Nonlinear parabolic equations and systems

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1 Introduction

The theory of nonlinear parabolic problems is so widely developed that it is impossible to give an overview in a few pages. Therefore in this chapter we consider only a specific class of equations and systems with a high degree of nonlinearity, that are called fully nonlinear because the nonlinearities involve the highest order derivatives of the unknowns appearing in the problems. For instance, a simple significant example is the Cauchy problem for a second order equation,

$$\begin{cases} u_t(t,x) = \Phi(D^2u(t,x)), \quad t \geq 0, \; x \in \overline{\Omega}, \\ \\ \Psi(Du(t,x)) = 0, \quad t \geq 0, \; x \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where the matrix $\partial\Phi/\partial q_{ij}(D^2u_0)$ is positive definite at each $x \in \overline{\Omega}$, and the vector with components $\partial\Psi/\partial p_i(Du_0)$ is nontangential at each $x \in \partial\Omega$. Ω is an open set in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, D and D^2 denote the gradient and the matrix of the second order derivatives with respect to the space variables x.

The theory of fully nonlinear parabolic equations and systems is not new. After a few papers in the sixties and seventies dealing mainly with local existence and uniqueness of regular solutions ([23, 38, 26, 27]), big improvements came in the eighties with the papers of Krylov about *a priori* estimates and existence in the large for second order equations (see e.g. the book [25]), and of Da Prato and Grisvard who initiated the theory of abstract fully nonlinear parabolic equations in Banach spaces, in [12].

This paper was followed by a series of works about geometric theory of fully nonlinear parabolic equations, an account of which up to 1995 may be found in chapters 8 and 9 of the book [33].

In the last years a further impulse to the general theory was given by the study of multidimensional parabolic free boundary problems, that can be transformed to fixed boundary ones by suitable changements of coordinates, and the resulting final systems are fully nonlinear. See [6, 9, 7, 8, 10, 31, 16, 18, 19, 20].

Let us describe the contents of this chapter.

In section 2 we give an overview on the theory of fully nonlinear parabolic equations in Banach spaces, including the discussion about stability, instability, and invariant manifolds of stationary solutions. Problems like (1.1) with the nonlinear boundary condition $\Psi(Du(t,x)) = 0$ replaced by a linear one, such as $u = 0$ or $\partial u/\partial \nu = 0$, may be turned into evolution equations in Banach spaces in a standard way. The function u is seen as a function of the only variable t with values in a Banach space X of functions defined in Ω, i.e. setting $U(t) = u(t,\cdot)$ we rewrite (1.1) as

$$\begin{cases} U'(t) = F(U(t)), \quad t \geq 0, \\ \\ U(0) = u_0, \end{cases}$$

where $F(U) = \Phi(D^2U)$ is defined in (an open subset of) a Banach space $D \subset X$, and the linear operator $A = F'(u_0) : D \mapsto X$ is the generator of an analytic semigroup. The difference between the above problem and the more popular semilinear problems treated for instance in [22] is that the nonlinearity is defined in the domain of A and not in some intermediate space between X and $D(A)$. This gives several technical difficulties that will be described in section 2.

In section 3 we see in details second order equations in smooth domains of $\mathbb{R}^N$ with fully nonlinear boundary conditions such as (1.1) and its generalizations, that cannot be treated as immediate applications of the abstract theory. In section 4 we discuss the principle of linearized stability and the construction of invariant manifolds near stationary solutions of these equations.

In section 5 we show how the general theory may be applied to different parabolic free boundary problems. As model problems we consider the free boundary heat equation, arising in combustion theory,

$$\begin{cases} u_t = \Delta u, & t > 0,\ x \in \Omega_t, \\ \\ u = 0,\ \dfrac{\partial u}{\partial n} = -1, & t > 0,\ x \in \partial\Omega_t, \end{cases} \tag{1.2}$$

and Stefan type problems like the Hele-Shaw flow,

$$\begin{cases} \Delta u = 0, & t > 0,\ x \in \Omega_t, \\ \\ u = 0,\ \ V = -\dfrac{\partial u}{\partial n}, & t > 0,\ x \in \Gamma_t, \\ \\ \dfrac{\partial u}{\partial n} = b, & t > 0,\ x \in J, \end{cases} \tag{1.3}$$

where the unknowns are the open sets $\Omega_t \subset \mathbb{R}^N$ for $t > 0$ and the function $u(t, x)$ for $t > 0$, $x \in \Omega_t$. In the boundary conditions, $n = n(t, x)$ is the exterior normal vector to $\partial\Omega_t$ at $x \in \partial\Omega_t$. In the case of problem (1.3) we have $N = 2$ and the boundary of Ω_t is made by a fixed known interior component J and a moving unknown exterior component Γ_t, V represents the normal velocity of the free boundary Γ_t, in such a way that V is positive for expanding curves, and $b \geq 0$.

Our procedure is to reduce the free boundary problems to fixed boundary ones by suitable changements of coordinates, and then to eliminate one of the unknowns (either u or the free boundary), expressing it in terms of the other unknown, to get a final problem for only one unknown. In both cases the final problem will be fully nonlinear, and nonlocal. In the case of problem (1.2) we eliminate the free boundary and the final problem is studied with the methods of sections 3 and 4. In problem (1.3) we eliminate u and the final problem is studied with the methods of parabolic evolution equations in Banach spaces of section 2.

2 Abstract parabolic problems

Let D, X be Banach spaces with respective norms $\|\cdot\|_D$, $\|\cdot\|$, and such that D is continuously embedded in X. We shall discuss the problem

$$\begin{cases} u'(t) = F(t, u(t)), & t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $F : [0,T] \times \mathcal{O} \mapsto X$ is a sufficiently smooth function, $T > 0$, and $\mathcal{O}$ is a neighborhood of the initial datum $u_0 \in D$.

The abstract parabolicity assumption near u_0 is that the operator $A : D(A) = D \mapsto X$, $A = F_u(0, u_0)$ is sectorial.

Definition 2.1 *A linear operator $A : D(A) \mapsto X$ is called sectorial if the resolvent set $\rho(A)$ contains a sector $\Sigma = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$ with $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and there is $M > 0$ such that $\|(\lambda - \omega)R(\lambda, A)\|_{L(X)} \leq M$ for each $\lambda \in \Sigma$.*

We recall that if A is sectorial, the analytic semigroup generated by A is defined by

$$e^{0A} = I, \quad e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0, \tag{2.2}$$

where $r > 0$, $\eta \in (\pi/2, \theta)$, and γ is the curve $\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta, \; |\lambda| \geq r\}$ $\cup \; \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \eta, \; |\lambda| = r\}$, oriented counterclockwise. A summary of the general theory of sectorial operators and analytic semigroups in Banach spaces may be found in chapter 2 of [33].

It is easy to see that if $A : D(A) = D \mapsto X$ is sectorial and $B \in L(D, X)$ is small enough, then $A + B : D(A + B) = D \mapsto X$ is still sectorial. So we may assume without much loss of generality that $F_u(t, x) : D \mapsto X$ is sectorial for each $t \in [0, T]$ and $x \in \mathcal{O}$.

As a first step we look for a local solution $u \in C([0, a]; D) \cap C^1([0, a]; X)$ for some $a \in (0, T]$. The most natural way to solve problem (2.1), at least locally, is by linearization near u_0. Setting $G(t, x) = F(t, x) - Ax = F(t, x) - F_u(0, u_0)x$ for $t \in [0, T]$, $x \in \mathcal{O}$, we rewrite problem (2.1) in the form

$$\begin{cases} u'(t) = Au(t) + G(t, u(t)), \quad t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{2.3}$$

and we try to solve it by a fixed point argument, i.e., we look at a solution as a fixed point of the operator Γ defined by $\Gamma u = v$ where v is the solution to the linear problem

$$\begin{cases} v'(t) = Av(t) + G(t, u(t)), \quad 0 \leq t \leq a, \\ \\ v(0) = u_0. \end{cases} \tag{2.4}$$

The simplest space where to set the fixed point would be (a closed ball in) $C([0, a]; D)$. In this case, for every u in the ball the function $f(t) = G(t, u(t))$

is in $C([0,a];X)$, and unless X and D are special spaces, well known counterexamples show that in general v does not belong to $C([0,a];D)$, and Γ cannot map a ball of $C([0,a];D)$ into itself. Therefore, we turn to Hölder spaces where optimal regularity results and estimates for linear problems are available.

Such optimal regularity results, needed to solve locally (2.1) and to describe the properties of the solution, are stated in the next section.

Note that these difficulties do not arise in semilinear equations, i.e., equations of the type (2.4) where G is defined in $[0,a]\times Y$, Y being an intermediate space between X and $D(A)$. This is because if $f(t)=G(t,u(t))$ is in $C([0,a];X)$, and $u_0\in D$, the solution v of (2.4) belongs to $C([0,a];Y)$, provided Y satisfies the interpolatory embedding property

$$\|y\|_Y \le C(\|y\|_X)^{1-\alpha}(\|y\|_D)^{\alpha},\ \ y\in Y,$$

for some $C>0$, $\alpha\in(0,1)$. In this case we have in fact $v\in C^{1-\alpha}([0,a];Y)$, see [33, Ch. 4].

2.1 Optimal regularity in linear problems

Let us consider the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & 0<t<a, \\ u(0) = u_0, \end{cases} \tag{2.5}$$

where A is a linear sectorial operator in general Banach space X and $f:[0,a]\mapsto X$ is (at least) continuous.

Definition 2.2 *A classical solution to problem (2.5) is a function* $u\in C([0,a];X)\cap C((0,a];D(A))\cap C^1((0,a];X)$ *that satisfies* $u'(t)=Au(t)+f(t)$ *for* $0<t\le a$ *and* $u(0)=u_0$. *A strict solution is a function* $u\in C([0,a];D)\cap C^1([0,a];X)$ *that satisfies* $u'(t)=Au(t)+f(t)$ *for* $0\le t\le a$ *and* $u(0)=u_0$.

It is well known that if problem (2.5) has a classical solution, then it is unique and it is given by the variation of constants formula,

$$u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s)ds,\ \ 0\le t\le a.$$

We state below two optimal regularity results in Hölder spaces, whose proofs are due to [32], [36] respectively, and may be found in [33, Ch. 4].

We need to introduce a class of real interpolation spaces between X and $D(A)$. For $0 < \alpha < 1$ the real interpolation space $D_A(\alpha,\infty) := (X, D(A))_{\alpha,\infty}$ is characterized by

$$\begin{cases} D_A(\alpha,\infty) = \{x \in X \;:\; t \mapsto v(t) = \|t^{1-\alpha}Ae^{tA}x\| \in L^\infty(0,1)\}, \\ \\ \|x\|_{D_A(\alpha,\infty)} \quad = \|x\| + [x]_{D_A(\alpha,\infty)} = \|x\| + \|v\|_\infty. \end{cases} \tag{2.6}$$

The weighted Hölder space $C^\alpha_\alpha((a,b];X)$ is defined as the set of all bounded functions $f : (a,b] \mapsto X$ such that $t \mapsto (t-a)^\alpha f(t)$ is α-Hölder continuous in $(a,b]$. The norm is $\|f\|_{C^\alpha_\alpha((a,b];X)} = \|f\|_\infty + \|(\cdot - a)^\alpha f(\cdot)\|_{C^\alpha((a,b];X)}$.

Theorem 2.3 *Let $0 < \alpha < 1$, $f \in C^\alpha_\alpha((0,T];X)$, $u_0 \in D(A)$. Then problem (2.5) has a classical solution u such that u' and Au belong to $C^\alpha_\alpha((0,T];X)$, $t \mapsto t^\alpha u'(t)$ is bounded with values in $(X, D(A))_{\alpha,\infty}$, and there is $C = C(T) > 0$, increasing in T, such that*

$$\begin{aligned} &\|u'\|_{C^\alpha_\alpha((0,T];X)} + \|Au\|_{C^\alpha_\alpha((0,T];X)} + \sup\nolimits_{0<t<T} \|t^\alpha u'(t)\|_{(X,D(A))_{\alpha,\infty}} \\ &\qquad\qquad \le C(\|f\|_{C^\alpha_\alpha((0,T];X)} + \|u_0\|_{D(A)}); \end{aligned} \tag{2.7}$$

If in addition $f \in C([0,T];X)$, and $Au_0 + f(0) \in \overline{D(A)}$, then u', $Au \in C([0,T];X)$, and u is a strict solution to problem (2.5).

Theorem 2.4 *Let $0 < \alpha < 1$, $f \in C^\alpha([0,T],X)$, $u_0 \in D(A)$ be such that $Au_0 + f(0) \in (X, D(A))_{\alpha,\infty}$. Then both u' and Au belong to $C^\alpha([0,T],X)$, u' is bounded with values in $(X, D(A))_{\alpha,\infty}$, and there is $C = C(T) > 0$, increasing in T, such that*

$$\begin{aligned} &\|u\|_{C^{1+\alpha}([0,T],X)} + \|Au\|_{C^\alpha([0,T],X)} + \sup\nolimits_{0<t<T} \|u'(t)\|_{(X,D(A))_{\alpha,\infty}} \\ &\quad \le C(\|f\|_{C^\alpha([0,T],X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{(X,D(A))_{\alpha,\infty}}). \end{aligned} \tag{2.8}$$

We emphasize that we need Hölder spaces because of the lack of optimal regularity in spaces of continuous functions, in the sense that if $f : [0,T] \mapsto X$ is continuous, in general it is not true that the u' and Au in (2.5) are continuous with values in X. However, we have optimal regularity in spaces of continuous functions if we replace X and $D(A)$ by the continuous interpolations spaces $E_0 = D_A(\alpha)$, $E_1 = D_A(\alpha+1)$, respectively.

For $0 < \alpha < 1$, the space $D_A(\alpha) := (X, D(A))_\alpha$ is defined as the closure of $D(A)$ in $(X, D(A))_{\alpha,\infty}$. It may be characterized by

$$(X, D(A))_\alpha = \{x \in (X, D(A))_{\alpha,\infty} : \lim_{t\to 0} t^{1-\alpha} A e^{tA} x = 0\},$$

and it is a Banach space under the norm of $(X, D(A))_{\alpha,\infty}$. The space $D_A(\alpha+1)$ is the domain of the part of A in $(X, D(A))_\alpha$:

$$D_A(\alpha + 1) = \{x \in D(A) : \; Ax \in D_A(\alpha)\},$$

and it is endowed with the graph norm $\|x\|_{D_A(\alpha+1)} = \|x\|_{D_A(\alpha)} + \|Ax\|_{D_A(\alpha)}$.

The following is a well known result due to Da Prato and Grisvard ([12]).

