangle\mu(dx) = \lim_{\delta\to 0}\int_{\Omega_\delta} \langle Du(x), D\psi(x)\rangle\mu(dx),$$

and for each $\delta\in(0,r)$, denoting by n_δ the normal exterior vector field to $\partial\Omega_\delta$, we have

$$\begin{aligned}
&\int_{\Omega_\delta} (\mathcal{A}u)(x)\psi(x)e^{-2U(x)}\,dx + \frac{1}{2}\int_{\Omega_\delta}\langle Du(x), D\psi(x)\rangle e^{-2U(x)}\,dx \\
&\quad= \frac{1}{2}\int_{\partial\Omega_\delta} \frac{\partial u}{\partial n_\delta}(x)\psi(x)e^{-2U(x)}\,d\sigma_x \\
&\quad= \frac{1}{2}\int_{\Omega'_r}\left(\left\langle D_{x'}u\big(x', g(x')+\delta\big), Dg(x')\right\rangle - \frac{\partial u}{\partial x_N}\big(x', g(x')+\delta\big)\right) \\
&\qquad\times \psi\big(x', g(x')+\delta\big)e^{-2U(x', g(x')+\delta)}\,dx'.
\end{aligned} \tag{8}$$

See [8, Theorem 1.5.3.1]. The integrals over Ω_δ converge to the respective integrals over the whole Ω. By difference, the last integral has finite limit as $\delta\to 0$.

Now we choose as ψ an extension of $\frac{\partial u}{\partial n}(x', g(x'))\sqrt{1+|Dg(x')|^2}$ multiplied by a cutoff function, precisely $\psi = u_{\mathcal{N}}\chi$ where

$$u_{\mathcal{N}}(x) = \left\langle D_{x'}u(x), Dg(x')\right\rangle - \frac{\partial u}{\partial x_N}(x),$$

and $\chi\in C_0^\infty(\mathbb{R}^N)$, $\chi\equiv 1$ in $B(x_0, 3r/4)$, $\chi\equiv 0$ outside $B(x_0,r)$, $0\leqslant\chi(x)\leqslant 1$ for each x. The extension $u_{\mathcal{N}}$ enjoys the following property: if $x=(x', g(x')+\delta)\in\partial\Omega_\delta$, then $u_{\mathcal{N}}(x) = \partial u/\partial n_\delta(x)\sqrt{1+|Dg(x')|^2}$. Using formula (8) we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega'_r}\left(\langle D_{x'}u, Dg\rangle - \frac{\partial u}{\partial x_N}\big(x', g(x')+\delta\big)\right)^2\chi\big(x', g(x')+\delta\big)e^{-2U(x', g(x')+\delta)}\,dx' \\
&\quad\leqslant \|\mathcal{A}u\|_{L^2}^2\|u_{\mathcal{N}}\|_{L^2}^2 + \frac{1}{2}\big\||Du|\big\|_{L^2}^2\big\||Du_{\mathcal{N}}|\big\|_{L^2}^2 \leqslant C\|u\|_{H^1}^2\big(\|\mathcal{A}u\|_{L^2}^2 + \|u\|_{H^2}^2\big),
\end{aligned} \tag{9}$$

where $L^2 = L^2(\Omega\cap B(x_0,r), e^{-2U(x)}\,dx)$, $H^k = H^k(\Omega\cap B(x_0,r), e^{-2U(x)}\,dx)$, $k=0,1$. Set $\Omega'_{r/2} = \{x'\in\mathbb{R}^{N-1}\colon \exists x_N\in\mathbb{R}, (x', x_N)\in\Omega\cap B(x_0, r/2)\}$. If $\delta < r/4$, the point $(x', g(x')+\delta)$ belongs to $B(x_0, 3r/4)$ for each $x'\in\Omega'_{r/2}$, and since $\chi\equiv 1$ in $B(x_0, 3r/4)$, then

$$\int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x')+\delta\big)^2 e^{-2U(x', g(x')+\delta)} \leqslant 2C\|u\|_{H^1}^2\big(\|\mathcal{A}u\|_{L^2}^2 + \|u\|_{H^2}^2\big) := \mathcal{K}, \tag{10}$$

which is independent of δ. A sequence $(u_{\mathcal{N}}(x', g(x') + \delta_n)e^{-U(x', g(x')+\delta_n)})_n \in \mathbb{N}$ converges weakly to a limit function $v \in L^2(\Omega'_{r/2}, dx')$. The norm of v still satisfies

$$\|v\|^2_{L^2(\Omega'_{r/2}, dx')} \leqslant \mathcal{K},$$

where $\mathcal{K}$ is the constant in (10). We shall prove that

$$v(x') = \begin{cases} u_{\mathcal{N}}(x', g(x'))e^{-U(x', g(x'))} = \frac{\partial u}{\partial n}(x', g(x'))\sqrt{1 + |Dg(x')|^2}e^{-U(x', g(x'))}, \\ \quad \text{if } (x', g(x')) \in \Gamma_1, \\ 0, \quad \text{if } (x', g(x')) \in \Gamma_\infty, \end{cases} \tag{11}$$

showing that for each $\phi \in C_0^\infty(\Omega'_{r/2})$ with support in $\Gamma'_1 := \{x' \in \Omega'_{r/2}\colon (x', g(x')) \in \Gamma_1\}$ we have

$$\lim_{\delta \to 0} \int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\phi(x')e^{-U(x', g(x')+\delta)}\, dx' = \int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x')\big)\phi(x')e^{-U(x', g(x'))}\, dx', \tag{12}$$

and for each $\phi \in C_0^\infty(\Omega'_{r/2})$ with support in $\Gamma'_\infty := \{x' \in \Omega'_{r/2}\colon (x', g(x')) \in \Gamma_\infty\}$ we have

$$\lim_{\delta \to 0} \int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\phi(x')e^{-U(x', g(x')+\delta)}\, dx' = 0. \tag{13}$$

The verification of (12) and of (13) is postponed to the end of the proof. Once (11) is established, the first claim in the statement follows. This is because $v \in L^2(\Omega'_{r/2}, dx')$ implies that $z \in L^2(\Gamma \cap B(x_0, r/2))$; since $\|v\|^2_{L^2(\Omega'_{r/2}, dx')} \leqslant \mathcal{K}$, then $\|z\|_{L^2(\Gamma \cap B(x_0, r/2))} \leqslant \text{const} \times \mathcal{K}$. Covering Γ by a sequence (or by a finite number, if Ω is bounded) of balls $B(x_0, r/2)$, such that the distance between any two centers is greater than a fixed $\delta_0 > 0$, and summing up, we find $z \in L^2(\Gamma)$, $\|z\|_{L^2(\Gamma)} \leqslant C(\|u\|_{H^2(\Omega,\mu)} + \|\mathcal{A}u\|_{L^2(\Omega,\mu)})$.

We come back to the limit

$$\lim_{\delta \to 0} \int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\psi\big(x', g(x') + \delta\big)e^{-2U(x', g(x')+\delta)}\, dx'$$

where ψ is any function in $H^1(\Omega, \mu) \cap L^\infty(\Omega)$ with support contained in the ball $B(x_0, r/2)$. We shall show that the above limit is equal to

$$\int_{\Omega'_{r/2}} v(x')\psi\big(x', g(x')\big)e^{-U(x', g(x'))}\, dx' = \int_{\Gamma_1 \cap B(x_0, r/2)} \frac{\partial u}{\partial n}(x)\psi(x)e^{-2U(x)}\, d\sigma_x, \tag{14}$$

where v is the function defined in (11). Note that the integral is meaningful because v is in $L^2(\Omega'_{r/2}, dx')$ and $\psi(\cdot, g(\cdot))e^{-U(\cdot,g(\cdot))}$ is bounded.

Once (14) is proved the statement follows, since any $\psi \in H^1(\Omega, \mu) \cap L^\infty(\Omega)$ with compact support may be written as a finite sum $\psi = \sum_{k=0}^m \psi_k$ where $\psi_k \in H^1(\Omega, \mu) \cap L^\infty(\Omega)$, the support of ψ_0 is contained in Ω and the supports of ψ_k, $k = 1, \dots, m$ are contained in balls $B(x_k, r/2)$, $x_k \in \Gamma$.

To prove (14), we split the integral over the region $\Omega'_{r/2}$ as the sum of the integral over Γ'_∞, the integral over a region with small measure, and the integral over a region where $e^{-2U(x',g(x')+\delta)}$ is bounded away from zero by a constant independent of δ.

Concerning the integral over Γ'_∞, we have

$$\left| \int_{\Gamma'_\infty} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\psi\big(x', g(x' + \delta)\big)e^{-U(x',g(x')+\delta)}\, dx' \right|$$

$$\leqslant \bigg(\int_{\Gamma'_\infty} \big(u_{\mathcal{N}}\big(x', g(x') + \delta\big)\big)^2 e^{-2U(x',g(x')+\delta)}\, dx' \bigg)^{1/2} \bigg(\int_{\Gamma'_\infty} e^{-2U(x',g(x')+\delta)}\, dx' \bigg)^{1/2} \|\psi\|_\infty$$

$$\leqslant \mathcal{K}\bigg(\int_{\Gamma'_\infty} e^{-2U(x',g(x')+\delta)}\, dx' \bigg)^{1/2} \|\psi\|_\infty,$$

where $\mathcal{K}$ is the constant in (10). For each in $x' \in \Gamma'_\infty$, $e^{-2U(x',g(x')+\delta)}$ goes to zero as $\delta \to 0$ and it does not exceed $e^{-2\inf U}$. Consequently

$$\lim_{\delta\to 0} \int_{\Gamma'_\infty} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\psi\big(x', g(x' + \delta)\big)e^{-U(x',g(x')+\delta)}\, dx' = 0. \tag{15}$$

Now for any $\varepsilon > 0$ we consider an open set $A_\varepsilon \subset \Omega'_{r/2}$, containing $\Gamma'_2 \cup \partial\Gamma'_\infty$ and with $(N-1)$-dimensional measure $\leqslant \varepsilon$.

For each $\delta \in (0, r/2)$ we have

$$\left| \int_{A_\varepsilon} u_{\mathcal{N}}\big(x', g(x') + \delta\big)\psi\big(x', g(x') + \delta\big)e^{-2U(x',g(x')+\delta)}\, dx' \right|$$

$$\leqslant \bigg(\int_{\Omega'_{r/2}} u_{\mathcal{N}}\big(x', g(x') + \delta\big)^2 e^{-2U(x',g(x')+\delta)}\, dx' \bigg)^{1/2} \bigg(\int_{A_\varepsilon} e^{-2U(x',g(x')+\delta)}\, dx' \bigg)^{1/2} \|\psi\|_\infty$$

$$\leqslant \mathcal{K}\|\psi\|_\infty \bigg(\int_{A_\varepsilon} e^{-2\inf U}\, dx' \bigg)^{1/2} := \eta(\varepsilon) \tag{16}$$

with $\eta(\varepsilon)$ independent of δ and going to zero as $\varepsilon \to 0$. Similarly, we have

$$\begin{aligned}
&\left|\int\limits_{A_\varepsilon} v(x')\psi\big(x', g(x')\big)e^{-U(x',g(x'))}\,dx'\right| \\
&\quad\leqslant \bigg(\int\limits_{\Omega'_{r/2}} v(x')^2\,dx'\bigg)^{1/2}\bigg(\int\limits_{A_\varepsilon} e^{-2U(x',g(x'))}\,dx'\bigg)^{1/2}\|\psi\|_\infty \\
&\quad\leqslant \mathcal{K}\|\psi\|_\infty\bigg(\int\limits_{A_\varepsilon} e^{-2\inf U}\,dx'\bigg)^{1/2} = \eta(\varepsilon).
\end{aligned} \tag{17}$$

Concerning the integral over $B_\varepsilon := \Gamma'_1 \setminus A_\varepsilon$, we remark that there exists $b = b(\varepsilon) \in \mathbb{R}$ such that $U(x', g(x') + s) \leqslant b$ for each x' in B_ε and for each $s \in (0, r/2)$. This implies that the Lebesgue measure and $e^{-2U(x)}\,dx$ are equivalent in $B_\varepsilon \times (0, r/2)$. We write

$$\begin{aligned}
&u_{\mathcal{N}}\big(x', g(x')+\delta\big)\psi\big(x', g(x')+\delta\big)e^{-U(x',g(x')+\delta)} - u_{\mathcal{N}}\big(x', g(x')\big)\psi\big(x', g(x')\big)e^{-U(x',g(x'))} \\
&\quad= \big(u_{\mathcal{N}}\big(x', g(x')+\delta\big)\psi\big(x', g(x')+\delta\big) - u_{\mathcal{N}}\big(x', g(x')\big)\psi\big(x', g(x')\big)\big)e^{-U(x',g(x')+\delta)} \\
&\qquad+ u_{\mathcal{N}}\big(x', g(x')\big)\psi\big(x', g(x')\big)\big(e^{-U(x',g(x')+\delta)} - e^{-U(x',g(x'))}\big)
\end{aligned}$$

and we estimate, for small δ,

$$\begin{aligned}
&\left|\int\limits_{B_\varepsilon} \big(u_{\mathcal{N}}\big(x', g(x')+\delta\big)\psi\big(x', g(x')+\delta\big) - u_{\mathcal{N}}\big(x', g(x')\big)\psi\big(x', g(x')\big)\big)e^{-U(x',g(x')+\delta)}\,dx'\right| \\
&\quad= \left|\int\limits_{B_\varepsilon}\int\limits_0^\delta \frac{\partial}{\partial x_N}(u_{\mathcal{N}}\psi)\big(x', g(x')+s\big)e^{-U(x',g(x')+s)}e^{-U(x',g(x')+\delta)+U(x',g(x')+s)}\,ds\,dx'\right| \\
&\quad\leqslant C\bigg(\int\limits_{\Omega} \big|D(u\psi)(x)\big|^2 e^{-2U(x)}\,dx\bigg)^{1/2} e^{b-\inf U}\bigg(\int\limits_{\Omega'_{r/2}\times(0,\delta)} dx\bigg)^{1/2} \\
&\quad\leqslant C_1\|u\|_{H^2(\Omega,\mu)}\|\psi\|_{H^1(\Omega,\mu)}\bigg(\int\limits_{\Omega'_{r/2}\times(0,\delta)} dx\bigg)^{1/2}
\end{aligned}$$