Theorem 2.5 *Let $0 < \alpha < 1$, and let $f \in C([0, T], D_A(\alpha))$, $u_0 \in D_A(\alpha + 1)$. Then the solution u of problem (2.5) belongs to $C([0, T], D_A(\alpha + 1)) \cap C^1([0, T], D_A(\alpha))$, and there is C such that*

$$\begin{aligned} &\|u'\|_{C([0,T];D_A(\alpha,\infty))} + \|Au\|_{C([0,T];D_A(\alpha,\infty))} + \|Au\|_{C^\alpha([0,T];X)} \\ &\qquad \le C(\|f\|_{C([0,T];D_A(\alpha,\infty))} + \|u_0\|_{D_A(\alpha+1,\infty)}). \end{aligned} \tag{2.9}$$

2.2 The nonlinear problem

Here we collect several results about problem (2.1) that are proved using Theorems 2.3 and 2.4 as main tools.

The minimal assumptions on $F : [0, T] \times \mathcal{O} \mapsto X$, $\mathcal{O}$ being an open set in D, are the following.

(H1) The function $(t, u) \mapsto F(t, u)$ is continuous with respect to (t, u) and it is Fréchet differentiable with respect to u. There exists $\alpha \in (0, 1)$ such that for all $\overline{u} \in \mathcal{O}$ there are $R = R(\overline{u})$, $L = L(\overline{u})$, $K = K(\overline{u}) > 0$ verifying

$$\begin{cases} \|F_u(t, v) - F_u(t, w)\|_{L(D,X)} \le L\|v - w\|_D, \\ \\ \|F(t, u) - F(s, u)\| + \|F_u(t, u) - F_u(s, u)\|_{L(D,X)} \le K|t - s|^\alpha, \end{cases} \tag{2.10}$$

for all t, $s \in [0, T]$, u, v, $w \in B(\overline{u}, R) \subset D$.

(H2) For every $t \in [0, T]$ and $v \in \mathcal{O}$ the Fréchet derivative $F_v(t, v)$ is sectorial in X and its graph norm is equivalent to the norm of D.

We now state the main local existence theorem.

Theorem 2.6 *Let $\mathcal{O} \subset D$ be an open set. Let $F : [0,T] \times \mathcal{O} \mapsto X$ satisfy assumptions (H1) and (H2). Let $u_0 \in \mathcal{O}$ be such that $F(0,u_0) \in \overline{D}$. Then there is a maximal $\tau = \tau(u_0) > 0$ such that problem (2.1) has a solution $u \in C([0,\tau);D) \cap C^1([0,\tau);X)$ with the following properties.*

(i) For every $\varepsilon \in (0,\tau)$, $u \in C^{\alpha}_{\alpha}((0,\tau-\varepsilon];D)$, $u' \in C^{\alpha}_{\alpha}((0,\tau-\varepsilon];X)$ and $t^{\alpha}u'(t)$ is bounded in $[0,\tau-\varepsilon]$ with values in $(X,D)_{\alpha,\infty}$.

(ii) u is the unique solution of (2.1) belonging to

$$\bigcup_{0<\beta<1} C^{\beta}_{\beta}((0,\tau-\varepsilon];D) \cap C([0,\tau-\varepsilon];D)$$

for each $\varepsilon \in (0,\tau)$.

(iii) u depends continuously on u_0, in the sense that for each $\overline{u} \in \mathcal{O}$ such that $F(0,\overline{u}) \in \overline{D}$, and for each $\overline{\tau} \in (0,\tau(\overline{u}))$ there are $\epsilon = \epsilon(\overline{u},\overline{\tau}) > 0$, $H = H(\overline{u},\overline{\tau}) > 0$ such that if

$$u_0 \in \mathcal{O}, \ F(0,u_0) \in \overline{D}, \ \|u_0 - \overline{u}\|_D \leq \epsilon,$$

then $\tau(u_0) \geq \overline{\tau}$ and

$$\|u(\cdot;u_0) - u(\cdot;\overline{u})\|_{C^{\alpha}_{\alpha}((0,\overline{\tau}];D)} + \|u_t(\cdot;u_0) - u_t(\cdot;\overline{u})\|_{C^{\alpha}_{\alpha}((0,\overline{\tau}];X)}$$

$$+\sup\{t^{\alpha}\|u_t(t,u_0) - u_t(t;\overline{u})\|_{(X,D)_{\alpha,\infty}} : 0 < t \leq \overline{\tau}\} \leq H\|u_0 - \overline{u}\|_D.$$

(iv) If in addition $F(0,u_0) \in (X,D)_{\alpha,\infty}$, then u is more regular up to $t=0$, precisely $u \in C^{\alpha}([0,\tau-\varepsilon];D) \cap C^{1+\alpha}([0,\tau-\varepsilon];X)$ for each $\varepsilon \in (0,\tau)$.

It is possible to obtain several further regularity results, as well as results of dependence on parameters, stability of stationary solutions and of periodic orbits. See [33, Ch. 8].

The uniqueness part of the statement of Theorem 2.6 is not completely satisfactory. In a sense it is natural, becuse we get uniqueness in the same space where we prove existence of the solution. But a theorem of uniqueness of the strict solution, i.e., uniqueness in $C([0,a];D) \cap C^1([0,a];D)$, is not available (except in special cases, of course), and uniqueness of the strict solution is still an open problem.

Applying Theorem 2.5 gives wellposedness results for fully nonlinear problems in continuous interpolation spaces.

Let $E_1 \subset E_0 \subset X$ be Banach spaces, let $\mathcal{O}$ be an open subset of E_1, and let $0 < \theta < 1$, $T > 0$. $F : [0,T] \times \mathcal{O} \mapsto E_0$ is a nonlinear function such that

(H3) F and F_x are continuous in $[0,T]\times\mathcal{O}$, for every $(\overline{t},\overline{u})\in[0,T]\times\mathcal{O}$ the operator $F_x(\overline{t},\overline{u}):\ E_1\mapsto E_0$ is the part in E_0 of a sectorial operator $L: D(L)\subset X\mapsto X$, such that $D_L(\theta)=E_0$, $D_L(\theta+1)=E_1$ with equivalence of the respective norms.

Theorem 2.7 *Let F satisfy assumption (H3), and let $u_0\in\mathcal{O}$. Then there is a maximal $\tau=\tau(u_0)>0$ such that problem (2.1) has a unique solution $u\in C([0,\tau);E_1)\cap C^1([0,\tau);E_0)$.*

The solution depends continuously on u_0, in the sense that for each $\overline{u}\in\mathcal{O}$ and for each $\overline{\tau}\in(0,\tau(\overline{u}))$ there are $\epsilon=\epsilon(\overline{u},\overline{\tau})>0$, $H=H(\overline{u},\overline{\tau})>0$ such that if

$$u_0\in\mathcal{O},\ \ \|u_0-\overline{u}\|_{E_1}\le\epsilon,$$

then $\tau(u_0)\ge\overline{\tau}$ and

$$\|u(t;u_0)-u(t;\overline{u})\|_{E_1}+\|u_t(t;u_0)-u_t(t,\overline{u})\|_{E_0}\le H\|u_0-\overline{u}\|_{E_1}.$$

The original proof due to Da Prato and Grisvard was simplified and written clearly in [5]. See also [33, Ch. 8].

A geometric theory of fully nonlinear abstract evolution equations may be developed, see [33, Ch. 9]. Here we quote the principle of linearized stability and the construction of stable, unstable, and center manifolds of stationary solutions made in [13], that will be used in the applications to free boundary problems of section 5.

Without loss of generality, we assume that the stationary solution is 0. In next Theorems 2.8, 2.9, 2.10, and 2.11 we shall assume that $F:\mathcal{O}\mapsto E_0$ satisfies (H3), $\mathcal{O}$ being a neighborhood of 0 in E_1, and that $F(0)=0$. Moreover we set

$$A=F'(0).$$

Theorem 2.8 *The following statements hold true.*

(i) If $\omega_A:=\sup\{\mathrm{Re}\,\lambda:\ \lambda\in\sigma(A)\}<0$, then for every $\omega\in(0,\omega_0)$ there are r, M such that if $\|u_0\|_{E_1}\le r$ then the solution u of (2.1) is defined in $[0,+\infty)$, and

$$\|u(t)\|_{E_1}+\|u'(t)\|_{E_0}\le Me^{-\omega t}\|u_0\|_{E_1},\ \ t\ge0.$$

(ii) If $\omega_A > 0$ *and* $\inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A), \mathrm{Re}\lambda > 0\} > 0$*, then the null solution of* $u' = F(u)$ *is unstable in* E_1*. Specifically, there exist nontrivial backward solutions of* $u' = F(u)$ *going to* 0 *as* t *goes to* $-\infty$*.*

In the case where A is hyperbolic, i.e.,

$$\sigma(A) \cap i\mathbb{R} = \emptyset \tag{2.11}$$

a saddle point theorem may be shown. We denote by P the spectral projection associated to the subset $\sigma^+(A)$ of $\sigma(A)$ with positive real part,

$$P = \frac{1}{2\pi i}\int_C R(\lambda, A)d\lambda,$$

where C is any closed simple regular curve in $\{\mathrm{Re}\ \lambda > 0\}$ surrounding $\sigma^+(A)$.

Theorem 2.9 *Assume that (2.11) holds. Then there are positive numbers* r_0*,* r_1*, such that*
(i) There exists $R_0 > 0$ *and a Lipschitz continuous function*

$$\varphi : B(0, r_0) \subset P(E_0) \mapsto (I - P)(E_1),$$

differentiable at 0 *with* $\varphi'(0) = 0$*, such that for every* u_0 *belonging to the graph of* φ *problem (2.1) has a unique backward solution* v *in* $C((-\infty, 0]; E_1)$*, such that*

$$\sup_{t<0} \|v(t)\|_{E_1} \le R_0. \tag{2.12}$$

Moreover $\|v(t)e^{-\omega t}\|_{E_1} \to 0$ *as* $t \to -\infty$ *for every* $\omega \in (0, \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma^+(A)\})$*.*

Conversely, if problem (2.1) has a backward solution v *which satisfies (2.12) and* $\|Pv(0)\|_{E_0} \le r_0$*, then* $v(0) \in$ *graph* φ*.*
(ii) There exist R_1*,* $r_1 > 0$ *and a Lipschitz continuous function*

$$\psi : B(0, r_1) \subset (I - P)(E_1) \mapsto P(E_0),$$

differentiable at 0 *with* $\psi'(0) = 0$*, such that for every* u_0 *belonging to the graph of* ψ *problem (2.1) has a unique solution* u *in* $C([0, +\infty); E_1)$ *such that*

$$\sup_{t>0} \|u(t)\|_{E_1} \le R_1. \tag{2.13}$$

Moreover, $\|u(t)e^{\omega t}\|_{E_1} \to 0$ *as* $t \to +\infty$ *for every* $\omega \in (0, -\sup \sigma^-(A))$, *where* $\sigma^-(A) := \{\lambda \in \sigma(A),\ \mathrm{Re}\,\lambda < 0\}$.

Conversely, if problem (2.1) has a solution u *which satisfies (2.13), and* $\|(I-P)u(0)\|_{D_A(\theta+1,\infty)} \leq r_1$, *then* $u(0) \in graph\ \psi$.

(iii) If in addition $F \in C^k(\mathcal{O}; E_0)$ *and* $F^{(k)}$ *is Lipschitz continuous for some* $k \in \mathbb{N}$, *then* ψ *and* φ *are* k *times differentiable, with Lipschitz continuous* k*-th order derivatives.*

As in the case of ordinary differential equations, the construction of center manifolds, or center-unstable manifolds, is more delicate. In addition to (H3) and to $F(0) = 0$ we shall assume that the set $\{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda \geq 0\}$ consists of a finite number of isolated eigenvalues with finite algebraic multiplicity. We shall denote by P_0 the spectral projection associated to it. The fact that the range of P_0 is finite dimensional is of fundamental importance in the proofs.

Applying P_0 and $I - P_0$ we see that problem

$$u'(t) = F(u(t)),\ \ t \geq 0,$$

is equivalent to the system

$$\begin{cases} x'(t) = A_+x(t) + P_0F(x(t) + y(t)), \ \ t \geq 0, \\ \\ y'(t) = A_-y(t) + (I - P_0)F(x(t) + y(t)), \ \ t \geq 0, \end{cases} \tag{2.14}$$

with $x(t) = P_0u(t)$, $y(t) = (I - P_0)u(t)$, $A_+ = A_{|P_0(E_0)} : P_0(E_0) \mapsto P_0(E_0)$, $A_- = A_{|(I-P_0)(E_1)} : (I - P_0)(E_1) \mapsto (I - P_0)(E_0)$.

We modify F by introducing a smooth cutoff function $\rho : P_0(E_0) \mapsto \mathbb{R}$ such that

$$0 \leq \rho(x) \leq 1,\ \ \rho(x) = 1 \ \text{if} \ \|x\|_0 \leq 1/2,\ \ \rho(x) = 0 \ \text{if} \ \|x\|_0 \geq 1.$$

Since $P_0(E_0)$ is finite dimensional, such a ρ does exist. For small $r > 0$ we consider the system

$$\begin{cases} x'(t) = A_+x(t) + f(x(t), y(t)), \ \ t \geq 0, \\ \\ y'(t) = A_-y(t) + g(x(t), y(t)), \ \ t \geq 0, \end{cases} \tag{2.15}$$

with initial data

$$x(0) = x_0 \in P_0(E_0),\ \ y(0) = y_0 \in (I - P_0)(E_0), \tag{2.16}$$

where

$$f(x,y) = P_0F(\rho(x/r)x + y), \;\; g(x,y) = (I - P_0)F(\rho(x/r)x + y).$$

System (2.15) coincides with (2.14) if $\|x(t)\|_{E_0} \le r/2$, and it is possible to show that if r and the initial data are small enough, then the solution of (2.15)-(2.16) exists in the large.

A finite dimensional invariant manifold $\mathcal{M}$ for system (2.15) with small r may be constructed as the graph of a bounded, Lipschitz continuous function $\gamma : P_0(E_0) \mapsto (I - P_0)(E_1)$.

Theorem 2.10 *Under the above assumptions, there exists $r_1 > 0$ such that for $r \le r_1$ there is a Lipschitz continuous function $\gamma : P_0(E_0) \mapsto (I - P_0)(E_1)$ such that the graph of γ is invariant for system (2.15). If in addition F is k times continuously differentiable, with $k \ge 2$, then there exists $r_k > 0$ such that if $r \le r_k$ then $\gamma \in C^{k-1}$, $\gamma^{(k-1)}$ is Lipschitz continuous, and*

$$\gamma'(x)(A_- x + f(x,\gamma(x))) = A_+\gamma(x) + g(x,\gamma(x)), \;\; x \in P_0(X). \tag{2.17}$$

Then it is possible to see that the graph $\mathcal{M}$ of γ attracts exponentially all the orbits which start from an initial datum sufficiently close to $\mathcal{M}$. Moreover each one of these orbits decays exponentially to an orbit in $\mathcal{M}$, in the sense specified by the next theorem.