that goes to 0 as $\delta \to 0$. Moreover

$$\begin{aligned}
&\left|\int\limits_{B_\varepsilon} u_{\mathcal{N}}\big(x', g(x')\big)\psi\big(x', g(x')\big)\big(e^{-U(x',g(x')+\delta)} - e^{-U(x',g(x'))}\big)\,dx'\right| \\
&\quad= \bigg(\int\limits_{B_\varepsilon} \big(u_{\mathcal{N}}\big(x', g(x')\big)\big)^2\,dx'\bigg)^{1/2}\bigg(\int\limits_{B_\varepsilon} \big(e^{-U(x',g(x')+\delta)} - e^{-U(x',g(x'))}\big)^2\,dx'\bigg)^2\|\psi\|_\infty.
\end{aligned}$$

The first integral does not exceed a constant by the norm of $u_{\mathcal{N}}$ in $H^1(B_\varepsilon \times (0, r/2), dx)$; since the Lebesgue measure and $e^{-2U(x',g(x')+s)}\,dx'\,ds$ are equivalent in $B_\varepsilon \times (0, r/2)$ it does not exceed a constant by the norm of u in $H^2(\Omega,\mu)$. The second integral goes to 0 as δ goes to 0; indeed $|e^{-U(x',g(x')+\delta)} - e^{-U(x',g(x'))}|$ goes to 0 pointwise, because $\lim_{\delta\to 0} U(x', g(x')+\delta) = \liminf_{x\to(x',g(x'))} U(x) = U(x', g(x'))$, and it does not exceed $2e^{-\inf U}$.

Summing up,

$$\left|\int_{B_\varepsilon} \left[(u_{\mathcal N}\psi)\big(x', g(x')+\delta\big) - (u_{\mathcal N}\psi)\big(x', g(x')\big)\right] e^{-U(x',g(x')+\delta)}\, dx'\right| \leqslant C(\varepsilon,\delta) \tag{18}$$

where $\lim_{\delta\to 0} C(\varepsilon,\delta) = 0$.

Using (15)–(18) we get

$$\left|\int_{\Omega'_{r/2}} \left[(u_{\mathcal N}\psi)\big(x', g(x')+\delta\big)e^{-2U(x',g(x')+\delta)} - v(x')\psi\big(x', g(x')\big)e^{-U(x',g(x'))}\right] dx'\right|$$

$$\leqslant 2\eta(\varepsilon) + \int_{B(x'_0,r/2)\cap\Gamma'_\infty} \left|(u_{\mathcal N}\psi)\big(x', g(x')+\delta\big)e^{-2U(x',g(x')+\delta)}\right| dx' + C(\varepsilon,\delta)$$

and hence

$$\lim_{\delta\to 0} \int_{B(x'_0,r/2)} \left[(u_{\mathcal N}\psi)\big(x', g(x')+\delta\big)e^{-2U(x',g(x')+\delta)} - v(x')\psi\big(x', g(x')\big)e^{-U(x',g(x'))}\right] dx' = 0.$$

To finish the proof we need to show that (12) and (13) hold. This is obtained arguing as in estimates (18) (with the set B_ε replaced by the support of ϕ) and (15), and it is a bit simpler because the test function $\phi(x')$, that replaces $\psi(x', g(x')+\delta)$, does not depend on δ. We do not need to introduce the sets A_ε because in the proof of (12) the support of ϕ is contained in the open set $\Gamma'_1 \cap B(x'_0, r/2)$ and hence $(x', s) \mapsto U(x', g(x')+s)$ is bounded on $\operatorname{supp}\phi \times (0, r/2)$. □

Corollary 2.6. *For each $u \in D(A)$ and $\psi \in H^1(\Omega,\mu)$ we have*

$$\int_\Omega (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_\Omega \big\langle Du(x), D\psi(x)\big\rangle \mu(dx). \tag{19}$$

Proof. If $u \in D(A)$ and $\psi \in H^1(\Omega,\mu)$, we approach ψ by a sequence of functions ψ_n in $H^1(\Omega,\mu)$, with compact support, and bounded.

Since in this case the corresponding functions z are equal to zero, for each n we have

$$\int_\Omega (\mathcal{A}u)(x)\psi_n(x)e^{-2U(x)}\, dx = -\frac{1}{2}\int_\Omega \big\langle Du(x), D\psi_n(x)\big\rangle e^{-2U(x)}\, dx$$

and letting $n \to \infty$, (19) follows. □

Taking $\psi = u$ in (19) shows that A is symmetric.

Moreover, using the procedure of Corollary 2.6 we may see also that for each $u \in H^2(\Omega,\mu)$ such that $\mathcal{A}u \in L^2(\Omega,\mu)$, and for each $\psi \in H^1(\Omega,\mu)$, the boundary integral $\int_{\Gamma_1} \frac{\partial u}{\partial n}(x)\psi(x)e^{-2U(x)}\,d\sigma_x$ exists, at least as an improper integral, and it is equal to $2\int_\Omega [(\mathcal{A}u)(x)v(x) + \langle Du(x), Dv(x)\rangle]e^{-2U(x)}\,dx$.

3. Operators in the whole $\mathbb{R}^N$ with Lipschitz continuous coefficients

Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ be convex, with Lipschitz continuous first order derivatives, and satisfying (2). We shall consider the probability measure $\mu(dx) = e^{-2U(x)}\,dx / \int_{\mathbb{R}^N} e^{-2U(x)}\,dx$ and the space $L^2(\mathbb{R}^N,\mu)$.

We recall a result proved in [6] on the realization A of $\mathcal{A}$ in $L^2(\mathbb{R}^N,\mu)$. It is defined by

$$\begin{cases} D(A) = \{u \in H^2(\mathbb{R}^N,\mu)\colon\ \mathcal{A}u \in L^2(\mathbb{R}^N,\mu)\} \\ \qquad\;\; = \{u \in H^2(\mathbb{R}^N,\mu)\colon\ \langle DU, Du\rangle \in L^2(\mathbb{R}^N,\mu)\}, \\ (Au)(x) = \mathcal{A}u(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{20}$$

Theorem 3.1. *Let $U : \mathbb{R}^N \mapsto \mathbb{R}$ be as above. Then for every $u \in D(A)$ and for every $\psi \in H^1(\mathbb{R}^N,\mu)$ we have*

$$\int_{\mathbb{R}^N} (\mathcal{A}u)(x)\psi(x)\mu(dx) = -\frac{1}{2}\int_{\mathbb{R}^N} \bigl\langle Du(x), D\psi(x)\bigr\rangle \mu(dx). \tag{21}$$

Moreover, the resolvent set of A contains $(0,+\infty)$ and

$$\begin{cases} \text{(i)} \quad \bigl\|R(\lambda,A)f\bigr\|_{L^2(\mathbb{R}^N,\mu)} \leqslant \dfrac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^N,\mu)}, \\ \text{(ii)} \quad \bigl\|\bigl|DR(\lambda,A)f\bigr|\bigr\|_{L^2(\mathbb{R}^N,\mu)} \leqslant \sqrt{\dfrac{2}{\lambda}}\|f\|_{L^2(\mathbb{R}^N,\mu)}, \\ \text{(iii)} \quad \bigl\|\bigl|D^2R(\lambda,A)f\bigr|\bigr\|_{L^2(\mathbb{R}^N,\mu)} \leqslant 2\sqrt{2}\|f\|_{L^2(\mathbb{R}^N,\mu)}. \end{cases} \tag{22}$$

The proof of the following Poincaré and log-Sobolev inequalities may be found in [1].