Theorem 2.11 *Let F be twice continuously differentiable. For every $\omega > 0$ such that $\omega < -\sup\{\mathrm{Re}\;\lambda : \;\; \lambda \in \sigma(A), \mathrm{Re}\;\lambda < 0\}$ there are $r(\omega)$, $M(\omega)$ such that if $\|x_0\|_{E_0}$ and $\|y_0\|_{E_1}$ are sufficiently small, then the solution of (2.15)-(2.16) exists in the large and satisfies*

$$\|y(t) - \gamma(x(t))\|_{E_1} \le M(\omega)e^{-\omega t}\|y_0 - \gamma(x_0)\|_{E_1}, \;\; t \ge 0. \tag{2.18}$$

Moreover there is $C(\omega) > 0$ such that if $\|x_0\|_{E_0}$ and $\|y_0\|_{E_1}$ are small enough there exists $\overline{x} \in P_0(E_0)$ such that

$$\|x(t) - \overline{z}(t)\|_{E_1} + \|y(t) - \gamma(\overline{z}(t))\|_{E_1} \le C(\omega)e^{-\omega t}\|y_0 - \gamma(x_0)\|_{E_0}, \;\; t \ge 0, \tag{2.19}$$

where $\overline{z}(t) = z(t;\gamma,\overline{x})$ is the solution of

$$z' = A_+ z + f(z + \gamma(z)); \quad z(0) = \overline{x}. \tag{2.20}$$

As a consequence, the problem of the stability of the null solution to (2.1) in the critical case

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} = 0, \tag{2.21}$$

is reduced to the stability of the null solution to a finite dimensional system.

Corollary 2.12 *Let $\mathcal{O}$ be a neighborhood of 0 in E_1, and let $F : \mathcal{O} \mapsto E_0$ be a C^2 function satisfying (H3), with $F(0) = 0$. Assume that $A = F'(0)$ satisfies (2.21) and that $\sigma(A) \cap i\mathbb{R}$ consists of a finite number of isolated eigenvalues with finite algebraic multiplicity.*

Then the null solution of (2.1) is stable (respectively, asymptotically stable, unstable) in E_1 if and only if the null solution of the finite dimensional system (2.20) is stable (respectively, asymptotically stable, unstable).

These stability results (precisely, Theorems 2.8, 2.9, 2.10, 2.11, and Corollary 2.12) may be extended to the case where E_0 and E_1 are real interpolation spaces $D_L(\theta,\infty)$, $D_L(\theta+1,\infty)$ instead of continuous interpolation spaces. See [33, Ch. 9].

2.3 Applications and drawbacks

Let us describe the applicability of the abstract theory in a simple significant example, Ω being a bounded open set in $\mathbb{R}^N$ with regular boundary $\partial\Omega$:

$$\begin{cases} u_t(t,x) = \Phi(D^2u(t,x)), \quad t \geq 0, \; x \in \Omega, \\ \\ \mathcal{B}u(t,x) = 0, \quad t \geq 0, \; x \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{2.22}$$

Here $\mathcal{B}$ is a first order differential operator with regular coefficients,

$$\mathcal{B}u = \sum_{i=1}^{N} \beta_i(x) D_i u(x) + \gamma(x)u(x),$$

satisfying the nontangentiality condition

$$\sum_{i=1}^{N} \beta_i(x)\nu_i(x) \neq 0, \quad x \in \partial\Omega, \tag{2.23}$$

where $\nu(x)$ is the unit exterior normal vector to $\partial\Omega$ at x. We may consider also a Cauchy-Dirichlet problem,

$$\begin{cases} u_t(t,x) = \Phi(D^2u(t,x)), \;\; t \geq 0, \; x \in \Omega, \\ \\ u(t,x) = 0, \;\; t \geq 0, \; x \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \;\; x \in \Omega. \end{cases} \tag{2.24}$$

The initial datum u_0 is a regular (at least, C^2) function satisfying the natural compatibility condition $\mathcal{B}u_0 = 0$ at $\partial\Omega$ for problem (2.22), or $u_0 = 0$ at $\partial\Omega$ for problem (2.24). Moreover Φ is a regular nonlinear function defined in a neighborhood of the range of D^2u_0 in $\mathbb{R}^{N^2}$, satisfying an ellipticity assumption

$$\sum_{i,j=1}^{N} \frac{\partial\Phi}{\partial q_{ij}}(Q)\xi_i\xi_j \geq \nu|\xi|^2, \;\; x \in \overline{\Omega}, \tag{2.25}$$

and the symmetry condition

$$\Phi(Q) = \Phi(Q^*) \tag{2.26}$$

for every matrix Q with entries close to the range of D^2u_0.

Let us see how we can apply Theorem 2.6 to problem (2.22). The choice $X = L^p(\Omega)$, $D = \{\varphi \in W^{2,p}(\Omega) : \; \mathcal{B}\varphi = 0 \text{ at } \partial\Omega\}$ does not work, because the function

$$F(\varphi)(x) = \Phi(D^2\varphi(x)), \;\; x \in \Omega, \tag{2.27}$$

does not map D into X, unless Φ has (not more than) linear growth. For instance, if Φ is a quadratic polynomial then F maps D into $L^{p/2}(\Omega)$. Much worse, even if Φ has linear growth, F is not differentiable unless Φ is linear.

After L^p and $W^{2,p}$, the simplest choice for the spaces D and X seems to be $X = C(\overline{\Omega})$, the space of the continuous functions from $\overline{\Omega}$ to $\mathbb{R}$, and $D = \{\varphi \in C^2(\overline{\Omega}) : \; \mathcal{B}\varphi = 0 \text{ at } \partial\Omega\}$. If Φ is smooth enough, assumption (H1) is easily checked for the function F defined in (2.27).

But assumption (H2) does not hold, unless $N = 1$. Indeed, $F'(u_0)$ is the realization of the elliptic operator $\mathcal{A}$ in $C(\overline{\Omega})$ with the above boundary condition, where

$$(\mathcal{A}\varphi)(x) = \sum_{i,j=1}^{N} \frac{\partial\Phi}{\partial q_{ij}}(D^2u_0(x))D_i\varphi(x)D_j\varphi(x), \;\; x \in \Omega \tag{2.28}$$

which is sectorial in $C(\overline{\Omega})$ thanks to the Stewart's Theorems ([39, 40], [33, Ch. 3]), but whose domain contains properly $C^2(\overline{\Omega})$. This is not due to the lack of regularity of the coefficients, or to the boundary condition, but it is a structural well known difficulty, shared by all the elliptic operators including the Laplacian: if φ and $\Delta\varphi$ are continuous in some open set, φ is not necessarily a C^2 function. Of course this difficulty disappears in dimension 1. So, we may apply Theorem 2.6 with $X = C(\overline{\Omega})$ either in dimension $N = 1$, or for special nonlinearities, for example $F(u)(x) = \Psi(\Delta u(x))$, where we can take D as the domain of the Laplacian in X. For the details, see [33, Ch. 8].

If we replace $C(\overline{\Omega})$ by its subspace $C^\theta(\overline{\Omega})$ of the bounded and uniformly θ-Hölder continuous functions, and we take $D = \{\varphi \in C^{2+\theta}(\overline{\Omega}) : \mathcal{B}\varphi = 0$ at $\partial\Omega\}$, the function F defined in (2.27) satisfies (H1) if Φ is smooth enough, and the classical Schauder type theorems plus generation theorems in Hölder spaces (see e.g. [33, Ch. 3]) show that also assumption (H2) is satisfied. So, Theorem 2.6 may be applied, and if the compatibility condition $F(u_0) \in \overline{D}$ holds, we get a local existence and uniqueness result. It is well known that D is not dense in X, and that the closure of D in X is a set of little-Hölder continuous functions: $\overline{D} = h^\theta(\overline{\Omega})$.

The space $h^\theta(\overline{\Omega})$ may be characterized as the subset of $C^{2+\theta}(\overline{\Omega})$ consisting of the functions φ such that

$$\lim_{h \to 0} \sup_{x,y \in \Omega,\, 0 < |x-y| \leq h} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\theta} = 0,$$

and it coincides with the closure of $C^\infty(\overline{\Omega})$ in $C^\theta(\overline{\Omega})$ if $\partial\Omega$ is C^∞. Similarly, $h^{2+\theta}(\overline{\Omega})$ is the subset of $C^{2+\theta}(\overline{\Omega})$ consisting of the functions with second order derivatives in $h^\theta(\overline{\Omega})$, and it coincides with the closure of $C^\infty(\overline{\Omega})$ in $C^{2+\theta}(\overline{\Omega})$ if $\partial\Omega$ is C^∞.

The assumption $F(u_0) \in \overline{D}$ holds provided $u_0 \in h^{2+\theta}(\overline{\Omega})$. This extra regularity assumption on the initial datum is preserved throughout the evolution because Theorem 2.6 implies that $u'(t) = F(u(t))$ belongs to $(X, D)_{\alpha,\infty} \subset \overline{D}$ for $t > 0$. Therefore, the choice of working in Hölder spaces leads naturally to little-Hölder spaces, and we can choose $X = h^\theta(\overline{\Omega})$, $D = \{\varphi \in h^{2+\theta}(\overline{\Omega}) : \mathcal{B}\varphi = 0$ at $\partial\Omega\}$ from the very beginning. Indeed, a Schauder type theorem and a generation of analytic semigroups theorem hold in the space of the little-Hölder continuous functions, as follows (see e.g. [33, Ch. 3]):

Theorem 2.13 *Let* $0 < \theta < 1$ *and let* $\partial\Omega$ *be of class* $h^{2+\theta}$. *Assume that the coefficients* a_{ij}, b_i, c *are in* $h^\theta(\overline{\Omega})$ *and satisfy the ellipticity condition*

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j > 0, \quad x \in \overline{\Omega},$$

and that the coefficients β_i, γ *are in* $h^{1+\theta}(\partial\Omega)$ *and satisfy the nontangentiality condition (2.23). Let* $\mathcal{A}$, $\mathcal{B}$ *be the differential operators defined by* $\mathcal{A} = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij} + \sum_{i=1}^{N} b_i D_i + c$, *and* $\mathcal{B} = \sum_{i=1}^{N} \beta_i D_i + \gamma$.

If $\varphi \in \cap_{p>1} W^{2,p}(\Omega)$ *is such that* $\mathcal{A}\phi \in h^\theta(\overline{\Omega})$, $\mathcal{B}\varphi = 0$ *at* $\partial\Omega$, *then* $\varphi \in h^{2+\theta}(\overline{\Omega})$, *and there exists* $C > 0$, *independent of* ϕ, *such that*

$$\|\varphi\|_{C^{2+\theta}(\overline{\Omega})} \leq C(\|\mathcal{A}\varphi\|_{C^\theta(\overline{\Omega})} + \|\varphi\|_\infty).$$

The same conclusion holds if the boundary operator $\mathcal{B}$ *is replaced by the trace operator.*

Theorem 2.14 *Let the assumptions of Theorem 2.13 hold. Then the realization of the operator* $\mathcal{A}$ *in* $h^\theta(\overline{\Omega})$, *with domain* $\{\varphi \in h^{2+\theta}(\overline{\Omega}) : \mathcal{B}\varphi = 0$ *at* $\partial\Omega\}$, *is sectorial in* $h^\theta(\overline{\Omega})$.

The application of Theorem 2.6 gives the following result.

Theorem 2.15 *Let* $0 < \theta < 1$. *Assume that* $\partial\Omega$ *is of class* $h^{2+\theta}$, *let the coefficients* β_i, $\gamma \in h^{1+\theta}(\partial\Omega)$ *satisfy the nontangentiality condition (2.23), and let* $u_0 \in h^{2+\theta}(\overline{\Omega})$ *satisfy the compatibility condition*

$$\sum_{i=1}^{N} \beta_i(x) D_i u_0(x) + \gamma(x) u_0(x) = 0, \quad x \in \partial\Omega.$$

Let Φ *be a* C^3 *function defined in a neighborhood of the range of* $D^2 u_0$, *satisfying the ellipticity condition (2.25) and the symmetry condition (2.26).*

Then there exists a maximal $\tau > 0$ *such that problem (2.22) has a solution* $u(t,x)$ *such that* $t \mapsto u(t,x)$ *belongs to* $C([0,\tau); h^{2+\theta}(\overline{\Omega})) \cap C^1([0,\tau); h^\theta(\overline{\Omega}))$. *For every* $\varepsilon \in (0,\tau)$ *and* $\beta \in (0,1)$, $u(t,\cdot) \in C^\beta_\beta((0,\tau-\varepsilon]; h^{2+\theta}(\overline{\Omega}))$ *and* $u_t(t,\cdot) \in C^\beta_\beta((0,\tau-\varepsilon]; h^\theta(\overline{\Omega}))$. u *is the unique solution to (2.22) with such regularity properties. Moreover, it depends continuously on the initial datum* u_0 *in the sense specified by statement (iv) of Theorem 2.6.*

A further difficulty arises if the boundary operator $\mathcal{B}$ is replaced by the trace, i.e., if we consider problem (2.24) instead of (2.22). Simple counterexamples in dimension 1 show that the realizations of second order elliptic operators with smooth coefficients and Dirichlet boundary condition in Hölder and little-Hölder spaces are not sectorial in general. The difficulty is due to the Dirichlet boundary condition, and it may be avoided replacing $C^\theta(\overline{\Omega})$ or $h^\theta(\overline{\Omega})$ by their subspaces consisting of functions that vanish at the boundary. See [33, Ch. 3] for a discussion. In the case of the choice $X = h^\theta_0(\overline{\Omega}) = \{\phi \in h^\theta(\overline{\Omega}) : \ \phi = 0 \text{ at } \partial\Omega\}$, the domain of $F'(u_0)$ is the subset of $h^{2+\theta}(\overline{\Omega})$ consisting of the functions φ such that φ and $F'(u_0)\varphi$ vanish at $\partial\Omega$; in general it does not coincide with the domain of $F'(u_1)$ for $u_1 \neq u_0$. Therefore we are not able to find a common domain D to apply Theorem 2.6, unless Φ is of a special type. For instance, if instead of a function $\Phi = \Phi(D^2u)$ we have $\Phi = \Phi(D^2u, Du, u)$ and $\Phi(q,p,0) = 0$ for each q and p, we are done, and a theorem similar to (2.15) holds.

Now let us see how we can apply Theorem 2.7. The space E is still $C(\overline{\Omega})$, $F'(u_0)$ is the realization of the operator $\mathcal{A}$ defined in (2.28) with homogeneous boundary condition, and

$$D_{F'(u_0)}(\alpha) = h^{2\alpha}(\overline{\Omega}),$$

for $\alpha < 1/2$,

$$D_{F'(u_0)}(\alpha) = \{\varphi \in h^{2\alpha}(\overline{\Omega}) : \ \mathcal{B}\varphi = 0 \text{ at } \partial\Omega\},$$

for $\alpha > 1/2$. See [33, Ch. 3].

If u_0 and $\partial\Omega$ are smooth enough, that is $\partial\Omega \in h^{2+2\alpha}$, $u_0 \in h^{2+2\alpha}(\overline{\Omega})$, Theorem 2.13 yields

$$D_{F'(u_0)}(\alpha+1) = \{\varphi \in h^{2+2\alpha}(\overline{\Omega}) : \ \mathcal{B}\varphi = 0 \text{ at } \partial\Omega\},$$

for $\alpha < 1/2$,

$$D_{F'(u_0)}(\alpha+1) = \{\varphi \in h^{2+2\alpha}(\overline{\Omega}) : \ \mathcal{B}\varphi = 0, \ \mathcal{B}(F'(u_0)\varphi) = 0 \text{ at } \partial\Omega\},$$

for $\alpha > 1/2$.

Fixed $\theta \in (0,1)$ we take $\alpha = \theta/2 \in (0,1/2)$ and we may apply Theorem 2.7, with $E_0 = h^\theta(\overline{\Omega})$, $E_1 = \{\varphi \in h^{2+\theta}(\overline{\Omega}) : \ \mathcal{B}\varphi = 0 \text{ at } \partial\Omega\}$. It is easy to see that the regularity assumption in (H3) is satisfied if Φ is a C^2 function, and the other assumptions in (H3) are satisfied thanks to Theorems 2.13 and 2.14. The final result is the following.

Theorem 2.16 *Let $0 < \theta < 1$. Assume that $\partial\Omega$ is of class $h^{2+\theta}$, let the coefficients β_i, $\gamma \in h^{1+\theta}(\partial\Omega)$ satisfy the nontangentiality condition (2.23), and let $u_0 \in h^{2+\theta}(\overline{\Omega})$ satisfy the compatibility condition*

$$\sum_{i=1}^{N} \beta_i(x) D_i u(x) + \gamma(x) u(x) = 0, \quad x \in \partial\Omega.$$

Let Φ be a C^2 function defined in a neighborhood of the range of $D^2 u_0$, satisfying the ellipticity condition (2.25) and the symmetry condition (2.26).

Then there exists a maximal $\tau > 0$ such that problem (2.22) has a unique solution $u(t,x)$ such that $t \mapsto u(t,x)$ belongs to $C([0,\tau); h^{2+\theta}(\overline{\Omega})) \cap C^1([0,\tau); h^{\theta}(\overline{\Omega}))$. It depends continuously on the initial datum u_0 in the sense specified in Theorem 2.7.

So, there is not much difference between this theorem and Theorem 2.15. Here we do not get extra time regularity, but Φ can be taken of class C^2 instead of C^3. Substantially, the applications of Theorems 2.6 and 2.7 to problem (2.22) give the same results.

In any case, problems with nonlinear boundary condition of the type $G(Du(t,x)) = 0$ for $x \in \partial\Omega$, with nonlinear smooth G, cannot be treated by a direct application of Theorems 2.6 and 2.7, even in the case of linear Φ. First, the boundary condition has to be incorporated in the domain D (if we use Theorem 2.6) or in the domain E_1 (if we use Theorem 2.7), but D and E_1 have to be linear spaces and the boundary condition is nonlinear. Second, and more important, even if we rewrite the boundary condition as a linear boundary condition plus a rest and then try to get rid of the rest using suitable trace theorems, the domain of the linearized operator changes with u_0. Take for instance $\Phi(D^2u) = \Delta u$, $X = h^\theta(\overline{\Omega})$. Assuming that the linearized operator

$$\varphi \mapsto \sum_{i=1}^{N} D_i G(Du_0(x)) D_i \varphi(x), \quad x \in \partial\Omega$$

is nontangential, the realization of the Laplace operator in $h^\theta(\overline{\Omega})$, with domain $D = \{\varphi \in h^{2+\theta}(\overline{\Omega}) : \sum_{i=1}^{N} D_i G(Du_0(x)) D_i \varphi(x) = 0 \text{ at } \partial\Omega\}$ is in fact sectorial, but its domain strongly depends on u_0.

For this type of problems, a direct approach in Hölder spaces seems to be simpler and more fruitful than applying abstract results. This approach is described in next section.

3 Equations and systems in Hölder spaces

Throughout this section we shall consider an open set $\Omega \subset \mathbb{R}^N$ with uniformly $C^{2+\theta}$ boundary, $0 < \theta < 1$. This means that there is $r > 0$ such that for each $x_0 \in \partial\Omega$ there is a $C^{2+\theta}$ diffeomorphism φ from the open ball $B(x_0, r)$ centered at x_0 and radius r to the unit open ball in $\mathbb{R}^N$ with the property that $\varphi(\Omega \cap B(x_0, r)) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{N-1} : |(x, y)| < 1, \, x < 0\}$; moreover, the $C^{2+\theta}$ norm of the diffeomorphisms and of their inverses are bounded by a constant independent of x_0.

For $k \in \mathbb{N}$, $C_b^k(\overline{\Omega})$ denotes the space of the functions with continuous bounded derivatives in $\overline{\Omega}$, and for $k \in \mathbb{N}$, $0 < \theta < 1$, $C^{k+\theta}(\overline{\Omega})$ denotes the subspace of $C_b^k(\overline{\Omega})$ consisting of the functions with uniformly θ-Hölder continuous k-th order derivatives.

We shall use the parabolic Hölder spaces $C^{\theta/2,\theta}(I \times \overline{\Omega})$, $C^{1/2+\theta/2,1+\theta}(I \times \overline{\Omega})$, $C^{1+\theta/2,2+\theta}(I \times \overline{\Omega})$, I being a real interval, $0 < \theta < 1$, with the usual meanings and norms.

We recall that a function w belongs to $C^{\theta/2,\theta}(I \times \overline{\Omega})$ if and only if w is bounded and moreover

$$[w]_{C^{\theta/2,\theta}(I\times\overline{\Omega})} = \sup_{x\in\overline{\Omega}} [w(\cdot, x)]_{C^{\theta/2}(I)} + \sup_{t\in I} [w(t, \cdot)]_{C^{\theta}(\overline{\Omega})} < \infty.$$

In this case we set

$$\|w\|_{C^{\theta/2,\theta}(I\times\overline{\Omega})} = \|w\|_\infty + [w]_{C^{\theta/2,\theta}(I\times\overline{\Omega})}.$$

$w \in C^{1/2+\theta/2,1+\theta}(I \times \overline{\Omega})$ if and only if w and its first order space derivatives $D_i w$ are bounded and moreover

$$[w]_{C^{1/2+\theta/2,1+\theta}(I\times\overline{\Omega})} = \sup_{x\in\overline{\Omega}} [w(\cdot, x)]_{C^{1/2+\theta/2}(I)} + \sum_{i=1}^N \sup_{t\in I} [D_i w(t, \cdot)]_{C^{\theta}(\overline{\Omega})} < \infty.$$

In this case we set

$$\|w\|_{C^{1/2+\theta/2,1+\theta}(I\times\overline{\Omega})} = \|w\|_\infty + \sum_{i=1}^N \|D_i w\|_\infty + [w]_{C^{1/2+\theta/2,1+\theta}(I\times\overline{\Omega})}.$$

The space $C^{1+\theta/2,2+\theta}(I \times \overline{\Omega})$ is defined similarly. w belongs to $C^{1+\theta/2,2+\theta}(I \times \overline{\Omega})$ if and only if w is bounded, there exist the derivatives w_t, $D_{ij}w$ for i, $j = 1, \dots N$, and they belong to $C^{\theta/2,\theta}(I \times \overline{\Omega})$. It is easy to see that in this case w belongs also to $C^{1/2+\theta/2,1+\theta}(I \times \overline{\Omega})$. The norm is

$$\|w\|_{C^{1+\theta/2,2+\theta}(I\times\overline{\Omega})} = \|w\|_{C^{1/2+\theta/2,1+\theta}(I\times\overline{\Omega})} + \|w_t\|_{C^{\theta/2,\theta}(I\times\overline{\Omega})}$$

$$+ \sum_{i,j=1}^{N} \|D_{ij}w\|_{C^{\theta/2,\theta}(I\times\overline{\Omega})}.$$

The spaces $C^{\theta/2,\theta}(I \times \partial\Omega)$ and $C^{1/2+\theta/2,1+\theta}(I \times \partial\Omega)$ are defined similarly, by means of a smooth ($C^{2+\theta}$) atlas for $\partial\Omega$. There exists $C_\theta > 0$ such that for every $w \in C^{\theta/2,\theta}(I \times \overline{\Omega})$ we have $\|w_{|I\times\partial\Omega}\|_{C^{\theta/2,\theta}(I\times\partial\Omega)} \leq C_\theta\|w\|_{C^{\theta/2,\theta}(I\times\overline{\Omega})}$, and similarly for every $w \in C^{1/2+\theta/2,1+\theta}(I \times \overline{\Omega})$ we have $\|w_{|I\times\partial\Omega}\|_{C^{1/2+\theta/2,1+\theta}(I\times\partial\Omega)} \leq C_\theta\|w\|_{C^{1/2+\theta/2,1+\theta}(I\times\overline{\Omega})}$.

Let us come back to problem

$$\begin{cases} u_t(t,x) = \Phi(D^2u(t,x)), & t \geq 0, \ x \in \overline{\Omega}, \\ \Psi(Du(t,x)) = 0, & t \geq 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}, \end{cases}$$

where u_0 is a regular function defined in $\overline{\Omega}$, and Φ, Ψ are regular functions defined in a neighborhood of the range of D^2u_0 and of Du_0, respectively. We need also symmetry and ellipticity assumptions of the type (2.26), (2.25) on Φ, as well as a nontangentiality assumption on Ψ. Precisely, we assume that there are open sets $\mathcal{O}_1 \subset \mathbb{R}^{N^2}$, $\mathcal{O}_2 \subset \mathbb{R}^N$ such that

$$\sum_{i,j=1}^{N} \frac{\partial\Phi}{\partial q_{ij}}(Q)\xi_i\xi_j > 0, \quad Q \in \mathcal{O}_1, \ \xi \in \mathbb{R}^N, \tag{3.1}$$

$$\Phi(Q) = \Phi(Q^*), \quad Q \in \mathcal{O}_1, \tag{3.2}$$

and

$$\sum_{i=1}^{N} \frac{\partial\Psi}{\partial p_i}(p)\nu_i(x) \neq 0, \quad p \in \mathcal{O}_2, \ x \in \partial\Omega, \tag{3.3}$$

where $\nu(x)$ is the unit exterior normal vector to $\partial\Omega$ at x.

Under these conditions, problem (1.1) is the simplest significant example of a fully nonlinear parabolic problem with fully nonlinear boundary condition. We give a complete proof of the local existence theorem for (1.1) because it exhibits the typical difficulties of fully nonlinear problems, but the technical points are reduced to the minimum and it is easy to see to which extent the proof itself may be generalized.

Also in this case we need an optimal regularity theorem for linear equations, the popular Ladyzhenskaja – Solonnikov – Ural'ceva Theorem ([28, Ch. 4]). In next Theorem 3.1, $\mathcal{A}$ is a linear second order differential operator,

$$(\mathcal{A}v)(\xi) = \sum_{i,j=1}^{N} a_{ij}(\xi)D_{ij}v(\xi) + \sum_{i=1}^{N} b_i(\xi)D_i v(\xi) + c(\xi)v(\xi), \quad \xi \in \overline{\Omega}, \tag{3.4}$$

satisfying the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(\xi)\eta_i\eta_j \geq \nu|\eta|^2, \quad \xi \in \overline{\Omega},\ \eta \in \mathbb{R}^N, \tag{3.5}$$

for some $\nu > 0$, and $\mathcal{B}$ is a linear first order differential operator,

$$(\mathcal{B}v)(\xi) = \gamma(\xi)v(\xi) + \sum_{i=1}^{N} \beta_i(\xi)D_i v(\xi), \quad \xi \in \partial\Omega. \tag{3.6}$$

satisfying the nontangentiality condition

$$\sum_{i=1}^{N} \beta_i(\xi)\nu_i(\xi) \neq 0, \quad \xi \in \partial\Omega. \tag{3.7}$$

Theorem 3.1 *Fix $\theta \in (0,1)$ and $T > 0$. Let Ω be an open set in $\mathbb{R}^N$ with uniformly $C^{2+\theta}$ boundary. Let a_{ij}, b_i, $c \in C^\theta(\overline{\Omega})$ satisfy (3.5), and let β_i, $\gamma \in C^{1/2+\theta/2,1+\theta}([0,T] \times \partial\Omega)$ satisfy (3.7). Define the operators $\mathcal{A}$ and $\mathcal{B}$ by (3.4), (3.6), respectively. Then for every $f \in C^{\theta/2,\theta}([0,T] \times \overline{\Omega})$, $g \in C^{1/2+\theta/2,1+\theta}([0,T] \times \partial\Omega)$ satisfying the compatibility condition*

$$\mathcal{B}w_0(0,\cdot) = g(0,\cdot) \ \text{ in } \partial\Omega, \tag{3.8}$$

the problem

$$\begin{cases} w_t = \mathcal{A}w + f, \ \ 0 \leq t \leq T,\ \xi \in \overline{\Omega}, \\ \mathcal{B}w = g, \ \ 0 \leq t \leq T,\ \xi \in \partial\Omega, \\ w(0,x) = w_0, \ \ \xi \in \overline{\Omega}, \end{cases} \tag{3.9}$$

has a unique solution $w \in C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$. *Moreover there exists* $C = C(T) > 0$, *increasing with respect to* T, *such that*

$$\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \leq$$
$$\leq C(\|f\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} + \|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)} + \|w_0\|_{C^{2+\theta}(\overline{\Omega})}). \tag{3.10}$$

Now we are ready for the proof of the local existence and uniqueness theorem.

Theorem 3.2 *Let* Ω *be an open set in* $\mathbb{R}^N$ *with uniformly* $C^{2+\theta}$ *boundary,* $0 < \theta < 1$. *Let* $\Phi : \mathcal{O}_1 \mapsto \mathbb{R}$ *be a* C^2 *function satisfying (3.2), (3.1), and let* $\Psi : \mathcal{O}_2 \mapsto \mathbb{R}$ *be a* C^3 *function satisfying (3.3).*

Then for each $u_0 \in C^{2+\theta}(\overline{\Omega})$ *such that the range of* D^2u_0 *is contained in* $\mathcal{O}_1$, *the range of* Du_0 *is contained in* $\mathcal{O}_2$, *and satisfying the compatibility condition*

$$\Psi(Du_0(x)) = 0, \quad x \in \partial\Omega,$$

there exist $T > 0$ *a unique* $u \in C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$ *that solves (1.1) in* $[0,T]\times\overline{\Omega}$.

Proof — Let $\mathcal{A}$ and $\mathcal{B}$ be the operators defined by

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} \frac{\partial\Phi}{\partial q_{ij}}(D^2u_0(x))D_{ij}\varphi(x), \quad x \in \overline{\Omega},$$

$$\mathcal{B}\varphi(x) = \sum_{i=1}^{N} \frac{\partial\Psi}{\partial p_i}(Du_0(x))D_i\varphi(x), \quad x \in \overline{\Omega}.$$

Moreover, set

$$\begin{aligned} Y &= \{u \in C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega}) :\ u(0,\cdot) = u_0, \\ &\qquad \|u - u_0\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \leq R\}, \end{aligned}$$

where the positive numbers T and R have to be chosen later.