Theorem 3.2. *Assume that $x \mapsto U(x) - \omega|x|^2/2$ is convex, for some $\omega > 0$. Then, setting $\bar{u} = \int_{\mathbb{R}^N} u(x)\mu(dx)$, we have*

$$\int_{\mathbb{R}^N} \bigl|u(x) - \bar{u}\bigr|^2 \mu(dx) \leqslant \frac{1}{2\omega}\int_{\mathbb{R}^N} \bigl|Du(x)\bigr|^2 \mu(dx),$$

$$\int_{\mathbb{R}^N} u^2(x)\log\bigl(u^2(x)\bigr)\mu(dx) \leqslant \frac{1}{\omega}\int_{\mathbb{R}^N} \bigl|Du(x)\bigr|^2 \mu(dx) + \overline{u^2}\log\bigl(\overline{u^2}\bigr),$$

for each $u \in H^1(\mathbb{R}^N,\mu)$ (we adopt the convention $0\log 0 = 0$).

4. The resolvent equation

We recall that for each $x \in \mathbb{R}^N$, the subdifferential $\partial U(x)$ of U at x is the set $\{y \in \mathbb{R}^N\colon U(\xi) \geqslant U(x) + \langle y, \xi - x\rangle,\ \forall \xi \in \mathbb{R}^N\}$. At each $x \in \Omega$, since U is real valued and continuous, $\partial U(x)$ is not empty and it has a unique element with minimal norm, that we denote by $DU(x)$. Of course if U is differentiable at x, $DU(x)$ is the usual gradient. At each $x \notin \overline{\Omega}$ and at each $x \in \Gamma_\infty$, $\partial U(x)$ is empty and $DU(x)$ is not defined.

We introduce now the main tool in our study, i.e. the *Moreau–Yosida approximations* of U,

$$U_\alpha(x) = \inf\left\{U(y) + \frac{1}{2\alpha}|x-y|^2\colon\ y \in \mathbb{R}^N\right\}, \quad x \in \mathbb{R}^N,\ \alpha > 0,$$

that are real valued on the whole $\mathbb{R}^N$ and enjoy good regularity properties: they are convex, differentiable, and we have (see e.g. [4, Propositions 2.6, 2.11])

$$U_\alpha(x) \leqslant U(x), \quad x \in \mathbb{R}^N, \qquad \left|DU_\alpha(x)\right| \leqslant \left|DU(x)\right|, \quad x \in \Omega,$$
$$\lim_{\alpha\to 0} U_\alpha(x) = U(x), \quad x \in \mathbb{R}^N,$$
$$\lim_{\alpha\to 0} DU_\alpha(x) = DU(x), \quad x \in \Omega; \qquad \lim_{\alpha\to 0}\left|DU_\alpha(x)\right| = +\infty, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

Moreover DU_α is Lipschitz continuous for each α, with Lipschitz constant $1/\alpha$. It is not hard to show that each U_α satisfies (2).

Once we have the integration formula (7) and the powerful tool of the Moreau–Yosida approximations at our disposal, the proof of the dissipativity of A is similar to the proof of Theorem 3.3 of [6].

Theorem 4.1. *The resolvent set of A contains $(0,+\infty)$. For every $\lambda > 0$ we have*

$$\begin{cases} \text{(i)} & \left\|R(\lambda, A)f\right\|_{L^2(\Omega,\mu)} \leqslant \dfrac{1}{\lambda}\|f\|_{L^2(\Omega,\mu)}, \\ \text{(ii)} & \left\|\left|DR(\lambda, A)f\right|\right\|_{L^2(\Omega,\mu)} \leqslant \dfrac{2}{\sqrt{\lambda}}\|f\|_{L^2(\Omega,\mu)}, \\ \text{(iii)} & \left\|\left|D^2R(\lambda, A)f\right|\right\|_{L^2(\Omega,\mu)} \leqslant 2\sqrt{2}\|f\|_{L^2(\Omega,\mu)}. \end{cases} \tag{23}$$

Moreover, $R(\lambda, A)$ preserves positivity, and $R(\lambda, A)\mathbb{1} = \mathbb{1}/\lambda$.

Proof. Let $\lambda > 0$, let $f \in L^2(\Omega,\mu)$, and consider the resolvent equation

$$\begin{cases} \lambda u - \mathcal{A}u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{at } \Gamma_1. \end{cases} \tag{24}$$

Uniqueness of the solution to (24) in $D(A)$ is easy. Indeed, if $\lambda u - \mathcal{A}u = 0$, then taking $\psi = u$ in (19) we get $\lambda\|u\|^2_{L^2(\Omega,\mu)} \leqslant 0$, and hence $u = 0$.

Now we show that (24) has in fact a solution $u \in D(A)$. Let U_α be the Moreau–Yosida approximation of U, and let the differential operator $\mathcal{A}_\alpha$ be defined by

$$(\mathcal{A}_\alpha u)(x) = \frac{1}{2}\Delta u(x) - \big\langle DU_\alpha(x), Du(x)\big\rangle, \quad x \in \mathbb{R}^N.$$

The function U_α satisfies the assumptions of Theorem 3.1. Set

$$Z_\alpha = \int\limits_{\mathbb{R}^N} \exp\big(-2U_\alpha(x)\big)\,dx, \qquad \mu_\alpha(dx) = \frac{1}{Z_\alpha}\exp\big(-2U_\alpha(x)\big)\,dx, \tag{25}$$

and let A_α be the realization of $\mathcal{A}_\alpha$ in $L^2(\mathbb{R}^N, \mu_\alpha)$ defined by

$$D(A_\alpha) = \big\{u \in H^2\big(\mathbb{R}^N, \mu_\alpha\big):\ \langle DU_\alpha, Du\rangle \in L^2\big(\mathbb{R}^N, \mu_\alpha\big)\big\}.$$