The solution to (1.1) is sought as a fixed point of the operator Γ defined in Y by $\Gamma u = w$, w being the solution of

$$\begin{cases} w_t(t,x) = \mathcal{A}w(t,x) + \Phi(D^2u(t,x)) - \mathcal{A}u(t,x), & 0 \leq t \leq T,\ x \in \overline{\Omega}, \\ \mathcal{B}w(t,x) = \mathcal{B}u(t,x) - \Psi(Du(t,x)), & 0 \leq t \leq T,\ x \in \partial\Omega, \\ w(0,x) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{3.11}$$

We have to choose T and R in such a way that Γ is well defined, it maps Y into itself, it is a contraction with constant less than 1, and the unique fixed point of Γ in Y is in fact the unique solution to (1.1).

For Γ be well defined, for every $u \in Y$ the ranges of $D^2u(t,\cdot)$ and of $Du(t,\cdot)$ need to be contained in $\mathcal{O}_1$ and in $\mathcal{O}_2$. If $\mathcal{O}_1$ contains the closure of the neighborhood of the range of D^2u_0 with radius r_1 and $\mathcal{O}_2$ contains the closure of the neighborhood of the range of Du_0 with radius r_2, we take T, R such that

$$T^{\theta/2}R \le r_1/2, \;\; T^{1/2+\theta/2}R \le r_2/2. \tag{3.12}$$

So, the compositions $\Phi(D^2u)$ and $\Psi(Du)$ are well defined for each $u \in Y$, and they belong to $C^{\theta/2,\theta}([0,T]\times\overline{\Omega})$, and to $C^{1/2+\theta/2,1+\theta}([0,T]\times\overline{\Omega})$, respectively. The compatibility condition (3.8) holds, and then by Theorem 3.1 problem (3.11) has a unique solution $w \in C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$.

Let us prove that Γ is a 1/2-contraction, for suitable T and R. The constants $C(T)$ given by Theorem 3.1 increase with T. So we take

$$T \le 1. \tag{3.13}$$

For each u, $v \in Y$, $\Gamma u - \Gamma v$ is the solution to (3.9) with $w_0 = 0$ and

$$\begin{aligned} f(t,x) &= \Phi(D^2u(t,x)) - \Phi(D^2v(t,x)) - \mathcal{A}(u-v)(t,x) \\ &= \int_0^1 \langle D\Phi((\sigma D^2u(t,x) + (1-\sigma)D^2v(t,x)) - D\Phi(D^2u_0(x)), \\ &\quad D^2u(t,x) - D^2v(t,x)\rangle d\sigma, \\ g(t,x) &= \mathcal{B}(u-v)(t,x) - \Psi(Du(t,x)) + \Psi(Dv(t,x)) \\ &= \int_0^1 \langle D\Psi(Du_0(x)) - D\Psi(\sigma Du(t,x) + (1-\sigma)Dv(t,x)), \\ &\quad Du(t,x) - Dv(t,x)\rangle d\sigma. \end{aligned}$$

Theorem 3.1 gives now,

$$\|\Gamma u - \Gamma v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le$$

$$\le C(\|f\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} + \|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\overline{\Omega})}),$$

where $C = C(1)$.

Let us estimate $\|f\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})}$. Since $u(0,\cdot)=v(0,\cdot)=u_0$, for each $t\in[0,T]$ we have

$$\|D^2u(t,\cdot)-D^2u_0\|_{L^\infty(\Omega)}\le T^{\theta/2}R,\quad \|D^2v(t,\cdot)-D^2u_0\|_{L^\infty(\Omega)}\le T^{\theta/2}R,$$

and

$$\|D^2u(t,\cdot)-D^2v(t,\cdot)\|_{L^\infty(\Omega)}\le T^{\theta/2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Therefore, for t, $s\in[0,T]$ and $x\in\overline{\Omega}$ we have

$$\begin{aligned}
&f(t,x)-f(s,x)=\\
&=\int_0^1 \langle D\Phi(\sigma D^2u(t,x)+(1-\sigma)D^2v(t,x))-D\Phi(D^2u_0(x)),\\
&\qquad D^2u(t,x)-D^2v(t,x)\rangle d\sigma\\
&-\int_0^1 \langle D\Phi(\sigma D^2u(s,x)+(1-\sigma)D^2v(s,x))-D\Phi(D^2u_0(x)),\\
&\qquad D^2u(s,x)-D^2v(s,x)\rangle d\sigma\\
&=\int_0^1 \langle D\Phi(\sigma D^2u(t,x)+(1-\sigma)D^2v(t,x))\\
&\qquad -D\Phi(\sigma D^2u(s,x)+(1-\sigma)D^2v(s,x)),D^2u(t,x)-D^2v(t,x)\rangle d\sigma\\
&+\int_0^1 \langle D\Phi(\sigma D^2u(s,x)+(1-\sigma)D^2v(s,x))-D\Phi(D^2u_0(x)),\\
&\qquad D^2u(t,x)-D^2v(t,x)-D^2u(s,x)+D^2v(s,x)\rangle d\sigma
\end{aligned}$$

Let L_1 be the supremum of $|D^2\Phi|$ in the neighborhood of the range of D^2u_0

with radius r_1. Then we have

$$|f(t,x) - f(s,x)| \leq$$

$$\leq \int_0^1 L_1(\sigma(t-s)^{\theta/2} + (1-\sigma)(t-s)^{\theta/2})\|D^2u(t,\cdot) - D^2v(t,\cdot)\|_{L^\infty(\Omega)}d\sigma$$

$$+ \quad \int_0^1 L_1(\sigma T^{\theta/2}R + (1-\sigma)T^{\theta/2}R)d\sigma(t-s)^{\theta/2}\cdot$$

$$\cdot[D^2u(\cdot,x) - D^2v(\cdot,x)\|_{C^{\theta/2}([0,T])}$$

$$\leq 2L_1T^{\theta/2}R(t-s)^{\theta/2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Recalling that $f(0,x) = 0$, so that $\|f(\cdot,x)\|_\infty \leq T^{\theta/2}[f(\cdot,x)]_{C^{\theta/2}([0,T])}$, we get for each $x \in \overline{\Omega}$

$$\|f(\cdot,x)\|_{C^{\theta/2}([0,T])} \leq 2L_1(T^{\theta/2} + T^\theta)R\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

To evaluate $[f(t,\cdot)]_{C^\theta(\overline{\Omega})}$ we recall that

$$[\varphi\psi]_{C^\theta(\overline{\Omega})} \leq \|\varphi\|_\infty[\psi]_{C^\theta(\overline{\Omega})} + [\varphi]_{C^\theta(\overline{\Omega})}\|\psi\|_\infty.$$

Therefore

$$[f(t,\cdot)]_{C^\theta(\overline{\Omega})} \leq$$

$$\leq \quad \int_0^1 \|D\Phi(\sigma D^2u(t,\cdot) + (1-\sigma)D^2v(t,\cdot)) - D\Phi(D^2u_0)\|_\infty d\sigma\,\cdot$$

$$\cdot[D^2u(t,\cdot) - D^2v(t,\cdot)]_{C^\theta(\overline{\Omega})}$$

$$+ \quad \int_0^1 [D\Phi(\sigma D^2u(t,\cdot) + (1-\sigma)D^2v(t,\cdot)) - D\Phi(D^2u_0)]_{C^\theta(\overline{\Omega})}d\sigma\cdot$$

$$\cdot\|D^2u(t,\cdot) - D^2v(t,\cdot)\|_\infty$$

$$\leq \int_0^1 L_1(\sigma t^{\theta/2}R + (1-\sigma)t^{\theta/2}R)d\sigma\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}$$

$$+\int_0^1 L_1(\sigma R + (1-\sigma)R)d\sigma\, T^{\theta/2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}$$

$$\leq 2L_1T^{\theta/2}R\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Summing up,

$$\|f\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} \le L_1(4T^{\theta/2} + 2T^{\theta})R\|u - v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}. \qquad (3.14)$$

Let us estimate $\|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)}$. We recall that there exists $C_\theta > 0$ such that $\|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)} \le C_\theta\|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\overline{\Omega})}$.

Let L_2, L_3 be the suprema of $|D^2\Psi|$, $|D^3\Psi|$ in the neighborhood of the range of Du_0 with radius r_2. For each $x \in \overline{\Omega}$, the estimate for the seminorm $\|g(\cdot,x)\|_{C^{1/2+\theta/2}([0,T])}$ is obtained as the estimate for $\|f(\cdot,x)\|_{C^{\theta/2}([0,T])}$. We get

$$\|g(\cdot,x)\|_{C^{1/2+\theta/2}([0,T])} \le 2L_2(T^{1/2+\theta/2} + T^{1+\theta})R\|u - v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

To estimate $\|g(t,\cdot)\|_{C^{1+\theta}(\overline{\Omega})}$ we write down the first order derivatives of g:

$$\begin{aligned}
&D_i g = \\
&\int_0^1 \sum_{k=1}^{N}(D_k\Psi(Du_0) - D_k\Psi((\sigma Du(t,x) + (1-\sigma)Dv)(D_{ik}u - D_{ik}v)d\sigma \\
&-\int_0^1 \sum_{j,k=1}^{N} \big(D_{kj}\Psi(\sigma Du + (1-\sigma)Dv)D_{ji}(\sigma u + (1-\sigma)v) \\
&\qquad -D_{kj}\Psi(Du_0)D_{ji}u_0\big)(D_k u - D_k v)d\sigma \\
&:= h_i(t,x) + m_i(t,x)
\end{aligned}$$

The functions h_i are estimated like f, and we get

$$\|h_i\|_\infty \le NL_2T^{1/2+\theta}R\,\|u - v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

$$[h_i]_{C^\theta(\overline{\Omega})} \le 2NL_2(T^{\theta/2} + T^{1/2+\theta/2})R\,\|u - v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Concerning the functions m_i, we recall that

$$\|D_k u - D_k v\|_\infty \le T^{1/2+\theta/2}\|u - v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

and from the inequality

$$[\varphi]_{C^\theta(\overline{\Omega})} \le K\|\varphi\|_\infty^{1-\theta}\|D\varphi\|_\infty^{\theta}$$

we get

$$
\begin{aligned}
&[D_k u(t,\cdot) - D_k v(t,\cdot)\|_{C^\theta(\overline{\Omega})}\\
&\le K\|Du(t,\cdot) - Dv(t,\cdot)\|_\infty^{1-\theta}\|D^2u(t,\cdot) - D^2v(t,\cdot)\|_\infty^\theta\\
&\le KT^{1/2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|D_{kj}\Psi(\sigma Du + (1-\sigma)Dv)D_{ji}(\sigma u + (1-\sigma)v) - D_{kj}\Psi(Du_0)D_{ji}u_0\|_{C^\theta(\overline{\Omega})}\\
\le C(L_3, R, \|u_0\|_{C^{2+\theta}(\overline{\Omega})}),
\end{aligned}
$$

with C increasing in all its arguments, we get

$$
\|m_i\|_{C^\theta(\overline{\Omega})} \le KC(L_3, R, \|u_0\|_{C^{2+\theta}(\overline{\Omega})})T^{1/2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.
$$

Summing up, we get

$$
\begin{aligned}
\|g\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\overline{\Omega})} \le\\
T^{\theta/2}K(L_2, L_3, T, R, \|u_0\|_{C^{2+\theta}(\overline{\Omega})})\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},
\end{aligned}
\tag{3.15}
$$

with $K(L_2, L_3, T, R, \|u_0\|_{C^{2+\theta}(\overline{\Omega})})$ positive and increasing with respect to all its arguments.

Taking into account (3.14) and (3.15) we obtain that Γ is a 1/2-contraction provided

$$
C\left[L_1(4T^{\theta/2} + 2T^\theta)R + T^{\theta/2}K(L_2, L_3, T, R, \|u_0\|_{C^{2+\theta}(\overline{\Omega})})\right] \le \frac{1}{2}. \tag{3.16}
$$

Now we check that Γ maps Y into itself if T, R are suitably chosen. For each $u \in Y$, we write $\Gamma u = \Gamma(u-u_0) + \Gamma u_0$. We already know that if (3.12), (3.13), (3.16) hold then

$$
\|\Gamma(u-u_0)\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le \frac{1}{2}\|u-u_0\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le \frac{R}{2}.
$$

Therefore, Γ maps Y into itself provided $\|\Gamma u_0 - u_0\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le R/2$. The function $w = \Gamma(u_0) - u_0$ is the solution to (3.9) with $f = \Phi(D^2u_0)$,

$g = -\Psi(Du_0)$, $w_0 = 0$, and its $C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$ norm is not small in general if T is small. We only have, by estimate (3.10),

$$\|\Gamma u_0 - u_0\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \leq$$

$$\leq C(\|\Phi(D^2u_0)\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} + \|\Psi(Du_0)\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)})$$

$$= C(\|\Phi(D^2u_0)\|_{C^{\theta}(\overline{\Omega})} + \|\Psi(Du_0)\|_{C^{1+\theta}([0,T]\times\partial\Omega)}),$$

with $C = C(1)$. Then Γ maps Y into itself if

$$R \geq 2C(\|\Phi(D^2u_0)\|_{C^{\theta}(\overline{\Omega})} + \|\Psi(Du_0)\|_{C^{1+\theta}([0,T]\times\partial\Omega)}). \tag{3.17}$$

In conclusion, if (3.12), (3.13), (3.16), (3.17) hold, Γ is a 1/2-contraction that maps Y into itself, so that it has a unique fixed point u in Y.

To finish the proof we have to show that u is the unique solution to (1.1) in $C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$. This is done in a standard way.

If (1.1) has two solutions u and v, set $t_0 = \sup\{t \in [0,T] : u = v$ in $[0,t]\times\overline{\Omega}\}$. If $t_0 = T$ then $u = v$ in the whole $[0,T]\times\overline{\Omega}$ and the proof is finished; if $t_0 < T$ we consider the initial-boundary value problem

$$\begin{cases} w_t(t,x) = \Phi(D^2w(t,x)), & t \geq t_0,\ x \in \overline{\Omega}, \\ \Psi(Dw(t,x)) = 0, & t \geq t_0,\ x \in \partial\Omega, \\ w(t_0,x) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{3.18}$$

where $w_0(x) = u(t_0,x) = v(t_0,x)$. The above proof shows that (3.18) has a unique solution in the set $Y' = \{w \in C^{1+\theta/2,2+\theta}([t_0,t_0+T']\times\overline{\Omega}) : w(t_0,\cdot) = w_0,\ \|w - w_0\|_{C^{1+\theta/2,2+\theta}([t_0,t_0+T']\times\overline{\Omega})} \leq R'\}$, provided R' is large enough and T' is small enough. Taking R' larger than $\|u - u(t_0,\cdot)\|_{C^{1+\theta/2,2+\theta}([t_0,T]\times\overline{\Omega})}$ and than $\|v - v(t_0,\cdot)\|_{C^{1+\theta/2,2+\theta}([t_0,T]\times\overline{\Omega})}$ we get $u = v$ in $[t_0,t_0+T']\times\overline{\Omega}$, and this contradicts the definition of t_0. Therefore, $t_0 = T$ and $u \equiv v$. □

With a little extra effort it is possible to prove that the solution depends continuously on the initial datum.

Corollary 3.3 *Under the assumptions of Theorem 3.2, fix any $u_0 \in C^{2+\theta}(\overline{\Omega})$ such that the range of D^2u_0 is contained in $\mathcal{O}_1$, the range of Du_0 is contained in $\mathcal{O}_2$, and $\Psi(Du_0(x)) = 0$ at $\partial\Omega$. Then there exist $r > 0$, $K > 0$ such that for each $v_0 \in C^{2+\theta}(\overline{\Omega})$ with $\|v_0 - u_0\|_{C^{2+\theta}(\overline{\Omega})} \leq r$ and satisfying* *the compatibility conditions* $\Psi(Dv_0(x)) = 0$ *at* $\partial\Omega$, *the solution* $v(t,x)$ *of problem (1.1) with initial datum* v_0 *is defined in* $[0,T]\times\overline{\Omega}$, *where* $T>0$ *is given by Theorem 3.2, and*

$$\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le K\|u_0-v_0\|_{C^{2+\theta}(\overline{\Omega})}.$$

Proof — We follow the notation of the proof of Theorem 3.2. If we take $r<r_1/2$, $r<r_2/2$ and we define

$$\begin{aligned} Y_1 &= \{u\in C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega}):\ u(0,\cdot)=v_0, \\ &\qquad \|u-v_0\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}\le R\}, \end{aligned}$$

with T and R chosen as in the proof of Theorem 3.2, then for each $u\in Y_1$ the ranges of D^2u and of Du are contained in the neighborhoods of the ranges of D^2u_0 and of Du_0 with radii r_1, r_2, respectively, thanks to (3.12). Since $u-v$ is a solution to problem (3.9) with $f(t,x)=\Phi(D^2u(t,x))-\Phi(D^2v(t,x))-\mathcal{A}(u-v)(t,x)$, $g(t,x)=\mathcal{B}(u-v)(t,x)-\Psi(Du(t,x))+\Psi(Dv(t,x))$, $w_0=u_0-v_0$, combining estimate (3.10) with the estimates of the proof of Theorem 3.2 we get

$$\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}\le C\left(\frac{1}{2}\|u-v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}+\|u_0-v_0\|_{C^{2+\theta}(\overline{\Omega})}\right),$$

and the statement follows with $K=2C$. □

Reading the proofs of Theorem 3.2 and of its corollary we can realize that they may be extended to more general situations.

First of all, we may allow more general nonlinearities, such as $\Phi=\Phi(t,x,u,Du,D^2u)$ and $\Psi=\Psi(t,x,u,Du)$. See e.g. [33, Ch. 8]. The nonlinear boundary condition may be replaced by a Dirichlet boundary condition, $u(t,x)=g(t,x)$ with $g\in C^{1+\theta/2,2+\theta}([0,T]\times\partial\Omega)$, and the proof comes out to be shorter.

Second, the nonlinearities may be also nonlocal: the essential property of Φ that we used in the proof was just that the function $F(u)=\Phi(D^2u)-\mathcal{A}u$ is differentiable near u_0, with locally Lipschitz continuous (null at u_0) derivative, as a function from $C^2_b(\overline{\Omega})$ to $C_b(\overline{\Omega})$, and from $C^{2+\theta}(\overline{\Omega})$ to $C^\theta(\overline{\Omega})$, and moreover that

$$\begin{aligned} &\|F'(u)v\|_{C^\theta(\overline{\Omega})}\le \\ &\|F'(u)\|_{L(C^2_b(\overline{\Omega}),C_b(\overline{\Omega}))}\|v\|_{C^{2+\theta}(\overline{\Omega})}+\|F'(u)\|_{L(C^{2+\theta}(\overline{\Omega}),C^\theta(\overline{\Omega}))}\|v\|_{C^2_b(\overline{\Omega})}. \end{aligned}$$

Third, the proof is not confined to a single second order equation but it works as well for higher order equations and systems. This is because optimal regularity theorems in parabolic Hölder spaces similar to Theorem 3.1 are available for higher order equations and systems ([37, 34]). A detailed proof for a general class of second order systems with Dirichlet boundary condition is in the paper [1].

A completely different approach is in the paper [24].

4 Existence in the large and stability

Existence in the large for arbitrary initial data is a hard task in the fully nonlinear case. The results available up to now concern only second order equations. The difficulty is due to the fact that we need a priori estimates in a very high norm, substantially in a $C^{1+\theta/2,2+\theta}$-norm, to get existence in the large; therefore the nonlinearities have to satisfy severe restrictions. See the books [25, 30] for further detailed discussion and comments.

On the other hand, existence in the large and stability for initial data close to stationary solutions or more generally to established given solutions, is a quite developed subject.

For initial data close to stationary solutions, the proof of the local existence Theorem 3.2 is easier, and it can be extended to a very general class of perturbations. We quote a result from [7], concerning problem

$$\begin{cases} u_t(t,\xi) = \mathcal{A}u + F(u(t,\cdot))(\xi), & \xi \in \overline{\Omega}, \\ \mathcal{B}u = G(u(t,\cdot))(\xi), & \xi \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), & \xi \in \overline{\Omega}. \end{cases} \tag{4.1}$$

Here the stationary solution is $u \equiv 0$. In [7] a bounded Ω is taken into consideration, but the proofs are easily extended to unbounded open sets. The assumptions on F and G are the following.

(H4) $F : B(0,R) \subset C^2_b(\overline{\Omega}) \to C_b(\overline{\Omega})$ is continuously differentiable with Lipschitz continuous derivative, $F(0) = 0$, $F'(0) = 0$ and the restriction of F to $B(0,R) \subset C^{2+\theta}(\overline{\Omega})$ has values in $C^{\theta}(\overline{\Omega})$ and is continuously differentiable; $G : B(0,R) \subset C^1_b(\overline{\Omega}) \to C_b(\partial\Omega)$ is continuously differentiable with Lipschitz continuous derivative, $G(0) = 0$, $G'(0) = 0$ and the restriction of G to $B(0,R) \subset C^{2+\theta}(\overline{\Omega})$ has values in $C^{1+\theta}(\partial\Omega)$ and is continuously differentiable.

Theorem 4.1 *Let Ω and the operators $\mathcal{A}$, $\mathcal{B}$ defined in (3.4), (3.6), satisfy the assumptions of Theorem 3.1. If (H4) holds, for every $T > 0$ there are r, $\rho > 0$ such that (4.1) has a solution $u \in C^{1+\theta/2,2+\theta}([0,T] \times \overline{\Omega})$ provided $\|u_0\|_{C^{2+\theta}(\overline{\Omega})} \leq \rho$. Moreover u is the unique solution in $B(0,r) \subset C^{1+\theta/2,2+\theta}([0,T] \times \overline{\Omega})$.*

Proof — Let $0 < r \leq R$, and set

$$K(r) = \sup\{\|F'(\varphi)\|_{L(C^{2+\theta}(\overline{\Omega}),C^{\theta}(\overline{\Omega}))} : \varphi \in B(0,r) \subset C^{2+\theta}(\overline{\Omega})\},$$

$$H(r) = \sup\{\|G'(\varphi)\|_{L(C^{2+\theta}(\overline{\Omega}),C^{1+\theta}(\partial\Omega))} : \varphi \in B(0,r) \subset C^{2+\theta}(\overline{\Omega})\}.$$

Since $F'(0) = 0$ and $G'(0) = 0$, $K(r)$ and $H(r)$ go to 0 as $r \to 0$. Let $L > 0$ be such that, for all φ, $\psi \in B(0,r) \subset C^2_b(\overline{\Omega})$ with small r,

$$\|F'(\varphi) - F'(\psi)\|_{L(C^2_b(\overline{\Omega}),C_b(\overline{\Omega}))} \leq L\|\varphi - \psi\|_{C^2_b(\overline{\Omega})},$$

$$\|G'(\varphi) - G'(\psi)\|_{L(C^1_b(\overline{\Omega}),C_b(\partial\Omega))} \leq L\|\varphi - \psi\|_{C^1_b(\overline{\Omega})}.$$

For every $0 \leq s \leq t \leq T$ and for every $w \in B(0,r) \subset C^{1+\theta/2,2+\theta}([0,T] \times \overline{\Omega})$ with r so small that $K(r), H(r) < \infty$, we have

$$\|F(w(t,\cdot))\|_{C^{\theta}(\overline{\Omega})} \leq K(r)\|w(t,\cdot)\|_{C^{2+\theta}(\overline{\Omega})}, \quad \|F(w(t,\cdot)) - F(w(s,\cdot))\|_{C_b(\overline{\Omega})} \leq$$

$$Lr\|w(t,\cdot) - w(s,\cdot)\|_{C^2_b(\overline{\Omega})} \leq Lr|t-s|^{\theta/2}\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

and similarly

$$\|G(w(t,\cdot))\|_{C^{1+\theta}(\partial\Omega)} \leq H(r)\|w(t,\cdot)\|_{C^{2+\theta}(\overline{\Omega})} \leq H(r)\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

$$\|G(w(t,\cdot)) - G(w(s,\cdot))\|_{C_b(\partial\Omega)} \leq$$

$$Lr\|w(t,\cdot) - w(s,\cdot)\|_{C^1_b(\overline{\Omega})} \leq Lr|t-s|^{1/2+\theta/2}\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Therefore, $(t,\xi) \to F(w(t,\cdot))(\xi)$ belongs to $C^{\theta/2,\theta}([0,T] \times \overline{\Omega})$, $(t,\xi) \to G(w(t,\cdot))(\xi)$ belongs to $C^{1/2+\theta/2,1+\theta}([0,T] \times \partial\Omega)$ and

$$\left\{\begin{array}{l} \|F(w)\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} \leq (K(r) + Lr)\|w\|_{C^{2+\alpha,1+\alpha/2}([0,T]\times\overline{\Omega})}, \\ \\ \|G(w)\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)} \leq (2H(r) + Lr)\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}. \end{array}\right. \tag{4.2}$$

So, if $\|w_0\|_{C^{2+\theta}(\overline{\Omega})}$ is small enough, we define a nonlinear map

$$\Gamma \ : \ \{w \in B(0,r) \subset C^{1+\theta/2,2+\theta}([0,T] \times \overline{\Omega}) : w(\cdot,0) = w_0\}$$

$$\mapsto C^{1+\theta/2,2+\theta}([0,T] \times \overline{\Omega}),$$

by $\Gamma w = v$, where v is the solution of

$$\begin{cases} v_t(x,t) = \mathcal{A}v + F(w(t,\cdot))(x), \ \ 0 \le t \le T, \ x \in \overline{\Omega}, \\ \mathcal{B}v = G(w(t,\cdot))(x), \ \ 0 \le t \le T, \ x \in \partial\Omega, \\ v(0,x) = w_0(x). \end{cases}$$

Actually, thanks to the compatibility condition $\mathcal{B}w_0 = G(w_0)$ and the regularity of $F(w)$ and $G(w)$, the range of Γ is contained in $C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})$. Moreover, Theorem 3.1 gives the estimate

$$\|v\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le$$

$$C(\|w_0\|_{C^{2+\theta}(\overline{\Omega})} + \|F(w)\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})} + \|G(w)\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)}),$$

with $C = C(T)$, so that

$$\|\Gamma(w)\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le$$

$$C(\|w_0\|_{C^{2+\theta}(\overline{\Omega})} + (K(r) + 2Lr + 2H(r))\|w\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}).$$

Therefore, if r is so small that

$$C(K(r) + 2Lr + 2H(r)) \le 1/2, \tag{4.3}$$

and w_0 is so small that

$$\|w_0\|_{C^{2+\theta}(\overline{\Omega})} \le Cr/2,$$

Γ maps the ball $B(0,r)$ into itself. Let us check that Γ is a 1/2-contraction. Let w_1, $w_2 \in B(0,r)$, $w_i(\cdot,0) = w_0$. Writing $w_i(t,\cdot) = w_i(t)$, $i = 1,2$, we have

$$\|\Gamma w_1 - \Gamma w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \le$$

$$C(\|F(w_1) - F(w_2)\|_{C^{\theta/2,\theta}([0,T]\times\overline{\Omega})}$$

$$+\|G(w_1) - G(w_2)\|_{C^{1/2+\theta/2,1+\theta}([0,T]\times\partial\Omega)}),$$

and, arguing as above, for $0 \le t \le T$,

$$\|F(w_1(t,\cdot)) - F(w_2(t,\cdot))\|_{C^\theta(\overline{\Omega})} \le$$

$$K(r)\|w_1(t,\cdot) - w_2(t,\cdot)\|_{C^{2+\theta}(\overline{\Omega})} \le K(r)\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

$$\|G(w_1(t,\cdot)) - G(w_2(t,\cdot))\|_{C^{1+\theta}(\partial\Omega)} \le$$

$$H(r)\|w_1(t,\cdot) - w_2(t,\cdot)\|_{C^{2+\theta}(\overline{\Omega})} \le H(r)\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

while for $0 \le s \le t \le T$

$$\|F(w_1(t,\cdot)) - F(w_2(t,\cdot)) - F(w_1(s,\cdot)) - F(w_2(s,\cdot))\|_{C_b(\overline{\Omega})} =$$

$$\Big\| \int_0^1 \quad F'(\sigma w_1(t,\cdot) + (1-\sigma)w_2(t,\cdot)(w_1(t,\cdot) - w_2(t,\cdot))$$

$$-F'(\sigma w_1(s,\cdot) + (1-\sigma)w_2(s,\cdot))(w_1(s,\cdot) - w_2(s,\cdot))d\sigma \Big\|_{C_b(\overline{\Omega})}$$

$$\le \int_0^1 \quad \|(F'(\sigma w_1(t,\cdot) + (1-\sigma)w_2(t,\cdot) - F'(\sigma w_1(s,\cdot) + (1-\sigma)w_2(s,\cdot))\cdot$$

$$\cdot(w_1(t,\cdot) - w_2(t,\cdot))d\sigma\|_{C_b(\overline{\Omega})}$$

$$+\int_0^1 \|F'(\sigma w_1(s,\cdot))(w_1(t,\cdot) - w_2(t,\cdot) - w_1(s,\cdot) + w_2(s,\cdot))\|_{C_b(\overline{\Omega})}$$

$$\le \quad \frac{L}{2}\,(\|w_1(t,\cdot) - w_1(s,\cdot)\|_{C_b^2(\overline{\Omega})} + \|w_2(t,\cdot) - w_2(s,\cdot)\|_{C^2(\overline{\Omega})})\cdot$$

$$\cdot\|w_1(t,\cdot) - w_2(t,\cdot)\|_{C_b^2(\overline{\Omega})}$$

$$+Lr\|w_1(t,\cdot) - w_2(t,\cdot) - w_1(s,\cdot) + w_2(s,\cdot))\|_{C_b^2(\overline{\Omega})}$$

$$\le 2Lr(t-s)^{\theta/2}\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

and similarly

$$\|G(w_1(t,\cdot)) - G(w_2(t,\cdot)) - G(w_1(s,\cdot)) - G(w_2(\cdot,s))\|_{C_b(\partial\Omega)}$$

$$\le 2Lr(t-s)^{1/2+\theta/2}\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}.$$

Therefore,

$$\|\Gamma w_1 - \Gamma w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}$$

$$\leq C(K(r) + 2Lr + 2H(r))\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})}$$

$$\leq \frac{1}{2}\|w_1 - w_2\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})},$$

the last inequality being a consequence of (4.3). The statement follows. □

In our example (4.1), assumption (H4) is satisfied with $F(\phi) = \Phi(D^2\phi) - \mathcal{A}\phi$, $G(\phi) = \Psi(D\phi) - \mathcal{B}\phi$, if Φ, Ψ satisfy the assumptions of Theorem 3.2, where $\mathcal{O}_1$, $\mathcal{O}_2$ are neighborhoods of 0 in $\mathbb{R}^{N^2}$, $\mathbb{R}^N$ respectively, $\Phi(0) = 0$, $\Psi(0) = 0$, and $\mathcal{A}$, $\mathcal{B}$ are the operators

$$\mathcal{A}\varphi(x) = \sum_{i,j=1}^{N} \frac{\partial\Phi}{\partial q_{ij}}(0) D_{ij}\varphi(x), \quad x \in \overline{\Omega},$$

$$\mathcal{B}\varphi(x) = \sum_{i=1}^{N} \frac{\partial\Psi}{\partial p_i}(0) D_i\varphi(x), \quad x \in \overline{\Omega}.$$

Theorem 4.1 says that if 0 is a stationary solution then the solution to (4.1) is defined in an arbitrary large time interval provided the initial datum is small enough. The next natural question is now the stability of the null solution. We shall see that the principle of linearized stability holds, and that in the hyperbolic case local stable and unstable manifolds may be constructed, just like in the case of ordinary differential equations. To do this we shall see again our nonlinear problem as a perturbation of a linear one, and the main tools will be optimal regularity / asymptotic behavior results for the linear case, stated in the next section.