Let $\tilde f$ be defined by $\tilde f(x) = f(x)$ for $x \in \Omega$, $\tilde f(x) = 0$ for x outside Ω. By Theorem 3.1, the problem

$$\lambda u - \mathcal{A}_\alpha u = \tilde f, \quad x \in \mathbb{R}^N, \tag{26}$$

has a unique solution $u_\alpha \in D(A_\alpha)$, which satisfies the estimates

$$\begin{cases} \|u_\alpha\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leqslant \dfrac{1}{\lambda}\|\tilde f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}, \\ \big\||Du_\alpha|\big\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leqslant \dfrac{2}{\sqrt{\lambda}}\|\tilde f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}, \\ \big\||D^2u_\alpha|\big\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leqslant 2\sqrt{2}\|\tilde f\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \end{cases} \tag{27}$$

due to (22). If in addition $f(x) \geqslant 0$ a.e., then $u_\alpha(x) \geqslant 0$ for each x. Since

$$\|\tilde f\|_{L^2(\mathbb{R}^N,\mu_\alpha)} = \bigg(\frac{1}{Z_\alpha}\int\limits_\Omega f^2 e^{-2U}\,dx\bigg)^{1/2} = \bigg(\frac{\int_\Omega e^{-2U}\,dx}{\int_{\mathbb{R}^N} e^{-2U_\alpha}\,dx}\bigg)^{1/2}\|f\|_{L^2(\Omega,\mu)}$$

remains bounded as $\alpha \to 0$, then u_α is bounded in $H^2(\mathbb{R}^N, \mu_\alpha)$ and the restriction $u_{\alpha|\Omega}$ is bounded in $H^2(\Omega, \mu)$. Up to a sequence, $u_{\alpha|\Omega}$ converges weakly in $H^2(\Omega, \mu)$ to a function $u \in H^2(\Omega, \mu)$ and it converges to u pointwise a.e. and in $H^{2-\varepsilon}(\Omega \cap B(x_0, r), dx)$ for every $\varepsilon \in (0, 2)$ and for every ball with closure that does not intersect $\Gamma_\infty \cup \Gamma_2$. Since $\lambda u_\alpha - \mathcal{A}_\alpha u_\alpha = f$ in Ω, then u satisfies $\lambda u - \mathcal{A}u = f$ almost everywhere in Ω. Since λu, Δu, f belong to $L^2(\Omega, \mu)$, then also $\langle DU, Du\rangle$ does.

If in addition $f(x) \geqslant 0$ a.e., then $u(x) \geqslant 0$ a.e. because it is the pointwise a.e. limit of $u_\alpha(x) \geqslant 0$, and since u is continuous by local elliptic regularity, $u(x) \geqslant 0$ for each $x \in \Omega$.

Let us prove that $\partial u/\partial n = 0$ at Γ_1. For each $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\Gamma \cap \operatorname{supp}\psi \subset \Gamma_1$ we have, by (21)

$$\int\limits_{\mathbb{R}^N} \mathcal{A}_\alpha u_\alpha \psi\,\mu_\alpha(dx) = -\frac{1}{2}\int\limits_{\mathbb{R}^N} \langle Du_\alpha, D\psi\rangle\,\mu_\alpha(dx).$$

On the other hand,

$$\int_{\mathbb{R}^N} \mathcal{A}_\alpha u_\alpha \psi \mu_\alpha(dx) = \int_{\mathbb{R}^N\setminus\Omega} \mathcal{A}_\alpha u_\alpha \psi \mu_\alpha(dx) + \int_{\Omega} \mathcal{A}_\alpha u_\alpha \psi \mu_\alpha(dx),$$

where

$$\int_{\mathbb{R}^N\setminus\Omega} \mathcal{A}_\alpha u_\alpha \psi \mu_\alpha(dx) = \lambda \int_{\mathbb{R}^N\setminus\Omega} u_\alpha \psi \mu_\alpha(dx),$$

because $\lambda u_\alpha - \mathcal{A}_\alpha u_\alpha = 0$ in $\mathbb{R}^N \setminus \Omega$, and

$$\begin{aligned}\int_{\Omega} \mathcal{A}_\alpha u_\alpha \psi \mu_\alpha(dx) &= -\frac{1}{2}\int_{\Omega} \langle Du_\alpha, D\psi\rangle \mu_\alpha(dx) + \frac{1}{2Z_\alpha}\int_{\Gamma} \frac{\partial u_\alpha}{\partial n}(x)\psi(x)e^{-2U_\alpha(x)}\,d\sigma_x \\ &= -\frac{1}{2}\int_{\Omega} \langle Du_\alpha, D\psi\rangle \mu_\alpha(dx) + \frac{1}{2Z_\alpha}\int_{\Gamma_1} \frac{\partial u_\alpha}{\partial n}(x)\psi(x)e^{-2U_\alpha(x)}\,d\sigma_x.\end{aligned}$$

The first equality follows from (7), and the second is true because ψ vanishes at $\Gamma \setminus \Gamma_1$. It follows that

$$\int_{\Gamma_1} \frac{\partial u_\alpha}{\partial n}(x)\psi(x)e^{-2U_\alpha(x)}\,d\sigma_x = -Z_\alpha \int_{\mathbb{R}^N\setminus\Omega} \langle Du_\alpha, D\psi\rangle \mu_\alpha(dx) - 2\lambda Z_\alpha \int_{\mathbb{R}^N\setminus\Omega} u_\alpha \psi \mu_\alpha(dx).$$

Note that $\|\psi\|_{H^1(\mathbb{R}^N\setminus\Omega,\mu_\alpha)}$ goes to 0 as $\alpha \to 0$. Since both u_α and $|Du_\alpha|$ are bounded in $L^2(\mathbb{R}^N, \mu_\alpha)$, and Z_α goes to $\int_\Omega e^{-2U(x)}\,dx$, the right-hand side goes to 0 as $\alpha \to 0$. On the other hand, since u_α goes to u in $H^{2-\varepsilon}(\Omega \cap \operatorname{supp}\psi, dx)$ for each $\varepsilon \in (0,2)$, then $\partial u_\alpha/\partial n$ goes to $\partial u/\partial n$ in $L^2(\Gamma \cap \operatorname{supp}\psi, d\sigma)$. This follows from the general theory of traces if $\Gamma \cap \operatorname{supp}\psi$ is smooth enough, say C^2. If $\Gamma \cap \operatorname{supp}\psi$ is just Lipschitz continuous, we use [8, Theorem 1.5.1.2] which implies that if $0 < \varepsilon < 1/2$, for each $i = 1, \dots, N$ the operator that maps v into the trace of $D_i v$ at $\Gamma \cap \operatorname{supp}\psi$ is continuous from $H^{2-\varepsilon}(\Omega \cap \operatorname{supp}\psi, dx)$ to $L^2(\Gamma \cap \operatorname{supp}\psi, d\sigma)$, and consequently $v \mapsto \partial v/\partial n$ is continuous from $H^{2-\varepsilon}(\Omega \cap \operatorname{supp}\psi, dx)$ to $L^2(\Gamma \cap \operatorname{supp}\psi, d\sigma)$. In any case, we have

$$\int_{\Gamma_1} \frac{\partial u}{\partial n}(x)\psi(x)e^{-2U(x)}\,d\sigma_x = \lim_{\alpha\to 0}\int_{\Gamma_1} \frac{\partial u_\alpha}{\partial n}(x)\psi(x)e^{-2U_\alpha(x)}\,d\sigma_x = 0.$$

This implies that $\partial u/\partial n = 0$ at Γ_1, and $u \in D(A)$. So, u is the unique solution in $D(A)$ of problem (24). Therefore $\rho(A) \supset (0, +\infty)$. Estimates (23) follow letting α to 0 in (27).