4.1 Asymptotic behavior in linear problems

Let us consider again the operators $\mathcal{A}$ and $\mathcal{B}$ defined in (3.4), (3.6). The realization A of $\mathcal{A}$ with homogeneous boundary conditions in $X = C(\overline{\Omega})$, i.e., the operator with domain

$$D(A) = \{\varphi \in C_b(\overline{\Omega}) \cap_{p>1} W^{2,p}_{loc}(\Omega) : \mathcal{A}\varphi \in C_b(\overline{\Omega}),\ \mathcal{B}\varphi(x) = 0,\ x \in \partial\Omega\}$$

$$A\varphi = \mathcal{A}\varphi, \quad \varphi \in D(A),$$

is a sectorial operator in $C_b(\overline{\Omega})$ thanks to [40]. We define

$$\sigma^-(A) = \{\lambda \in \sigma(A) : \text{ Re } \lambda < 0\}, \quad \sigma^+(A) = \{\lambda \in \sigma(A) : \text{ Re } \lambda > 0\}.$$

We shall consider the assumptions

$$\begin{cases} \text{(i)} \quad \sup\{\text{Re } \lambda : \lambda \in \sigma^-(A)\} < 0, \\ \\ \text{(ii)} \quad \inf\{\text{Re } \lambda : \lambda \in \sigma^+(A)\} > 0, \end{cases} \tag{4.4}$$

which are always true if Ω is bounded, because in this case the domain of A is compactly embedded in $C(\overline{\Omega})$, the resolvent operators $(\lambda I - A)^{-1}$ are compact, and the spectrum consists of a sequence of eigenvalues.

If (4.4)(ii) holds, $\sigma^+(A)$ is closed. We denote by P^+ the spectral projection associated to $\sigma^+(A)$, i.e.,

$$P^+ = \frac{1}{2\pi i} \int_C R(\lambda, A) d\lambda,$$

where C is any closed simple regular curve in $\{\text{Re } \lambda > 0\}$ surrounding $\sigma^+(A)$.

If (4.4)(i) holds, $\sigma^-(A)$ is closed, and we denote by P^- the spectral projection associated to $\sigma^+(A)$, i.e.,

$$P^- = I - P, \quad P = \frac{1}{2\pi i} \int_C R(\lambda, A) d\lambda,$$

where now C is any closed simple regular curve surrounding $\sigma(A) \setminus \sigma^-(A)$ with index 0 with respect to all points in $\sigma^-(A)$. If the spectrum of A does not intersect the imaginary axis, then $P^- = I - P^+$.

We also need a deeper insight into the solution to (3.9). We shall consider a representation formula for w, that is an extension of the well known Balakrishnan formula (see e.g. [33, p.200]):

$$\begin{aligned} w(t,\cdot) &= e^{tA}(w_0 - \mathcal{N}g(0,\cdot)) + \int_0^t e^{(t-s)A}[f(s,\cdot) + \mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ &\quad -A\int_0^t e^{(t-s)A}[\mathcal{N}g(s,\cdot) - \mathcal{N}g(0,\cdot)]ds + \mathcal{N}g(0,\cdot) \\ &= e^{tA}u_0 + \int_0^t e^{(t-s)A}[f(s,\cdot) + \mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ &\quad -A\int_0^t e^{(t-s)A}\mathcal{N}g(s,\cdot)ds, \quad 0 \le t \le T. \end{aligned} \tag{4.5}$$

Here $\mathcal{N}$ is any lifting operator such that

$$\begin{cases} \mathcal{N} \in L(C^\alpha(\partial\Omega), C^{\alpha+1}(\overline{\Omega})), \ \ 0 \le \alpha \le \theta + 1, \\ \\ \mathcal{B}\mathcal{N}g = g, \ \ g \in C_b(\partial\Omega). \end{cases} \tag{4.6}$$

For instance, we can take as $\mathcal{N}$ the operator given in Theorem 0.3.2 of [33]. Later we will need an explicit expression of $\mathcal{N}$, so we give some details above.

Lemma 4.2 *Let Ω be an open set in $\mathbb{R}^N$ with uniformly $C^{2+\theta}$ boundary, and let $\mathcal{B}$ satisfy the assumptions of Theorem 3.1. Then there exists a lifting operator satisfying (4.6).*

Proof — As a first step we construct $\mathcal{N}$ for $\Omega = \mathbb{R}^N_- = \{(x, y) : \ x < 0, \ y \in \mathbb{R}^{N-1}\}$. Fix a function $\varphi \in C^\infty(\mathbb{R}^{N-1})$, with compact support, and such that $\int_{\mathbb{R}^{N-1}} \varphi(\xi) d\xi = 1$. Fix moreover $\delta > 0$ and $\eta \in C^\infty((-\infty, 0])$ such that $\eta \equiv 0$ for $x \le -2\delta$, $\eta \equiv 1$ for $-\delta \le x \le 0$.

For each $k \in \mathbb{N}$ and $g \in L^\infty(\mathbb{R}^{N-1})$ set

$$Ng(x, y) = x\eta(x) \int_{\mathbb{R}^{N-1}} \varphi(\xi) g(y + \xi x) d\xi, \ \ x \le 0, \ y \in \mathbb{R}^{N-1}. \tag{4.7}$$

Then $N \in L(C^\alpha(\mathbb{R}^{N-1}); C^{1+\alpha}(\mathbb{R}^N_-))$, for each $k \in \mathbb{N}$, $\alpha \ge 0$, and for every $y \in \mathbb{R}^{N-1}$ it holds

$$\begin{cases} Ng(0, y) = 0, \\ \\ \dfrac{\partial}{\partial x} Ng(0, y) = \psi(y). \end{cases} \tag{4.8}$$

If $\mathcal{B}$ is the normal derivative, we are done. If $\mathcal{B}v = \beta(y)\partial/\partial x$ plus derivatives with respect to y, we define $(\mathcal{N}g)(x, y) = (Ng)(x, y)/\beta(y)$.

The case of a general open set with uniformly $C^{2+\theta}$ boundary is reduced to this one in a standard way, by locally stretching the boundary and using partitions of unity. See for instance [34], where more general lifting operators were constructed for systems of m boundary conditions. □

The following theorems were proved in [7].

Theorem 4.3 *Let assumption (4.4)(i) hold, and fix $\omega > 0$ such that $\omega < -\max\{\mathrm{Re}\,\lambda : \ \lambda \in \sigma^-(A)\}$. Let f be such that $(t, \xi) \to e^{\omega t} f(t, \xi) \in C^{\theta/2, \theta}([0, \infty) \times \overline{\Omega})$, let g be such that $(t, \xi) \to e^{\omega t} g(t, \xi) \in C^{1+\theta, 1/2+\theta/2}([0, \infty) \times \partial\Omega)$ and* *let* $w_0 \in C^{2+\theta}(\overline{\Omega})$ *satisfy the compatibility condition (3.8). Then* $v(t,\xi) = e^{\omega t}w(t,\xi)$ *is bounded in* $[0,+\infty) \times \overline{\Omega}$ *if and only if*

$$\begin{aligned} (I-P^-)w_0 &= -\int_0^{+\infty} e^{-sA}(I-P^-)[f(s,\cdot)+\mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ &\quad +L\int_0^{\infty} e^{-sA}(I-P^-)\mathcal{N}g(s,\cdot)ds. \end{aligned} \tag{4.9}$$

In this case, w *is given by*

$$\begin{aligned} & w(t,\cdot) = e^{tA}P^-u_0 \\ & +\int_0^t e^{(t-s)A}P^-[f(s,\cdot)+\mathcal{A}\mathcal{N}g(s,\cdot)]ds - A\int_0^t e^{(t-s)A}P^-\mathcal{N}g(s,\cdot)ds \\ & -\int_t^{+\infty} e^{(t-s)A}(I-P^-)[f(s,\cdot)+\mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ & +A\int_t^{+\infty} e^{(t-s)A}(I-P^-)\mathcal{N}g(s,\cdot)ds, \end{aligned} \tag{4.10}$$

and the function $v = e^{\omega t}w$ *belongs to* $C^{1+\theta/2,2+\theta}([0,\infty)\times\overline{\Omega})$, *with the estimate*

$$\|v\|_{C^{1+\theta/2,2+\theta}([0,\infty)\times\overline{\Omega})} \leq$$

$$\leq C(\|w_0\|_{C^{2+\theta}(\overline{\Omega})} + \|e^{\omega t}f\|_{C^{\theta/2,\theta}([0,\infty)\times\overline{\Omega})} + \|e^{\omega t}g\|_{C^{1/2+\theta/2,1+\theta}([0,\infty)\times\partial\Omega)}).$$

Theorem 4.3 has an important corollary in the stable case, when $\sigma(A) = \sigma^-(A)$.

Corollary 4.4 *Assume that* $\omega_A := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0$, *and fix* $\omega \in (0,\omega_0)$. *Let* f *be such that* $(t,\xi) \to e^{\omega t}f(t,\xi) \in C^{\theta/2,\theta}([0,\infty)\times\overline{\Omega})$, *let* g *be such that* $(t,\xi) \to e^{\omega t}g(t,\xi) \in C^{1+\theta,1/2+\theta/2}([0,\infty)\times\partial\Omega)$ *and let* $w_0 \in C^{2+\theta}(\overline{\Omega})$ *satisfy the compatibility condition (3.8). Then* $v(t,\xi) = e^{\omega t}w(t,\xi)$ *belongs to* $C^{1+\theta/2,2+\theta}([0,\infty)\times\overline{\Omega})$ *and*

$$\|v\|_{C^{1+\theta/2,2+\theta}([0,\infty)\times\overline{\Omega})}$$

$$\leq C(\|w_0\|_{C^{2+\theta}(\overline{\Omega})} + \|e^{\omega t}f\|_{C^{\theta/2,\theta}([0,\infty)\times\overline{\Omega})} + \|e^{\omega t}g\|_{C^{1/2+\theta/2,1+\theta}([0,\infty)\times\partial\Omega)}).$$

Let us now consider the backward problem

$$\begin{cases} w_t = \mathcal{A}u + f(t,\xi), \;\; t \le 0, \; \xi \in \overline{\Omega}, \\ \\ \mathcal{B}u = g(t,\xi), \;\; t \le 0, \; \xi \in \partial\Omega, \\ \\ w(0,\xi) = w_0(\xi), \;\; \xi \in \overline{\Omega}. \end{cases} \tag{4.11}$$

Theorem 4.5 *Let assumption (4.4)(ii) hold, with $\sigma^+(A) \neq \varnothing$, and fix $\omega > 0$ such that $\omega < \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma^+(A)\}$. Let f be such that $(t,\xi) \to e^{-\omega t}f(t,\xi) \in C^{\theta/2,\theta}((-\infty,0]\times\overline{\Omega})$ and let g be such that $(t,\xi) \to e^{-\omega t}g(t,\xi) \in C^{1/2+\theta/2,1+\theta}((-\infty,0] \times \partial\Omega)$, $u_0 \in C^{2+\theta}(\overline{\Omega})$.*

Then problem (4.11) has a solution w such that $v(t,\xi) = e^{-\omega t}w(t,\xi)$ is bounded in $(-\infty,0] \times \overline{\Omega}$ if and only if

$$\begin{aligned} (I-P^+)u_0 \;\; = \;\; & \int_{-\infty}^0 e^{-sA}(I-P^+)[f(s,\cdot) + \mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ & -A\int_{-\infty}^0 e^{-sA}(I-P^+)\mathcal{N}g(s,\cdot)ds. \end{aligned} \tag{4.12}$$

In this case, w is given by

$$\begin{aligned} w(t,\cdot) \;\; = \;\; & e^{tA}P^+u_0 + \int_0^t e^{(t-s)A}P^+[f(s,\cdot) + \mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ & -A\int_0^t e^{(t-s)A}P^+\mathcal{N}g(s,\cdot)ds \\ & +\int_{-\infty}^t e^{(t-s)A}(I-P^+)[f(s,\cdot) + \mathcal{A}\mathcal{N}g(s,\cdot)]ds \\ & -A\int_{-\infty}^t e^{(t-s)A}(I-P^+)\mathcal{N}g(s,\cdot)ds, \;\; t \le 0. \end{aligned} \tag{4.13}$$

Moreover, $v = e^{-\omega t}w$ belongs to $C^{1+\theta/2,2+\theta}((-\infty,0] \times \overline{\Omega})$ and

$$\|v\|_{C^{1+\theta/2,2+\theta}((-\infty,0]\times\overline{\Omega})} \le C(\|w_0\|_{C(\overline{\Omega})}$$

$$+\|e^{-\omega t}f\|_{C^{\theta/2,\theta}((-\infty,0]\times\overline{\Omega})} + \|e^{-\omega t}g\|_{C^{1/2+\theta/2,1+\theta}((-\infty,0]\times\partial\Omega)}).$$

4.2 Principle of linearized stability and local invariant manifolds

With the aid of Theorems 4.3, 4.5, and Corollary 4.4 we may show similar behaviors for the solutions to fully nonlinear problems with initial data close to stationary solutions, provided the linearized operator near the stationary solution under consideration satisfies assumption (4.4)(i) or (4.4)(ii). For the proofs see [7].