Finally, the equality $R(\lambda, A)\mathbb{1} = \mathbb{1}/\lambda$ is obvious. □

Corollary 4.2. *The operator A is self-adjoint and dissipative in $L^2(\Omega, \mu)$.*

Proof. The integration formula (19) implies that A is symmetric. By Theorem 4.1 the resolvent set of A is not empty, and then A is self-adjoint. Estimate (23)(i) implies that A is dissipative. $\square$

Many other properties of A are collected in the next section.

5. Consequences, remarks, open problems

Since A is self-adjoint and dissipative in $L^2(\Omega,\mu)$, it is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega,\mu)$. In this section we prove further properties of $T(t)$ and of A.

It is convenient to see A as the part in $L^2(\Omega,\mu)$ of the operator $A_0 : D(A_0) := H^1(\Omega,\mu) \mapsto (H^1(\Omega,\mu))'$, defined by

$$\langle A_0 u, f\rangle = -\frac{1}{2}\int_\Omega \big\langle Du(x), Df(x)\big\rangle \mu(dx), \quad f \in H^1(\Omega,\mu).$$

In other words, A_0 is the operator associated to the bilinear form

$$a(u,v) = -\frac{1}{2}\int_\Omega \big\langle Du(x), Dv(x)\big\rangle \mu(dx), \quad u, v \in H^1(\Omega,\mu),$$

and it generates an analytic semigroup in $(H^1(\Omega,\mu))'$. The domain of its part in $L^2(\Omega,\mu)$ consists of the functions $u \in H^1(\Omega,\mu)$ such that $A_0 u = f \in L^2(\Omega,\mu)$, in the sense that $\int_\Omega \langle Du(x), Dv(x)\rangle \mu(dx) = \int_\Omega f v \mu(dx)$ for every $v \in H^1(\Omega,\mu)$, and it coincides with $D(A)$. Note that, since $D(A_0) = H^1(\Omega,\mu)$, then the part of A_0 in $H^1(\Omega,\mu)$, which coincides with the part of A in $H^1(\Omega,\mu)$, generates an analytic semigroup in $H^1(\Omega,\mu)$. Among other consequences, this implies that the domain of A is dense in $H^1(\Omega,\mu)$.

We recall that a *symmetric Markov semigroup* is a semigroup of self-adjoint positivity preserving linear operators $S(t)$ in $L^2(\mathcal{O},\nu)$ that satisfy $\|S(t)f\|_\infty \leqslant \|f\|_\infty$ for each $f \in L^2(\mathcal{O},\nu) \cap L^\infty(\mathcal{O},\nu)$ and $t > 0$. Here $(\mathcal{O},\nu)$ is any measure space.

In the following proposition we list some properties of $T(t)$ and of A that follow in a standard way from Theorem 4.1 and from the above considerations.

Proposition 5.1. *The following statements hold.*

(i) *The measure μ is an infinitesimally invariant measure for A. Consequently,*

$$\int_\Omega T(t)f\,\mu(dx) = \int_\Omega f\,\mu(dx), \quad f \in L^2(\Omega,\mu),\ t > 0.$$

(ii) *The space $H^1(\Omega,\mu)$ is the domain of $(-A)^{1/2}$.*
(iii) *Every function in $D(A)$ is the limit in $H^2(\Omega,\mu)$ of a sequence of functions in $H^2(\Omega,\mu)$ that are restrictions to Ω of functions belonging to $C_b^2(\mathbb{R}^N)$.*
(iv) *$T(t)$ is a symmetric Markov semigroup in $L^2(\Omega,\mu)$.*
(v) *The kernel of A consists of constant functions.*

(vi) *For all* $f \in L^2(\Omega,\mu)$ *we have*

$$\lim_{t\to+\infty} T(t)f = \overline{f} := \int_\Omega f(y)\mu(dy) \quad \textit{in } L^2(\Omega,\mu).$$

Proof. (i) Taking $\psi \equiv 1$, formula (7) shows that $\int_\Omega \mathcal{A}u\mu(dx) = 0$ for each $u \in D(A)$, and therefore μ is an infinitesimally invariant measure for A.

Since A is the infinitesimal generator of $T(t)$, then for each $f \in L^2(\Omega,\mu)$ we have $T(t)f = \frac{1}{2\pi i}\int_\gamma e^{-\lambda t}R(\lambda,A)f\,d\lambda$ for any regular curve γ in the complex plane that joins $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, with $\theta > \pi/2$, and that does not intersect $(-\infty,0]$. By infinitesimal invariance, $\int_\Omega R(\lambda,A)f\mu(dx) = \lambda^{-1}\int_\Omega f\mu(dx)$ so that

$$\int_\Omega T(t)f\mu(dx) = \frac{1}{2\pi i}\int_\gamma \lambda^{-1}e^{-\lambda t}\,d\lambda \int_\Omega f\mu(dx) = \int_\Omega f\mu(dx).$$

(ii) Formula (7) implies that the seminorm $u \mapsto (\int_\Omega |Du|^2\mu(dx))^{1/2}$ is equivalent to the seminorm of $D((-A)^{1/2})$, $u \mapsto (\int_\Omega -Au\cdot u\mu(dx))^{1/2}$ on $D(A)$. Therefore, $D((-A)^{1/2})$ is the closure of the domain of A in the $H^1(\Omega,\mu)$ norm, that coincides with $H^1(\Omega,\mu)$ because $D(A)$ is dense in $H^1(\Omega,\mu)$.

(iii) Let $f \in C_0^\infty(\Omega)$. The functions u_α in the proof of Theorem 4.1 belong to $C_b^{2+\theta}(\mathbb{R}^N)$ for each $\theta \in (0,1)$, by the Schauder regularity result of [13], and their restrictions to Ω are bounded in $H^2(\Omega,\mu)$. The sequence $(u_{\alpha_k|\Omega})_{k\in\mathbb{N}}$ that is used in the proof of Theorem 4.1 converges weakly to $R(\lambda,A)f$ in $H^2(\Omega,\mu)$.

Let now $u \in D(A)$, fix $\lambda > 0$ and set $\lambda u - Au = f$. By Lemma 2.4 there is a sequence $(f_n)_{n\in\mathbb{N}}$ of $C_0^\infty(\Omega)$ functions that approach f in $L^2(\Omega,\mu)$, so that by Theorem 4.1 the functions $R(\lambda,A)f_n$ approach u in $H^2(\Omega,\mu)$. In its turn, each $R(\lambda,A)f_n$ is the weak limit of the above mentioned sequence $(u_{\alpha_k|\Omega})_{k\in\mathbb{N}}$. Therefore, the restrictions to Ω of u_{α_k} approach u weakly in $H^2(\Omega,\mu)$. Then there is a sequence of convex combinations of the functions $u_{\alpha_k|\Omega}$ that converges to u strongly in $H^2(\Omega,\mu)$, and statement (iii) follows.

(iv) Since A is self-adjoint, each $T(t)$ is self-adjoint. For each $\lambda > 0$, $R(\lambda,A)$ preserves positivity by Theorem 4.1, hence each $T(t)$ preserves positivity. Since $R(\lambda,A)\mathbb{1} = \mathbb{1}/\lambda$, then $T(t)\mathbb{1} = \mathbb{1}$. This easily yields that $\|T(t)f\|_\infty \leqslant \|f\|_\infty$ for each $f \in L^2(\Omega,\mu)\cap L^\infty(\Omega,\mu)$.

(v) If $u \in D(A)$ and $Au = 0$, then $0 = \int_\Omega Au\cdot u\mu(dx) = -\frac{1}{2}\int_\Omega |Du|^2\mu(dx)$; since Ω is connected then u is constant.

(vi) The function $t \to \varphi(t) = \int_\Omega (T(t)f)^2\mu(dx)$ is nonincreasing and bounded, then there exists the limit $\lim_{t\to+\infty}\varphi(t) = \lim_{t\to+\infty}\langle T(2t)f, f\rangle_{L^2(\Omega,\mu)}$. By a standard argument it follows that there exists a symmetric nonnegative operator $Q \in \mathcal{L}(L^2(\Omega,\mu))$ such that

$$\lim_{t\to+\infty} T(t)f = Qf, \quad f \in L^2(H,\mu).$$

On the other hand, using the Mean Ergodic Theorem in Hilbert space (see e.g. [15, p. 24]) we get easily

$$\lim_{t\to+\infty} T(t)f = P\left(\int_0^1 T(s)f\,ds\right),$$

where P is the orthogonal projection on the kernel of A. But the kernel of A consists of the constant functions, and the statement follows. □

Like all symmetric Markov semigroups, $T(t)$ may be extended in a standard way to a contraction semigroup (that we shall still call $T(t)$) in $L^p(\Omega,\mu)$, $1 \leqslant p \leqslant \infty$. $T(t)$ is strongly continuous in $L^p(\Omega,\mu)$ for $1 \leqslant p < \infty$, and it is analytic for $1 < p < \infty$. See e.g. [5, Chapter 1]. The infinitesimal generator of $T(t)$ in $L^p(\Omega,\mu)$ is denoted by A_p.

The characterization of the domain of A_p in $L^p(\Omega,\mu)$ for $p \neq 2$ is an interesting open problem. Even in the simplest case $\Omega = \mathbb{R}^N$, for $p \neq 2$ we know sufficient conditions in order that $D(A_p)$ be contained in $W^{2,p}(\mathbb{R}^N,\mu)$, but in these cases it coincides with $W^{2,p}(\mathbb{R}^N,\mu)$, see [14]. Of course the case where $u \in W^{2,p}(\mathbb{R}^N,\mu)$ does not imply $\langle DU, Du\rangle \in L^p(\mathbb{R}^N,\mu)$ is not covered.

Independently on the characterization of $D(A_p)$, an important optimal regularity result for evolution equations follows from [11].

Corollary 5.2. *Let $1 < p < \infty$, $T > 0$. For each $f \in L^p((0,T); L^p(\Omega,\mu))$ (i.e. $(t,x) \mapsto f(t)(x) \in L^p((0,T) \times \Omega; dt \times \mu)$) the problem*

$$\begin{cases} u'(t) = A_p u(t) + f(t), & 0 < t < T,\\ u(0) = 0, \end{cases}$$

has a unique solution $u \in L^p((0,T); D(A_p)) \cap W^{1,p}((0,T); L^p(\Omega,\mu))$.

Since A is self-adjoint and dissipative, the spectrum of A is contained in $(-\infty,0]$. By Proposition 5.1(iv), 0 is the maximum element in the spectrum of A.

If $D(A)$ were compactly embedded in $L^2(\Omega,\mu)$, then 0 would be an isolated simple eigenvalue of A. But in general $D(A)$ is not compactly embedded in $L^2(\Omega,\mu)$, as some conterexamples show. For instance, the counterexample in [12] (with Dirichlet boundary condition) acts as a counterexample also in our case.

Note that a necessary and sufficient condition for 0 to be isolated in the spectrum of A is that the Poincaré inequality holds, i.e.,

$$\int_\Omega |u-\bar{u}|^2 \mu(dx) \leqslant \frac{1}{2\omega} \int_\Omega |Du|^2 \mu(dx), \quad u \in H^1(\Omega,\mu), \tag{28}$$

for some $\omega > 0$. Here $\bar{u}$ is the mean value of u,

$$\bar{u} := \int_\Omega u(x)\mu(dx)$$

and in this case the mapping $u \mapsto \bar{u}$ coincides with the spectral projection onto the kernel of A.

More precisely, let us denote by $L^2_0(\Omega,\mu)$ the subspace of $L^2(\Omega,\mu)$ consisting of the functions with vanishing mean. It is the orthogonal complement of the kernel of A and it is invariant under $T(t)$. The part A_0 of A in $L^2_0(\Omega,\mu)$ is still a self-adjoint operator, then $\langle(-A_0 - \omega I)u, u\rangle \geqslant 0$ for each $u \in D(A_0)$ if and only if $\sigma(A_0 + \omega I) \subset (-\infty,0]$. In other words, (28) holds for each $u \in D(A_0)$ if and only if $\sigma(A_0) \subset (-\infty, -\omega]$. In its turn, $\sigma(A_0) \subset (-\infty, -\omega]$ if and only if $\sigma(A) \subset (-\infty, -\omega] \cup \{0\}$. In this case we have

$$\left\|T(t)f\right\|_{L^2(\Omega,\mu)} \leqslant e^{-\omega t}\|f\|_{L^2(\Omega,\mu)}, \quad t > 0,\ f \in L^2_0(\Omega, \mu). \tag{29}$$

Indeed, for each $t > 0$ and $f \in L^2_0(\Omega, \mu)$,

$$\frac{d}{dt}\left\|T(t)f\right\|^2 = \int_\Omega 2AT(t)f \cdot T(t)f\,\mu(dx) = -\left\|DT(t)f\right\|^2 \leqslant -2\omega\left\|T(t)f\right\|^2,$$

so that (29) holds.

Under suitable assumptions on U, it is possible to prove that $D(A)$ is compactly embedded in $L^2(\Omega, \mu)$. Therefore, (28) holds for some $\omega > 0$.

The following proposition is adapted from [12].

Proposition 5.3. *Assume that $U \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$\Delta U(x) \leqslant a\left|DU(x)\right|^2 + b, \quad x \in \Omega, \qquad \frac{\partial U}{\partial n} \geqslant 0, \quad x \in \partial\Omega,$$

for some $a < 2$, $b \in \mathbb{R}$. Then the map $u \mapsto |DU|u$ is bounded from $H^1(\Omega, \mu)$ to $L^2(\Omega, \mu)$. If, in addition, $|DU| \to +\infty$ as $|x| \to \infty$, then $H^1(\Omega, \mu)$ (hence $D(A)$) is compactly embedded in $L^2(\Omega, \mu)$.

Proof. We shall show that there is $C > 0$ such that

$$\left\||DU|u\right\|_{L^2(\Omega,\mu)} \leqslant C\|u\|_{H^1(\Omega,\mu)} \tag{30}$$

for every $u \in L^\infty(\Omega, \mu) \cap H^1(\Omega, \mu)$, with compact support. Such functions are dense in $H^1(\Omega, \mu)$ by Lemma 2.4. For such functions the integral $\int_\Omega u^2|DU|^2\mu(dx)$ is finite.

Integrating by parts and then using Young's inequality we get for every $\varepsilon > 0$ and for a suitable C_ε

$$\begin{aligned}
\int_\Omega u^2|DU|^2\mu(dx) &= -\frac{1}{2}\int_\Omega u^2\langle DU, De^{-2U}\rangle\,dx \\
&= \frac{1}{2}\int_\Omega u^2\Delta U e^{-2U}\,dx + \int_\Omega u\langle DU, Du\rangle e^{-2U}\,dx - \frac{1}{2}\int_{\partial\Omega}\frac{\partial U}{\partial n}u^2 e^{-2U}\,d\sigma_x \\
&\leqslant \frac{1}{2}(a+\varepsilon)\int_\Omega |u|^2|DU|^2\mu(dx) + C_\varepsilon\int_\Omega |Du|^2\mu(dx) + \frac{b}{2}\int_\Omega |u|^2\mu(dx).
\end{aligned}$$

Choosing ε such that $a + \varepsilon < 2$ estimate (30) follows.

If $\lim_{|x|\to\infty} |DU(x)| = +\infty$, then for each $\varepsilon > 0$ there is $R > 0$ such that $|DU| \geqslant 1/\varepsilon$ in $\Omega \setminus B(0, R)$. Moreover for every u in the unit ball B of $H^1(\Omega, \mu)$ we have

$$\frac{1}{\varepsilon^2} \int\limits_{\{x\in\Omega:|DU(x)|\geqslant\varepsilon\}} |u|^2\mu(dx) \leqslant \int\limits_{\Omega} |u|^2|DU|^2\mu(dx) \leqslant C^2,$$

where C is the constant in (30). The complement $\Omega_\varepsilon := \{x \in \Omega\colon |DU(x)| < \varepsilon\}$ is contained in $B(0, R)$ and it has positive distance from $\Gamma_2 \cup \Gamma_\infty$, therefore the Lebesgue measure is equivalent to μ on it. Since the embedding of $H^1(\Omega_\varepsilon, dx)$ into $L^2(\Omega_\varepsilon, dx)$ is compact, we can find $\{f_1, \dots, f_k\} \subset L^2(\Omega_\varepsilon, dx)$ such that the balls $B(f_i, \varepsilon) \subset L^2(\Omega_\varepsilon, dx)$ cover the restrictions of the functions of B to Ω_ε. Denoting by $\tilde{f}_i$ the extension of f_i to the whole Ω that vanishes outside Ω_ε, it follows that $B \subset \bigcup_{i=1}^k B(\tilde{f}_i, (C+1)\varepsilon)$ and the proof is complete. □

Another well-known sufficient condition for $H^1(\Omega, \mu)$ (and hence $D(A)$) be compactly embedded in $L^2(\Omega, \mu)$ is that a logarithmic Sobolev inequality holds:

$$\int\limits_{\Omega} u^2 \log\bigl(|u|^2\bigr)\mu(dx) \leqslant \frac{1}{\omega} \int\limits_{\Omega} |Du|^2\mu(dx) + \|u\|^2_{L^2(\Omega,\mu)} \log\bigl(\|u\|^2_{L^2(\Omega,\mu)}\bigr), \tag{31}$$

for all $u \in H^1(\Omega, \mu)$ and some $\omega > 0$ (where we set $0 \log 0 = 0$).

In what follows we give sufficient conditions for the validity of (31).

Proposition 5.4. *Assume that*

$$|DU| \in L^2(\Omega, \mu), \tag{32}$$

$$\exists \omega > 0 \textit{ such that } x \mapsto U(x) - \omega|x|^2/2 \textit{ is convex.} \tag{33}$$

Then (31) *and* (28) *hold.*

Proof. By statement (iii) of Lemma 2.4, the set of the functions in $H^1(\Omega, \mu)$ having compact support with positive distance from $\Gamma_\infty \cup \Gamma_2$ is dense in $H^1(\Omega, \mu)$. Therefore it is sufficient to prove that (31) holds for any u in such a set. Note that u may be extended to a function $\tilde{u} \in H^1(\mathbb{R}^N, dx)$ with compact support. This is because the support of u is far from $\Gamma_\infty \cup \Gamma_2$, hence it is contained in a region where μ is equivalent to the Lebesgue measure and we may apply extension operators for functions defined in bounded open sets with Lipschitz continuous boundaries, see e.g. [8, Theorem 1.4.3.1]. Then $\tilde{u}$ belongs to $H^1(\mathbb{R}^N, \mu_\alpha)$ for each $\alpha > 0$. For each α, U_α satisfies (33) with constant ω_α which goes to ω as α goes to 0, and DU_α is Lipschitz continuous. By Theorem 3.2,

$$\int\limits_{\mathbb{R}^N} \tilde{u}^2(x) \log\bigl(\tilde{u}^2(x)\bigr)\mu_\alpha(dx) \leqslant \frac{1}{\omega_\alpha} \int\limits_{\mathbb{R}^N} \bigl|D\tilde{u}(x)\bigr|^2\mu_\alpha(dx) + \|\tilde{u}\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \log\bigl(\|\tilde{u}\|_{L^2(\mathbb{R}^N,\mu_\alpha)}\bigr).$$

The right-hand side goes to

$$\frac{1}{\omega} \int\limits_{\Omega} \bigl|Du(x)\bigr|^2\mu(dx) + \|u\|_{L^2(\Omega,\mu)} \log\bigl(\|u\|_{L^2(\Omega,\mu)}\bigr)$$

as $\alpha \to 0$, by dominated convergence. Observing that

$$\int_{\Omega} u^2 \log(|u|^2)\mu_\alpha(dx) \leqslant \int_{\mathbb{R}^N} \tilde{u}^2 \log(\tilde{u}^2)\mu_\alpha(dx)$$

for each $\alpha > 0$, and letting again $\alpha \to 0$, (31) follows. The proof of (28) is similar. □

Corollary 5.5. *Under the assumptions of Proposition* 5.4*, $H^1(\Omega, \mu)$ is compactly embedded in $L^2(\Omega, \mu)$. Therefore, the spectrum of A consists of a sequence of semisimple isolated eigenvalues $\lambda_n \leqslant -\omega$, plus the simple eigenvalue* 0*. Moreover $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$ with $q(t) = 1 + e^{2\omega t}$, and*

$$\|T(t)f\|_{L^{q(t)}(\Omega,\mu)} \leqslant \|f\|_{L^2(\Omega,\mu)}, \quad t > 0,\ f \in L^2(\Omega, \mu). \tag{34}$$

Proof. The proof of the compact embedding is standard, see e.g. [12]. The fact that $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$, as well as estimate (34), follows from [9,10]. □

Propositions 5.3 and 5.4 give just sufficient conditions for $D(A)$ be compactly embedded in $L^2(\Omega, \mu)$. They are far from being necessary and sufficient. Finding more stringent conditions is another interesting open problem.

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