Theorem 4.6 *Let Ω be an open set in $\mathbb{R}^N$ with uniformly $C^{2+\theta}$ boundary, $0 < \theta < 1$, let the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of Theorem 3.1, and let F, G, satisfy assumption (H4).*

(i) If $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0$ then the stationary solution $u = 0$ of problem (4.1) is stable with respect to the $C^{2+\theta}(\overline{\Omega})$ norm. More precisely, for every $\omega \in (0, -\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\})$, there are C, $r > 0$ such that for every u_0 satisfying (3.8) and $\|u_0\|_{C^{2+\theta}(\overline{\Omega})} \leq r$, the solution of (4.1) with initial datum u_0 exists in the large and satisfies

$$\|u(t, \cdot)\|_{C^{2+\theta}(\overline{\Omega})} \leq Ce^{-\omega t}\|u_0\|_{C^{2+\theta}(\overline{\Omega})}, \quad t \geq 0.$$

(ii) If $\sigma(A)$ contains elements with positive real part and (4.4)(ii) holds, then $u = 0$ is unstable in $C^{2+\theta}(\overline{\Omega})$.

We recall that if Ω is bounded then (4.4)(ii) is satisfied, and Theorem 4.6 looks like the usual principle of linearized stability for ordinary differential equations. If Ω is unbounded, it may happen that (4.4)(ii) is not satisfied. However, it is still possible to give an instability result, relying on the next theorem taken from Henry's book [22].

Theorem 4.7 *Let X be a real Banach space. Let T be a map from a neighborhood of the origin in X with $T(0) = 0$, let M be a bounded linear operator on X with spectral radius r greater than 1, and*

$$T(x) = Mx + O(\|x\|^p), \quad \text{as } x \to 0,$$

for some constant $p > 1$. Then the origin is unstable for the iterates of T, i.e. there exists a constant $C > 0$ and there exists x_0 arbitrarily close to 0 such that if $x_{n+1} = T(x_n) = T^{n+1}(x_0)$ for $n \in \mathbb{N}$ then for some N (depending on x_0), the sequence $x_1, x_2, \ldots, x_N$ is well defined and $\|x_N\| \geq C$.

If $G \equiv 0$ in problem (4.1), we can apply Theorem 4.7 with $X = \{u_0 \in C^{2+\theta}(\overline{\Omega}) : \mathcal{B}u_0 = 0 \text{ at } \partial\Omega\}$, $T(u_0) = u(1; u_0)$ is the solution of (4.1) with initial datum u_0, evaluated at time $t = 1$ (the lifetime of the solution is bigger than 1 provided $\|u_0\|_{C^{2+\theta}(\overline{\Omega})}$ is small enough, thanks to Theorem 4.1). Then $T^n u_0 = u(n; u_0)$, $M = e^A$, $p = 2$, and the spectral radius of M is equal to $\exp(\omega_A)$ where $\omega_A = \sup\{\text{ Re}\lambda : \lambda \in \sigma(A)\}$ because the spectral mapping theorem holds for analytic semigroups. So, if $\omega > 0$ Theorem 4.7 implies that the null solution of (4.1) is unstable in $C^{2+\theta}(\overline{\Omega})$.

For a nonvanishing function G the set $\mathcal{I}$ of admissible initial data for problem (4.1), $\mathcal{I} = \{u_0 \in C^{2+\theta}(\overline{\Omega}) : \|u_0\|_{C^{2+\theta}(\overline{\Omega})} \leq r, \ \mathcal{B}u_0 = G(u_0)\}$ is not a neighborhood of 0 in a linear space. However, we shall see in next Lemma 4.8 that it is the graph of a regular function defined in a neighborhood of 0 in $D(A_\theta)$, where $D(A_\theta)$ is the domain of the part of A in $C^{2+\theta}(\overline{\Omega})$:

$$D(A_\theta) = \{u_0 \in C^{2+\theta}(\overline{\Omega}) : \mathcal{B}u_0 = 0\}.$$

The already mentioned lifting operator $\mathcal{N}$ is a right inverse of the function $C^{2+\theta}(\overline{\Omega}) \mapsto C^{1+\theta}(\partial\Omega)$, $u \mapsto \mathcal{B}u$, so that $C^{2+\theta}(\overline{\Omega})$ is the direct sum $D(A_\theta) \oplus (I - \Pi)(C^{2+\theta}(\overline{\Omega}))$, where Π is the projection on $D(A_\theta) = \text{Ker } \mathcal{B}$ given by

$$\Pi u = u - \mathcal{N}\mathcal{B}u.$$

Lemma 4.8 *There is a neighborhood $\mathcal{O}$ of 0 in $C^{2+\theta}(\overline{\Omega})$ such that $\mathcal{I} \cap \mathcal{O}$ is the graph of a smooth function*

$$H : B(0, \rho) \subset D(A_\theta) \mapsto (I - \Pi)(C^{2+\theta}(\overline{\Omega}))$$

with $\rho > 0$. Moreover $H'(0) = 0$.

Proof — Define $J : B(0, r) \subset C^{2+\theta}(\overline{\Omega}) \mapsto C^{1+\theta}(\partial\Omega)$,

$$J(\varphi) = \mathcal{B}\varphi - G(\varphi),$$

with $r < R$, see assumption (H4). Then J is smooth and $J'(0) = \mathcal{B}$ is an isomorphism from $(I - \Pi)(C^{2+\theta}(\overline{\Omega}))$ to $C^{1+\theta}(\partial\Omega)$. Moreover $\mathcal{B}_{|D(A_\theta)} = 0$. It is sufficient now to apply the Implicit Function Theorem. $\square$

Corollary 4.9 *Under the assumptions of Theorem 4.6, if $\sigma(A)$ contains elements with positive real part then the origin is unstable in $C^{2+\theta}(\overline{\Omega})$.*

Proof — Theorem 4.7 is applied to the map

$$T : B(0,\rho) \subset D(A_\theta) \mapsto D(A_\theta), \quad T(x_0) = \Pi(u(1; x_0 + H(x_0))),$$

where ρ is given by Theorem 4.1 with $T = 1$, $u(1, x_0 + H(x_0))$ is the solution of (4.1) with initial condition $u(0) = x_0 + H(x_0)$.

We shall show that the derivative of T at $x_0 = 0$ is $M = e^{A_\theta}{}_{|D(A_\theta)}$, and that

$$\|T(x_0) - e^{A_\theta} x_0\|_{C^{2+\theta}(\overline{\Omega})} \leq C\|x_0\|^2_{C^{2+\theta}(\overline{\Omega})}, \tag{4.14}$$

so that the assumptions of Theorem 4.7 are satisfied with $p = 2$ and $r = e^{\omega_A}$. Applying Theorem 4.7 gives immediately instability of the null solution to (4.1).

Estimate (4.14) is a consequence of the construction of the solution to (4.1), as a fixed point of the operator Γ, see Theorem 4.1. Indeed, from the representation formula (4.5) and estimates (4.2) it follows that

$$\|\Gamma u - e^{tA}(u_0 - \mathcal{N}G(u_0))\|_{C^{1+\theta/2,2+\theta}([0,1]\times\overline{\Omega})} \leq \frac{1}{2}\|u\|^2_{C^{1+\theta/2,2+\theta}([0,1]\times\overline{\Omega})}, \tag{4.15}$$

for every $u_0 \in \mathcal{I} \cap B(0,\rho)$ with ρ small enough, which implies for the fixed point u

$$\|u\|_{C^{1+\theta/2,2+\theta}([0,T]\times\overline{\Omega})} \leq C\|u_0\|_{C^{2+\theta}(\overline{\Omega})}.$$

Replacing in (4.15) and then taking $t = 1$ we obtain

$$\|u(1,\cdot) - e^{A_\theta}(u_0 - \mathcal{N}G(u_0))\|_{C^{2+\theta}(\overline{\Omega})} \leq C\|u_0\|^2_{C^{2+\theta}(\overline{\Omega})}$$

which implies, for $u_0 = x_0 + H(x_0)$,

$$\|\Pi u(1,\cdot) - e^{A_\theta} x_0\|_{C^{2+\theta}(\overline{\Omega})} \leq C'\|x_0 + H(x_0)\|^2_{C^{2+\theta}(\overline{\Omega})} \leq C''\|x_0\|^2_{C^{2+\theta}(\overline{\Omega})},$$

and (4.14) follows. □

Corollary 4.9 improves part (ii) of Theorem 4.6; however it is not completely satisfactory because the $C^{2+\theta}(\overline{\Omega})$ norm is very strong and consequently the instability result is rather weak. We can improve the instability result using a refinement of Theorem 4.7 whose proof is in [9].

Theorem 4.10 *Let the conditions of Theorem 4.7 be satisfied, and assume in addition that the spectral radius r of M is an eigenvalue. Let $\overline{u} \in X$ be an* *eigenfunction and $x' \in X'$ (the space of all linear continuous functions from X to $\mathbb{R}$) be such that $x'(\overline{u}) \neq 0$. Then there are $C' > 0$ and initial data x_0 arbitrarily close to 0 such that if $x_{n+1} = T(x_n) = T^{n+1}(x_0)$ for $n \in \mathbb{N}$ then for some N (depending on x_0), the sequence $x_1, x_2, \dots, x_N$ is well defined, $x'(x_N)$ has the same sign of $x'(\overline{u})$, and $|x'(x_N)| \geq C'$.*

Corollary 4.11 *Under the assumptions of Theorem 4.6, suppose moreover that ω_A is an eigenvalue of A, and let $\overline{u}$ be an eigenvector. If $x' \in (C^{2+\theta}(\overline{\Omega}))'$ is such that $x'(\overline{u}) \neq 0$, then there is $C' > 0$ such that for every $\delta > 0$ there are $u_0 \in \mathcal{I}$ with norm less or equal to δ, and $N \in \mathbb{N}$ such that the corresponding solution u of (4.1) is defined at $T = N$ and $|x'(\Pi u(N, \cdot))| \geq C'$.*

The element $x' \in (C^{2+\theta}(\overline{\Omega}))'$ may be, for instance, the evaluation of φ or of some first or second order derivative of φ at some point. In this case Corollary 4.11 gives pointwise instability. Let us show this in a simple example.

Example 4.12 Consider the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + au(t,x) + \mathcal{F}(D^2u(t,x)), \quad t \geq 0, \ x \in \overline{\Omega}, \\ \\ \dfrac{\partial u}{\partial n}(t,x) = \mathcal{G}(Du(t,x)), \quad t \geq 0, \ x \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \end{cases} \tag{4.16}$$

where Ω is either a bounded open set with $C^{2+\theta}$ boundary, or a halfplane, $\mathcal{F}$, $\mathcal{G}$ are smooth functions defined in a neighborhood of 0 in $\mathbb{R}^{N^2}$, $\mathbb{R}^N$, respectively, vanishing at 0 with all their first order derivatives, and $a > 0$.

Then $\omega_A = a > 0$, and Corollary 4.9 implies that the null solution is unstable in $C^{2+\theta}(\overline{\Omega})$.

We get a much better instability result using Corollary 4.11. ω_A is an eigenvalue of A with constant eigenfunctions. Therefore for each $x_0 \in \overline{\Omega}$ the mapping $\varphi \mapsto \varphi(x_0)$ is an element of $(C^{2+\theta}(\overline{\Omega}))'$ that does not vanish on the eigenfunction 1. Corollary 4.11 implies that there is $C' > 0$ such that for every $\delta > 0$ there are $u_0 \in \mathcal{I}$ with norm less or equal to δ, and $N \in \mathbb{N}$ such that the solution u of (4.1) is defined at $T = N$ and $|(\Pi u(N, \cdot))(x_0)| \geq C'$. Since $\Pi u = u - \mathcal{N}\mathcal{B}u$, if $\mathcal{N}$ satisfies $(\mathcal{N}g)(x_0) = 0$ for each g, we have $(\Pi u(N, \cdot))(x_0) = u(N, x_0)$, and hence

$$|u(N, x_0)| \geq C'. \tag{4.17}$$

From the construction of $\mathcal{N}$ in Lemma 4.2 we know that $\mathcal{N}g$ vanishes at $\partial\Omega$ for every g. If $x_0 \in \Omega$, taking δ small enough in the proof of Lemma 4.2 we may let $(\mathcal{N}g)(x_0) = 0$ for each g, and (4.17) follows.

Now we go further in the description of the behavior of the solutions for small initial data, showing the existence of the local stable and unstable invariant manifolds. The following results were proved in [7].

Theorem 4.13 *Let the assumptions of Theorem 4.6 hold.*
(i) Assume that $\sigma^+(A) \neq \varnothing$ has positive distance from the imaginary axis, and fix $\omega \in (0, \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma_+(A)\})$. Then there exist R_0, $r_0 > 0$ and a Lipschitz continuous function

$$\varphi : B(0, r_0) \subset P^+(C_b(\overline{\Omega})) = P^+(D(A_\theta)) \to (I - P^+)(C^{2+\theta}(\overline{\Omega})),$$

differentiable at 0 with $\varphi'(0) = 0$, such that for every u_0 belonging to the graph of φ, problem (4.1) has a unique backward solution v such that $\tilde{v}$ defined by $\tilde{v}(t,\xi) = e^{-\omega t} v(t,\xi)$, belongs to $C^{1+\theta/2,2+\theta}((-\infty,0]\times\overline{\Omega})$ and satisfies

$$\|\tilde{v}\|_{C^{1+\theta/2,2+\theta}((-\infty,0]\times\overline{\Omega})} \leq R_0. \tag{4.18}$$

Moreover, for every $\omega' \in (0, \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma^+(A)\})$ we have $(t,\xi) \to e^{-\omega' t} v(t,\xi) \in C^{1+\theta/2,2+\theta}((-\infty,0] \times \overline{\Omega}))$. Conversely, if problem (4.1) has a backward solution v which satisfies (4.18) and $\|P^+ v(0,\cdot)\| \leq r_0$, then $v(0,\cdot) \in$ graph φ.
(ii) Assume that $\sigma^-(A)$ has positive distance from the imaginary axis, and fix $\omega \in (0, -\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma^-(A)\})$. Then there exist R_1, $r_1 > 0$ and a Lipschitz continuous function

$$\psi : B(0, r_1) \subset P^-(D(A_\theta)) \to (I - P^-)(C^{2+\theta}(\overline{\Omega})),$$

differentiable at 0 with $\psi'(0) = 0$, such that for every u_0 belonging to the graph of ψ, problem (4.1) has a unique solution w such that $\widetilde{w}$ defined by $\widetilde{w}(t,\xi) = e^{\omega t} w(t,\xi)$ belongs to $C^{1+\theta/2,2+\theta}([0,\infty) \times \overline{\Omega})$ and

$$\|\widetilde{w}\|_{C^{1+\theta/2,2+\theta}([0,\infty)\times\overline{\Omega})} \leq R_1. \tag{4.19}$$

Moreover, for every $\omega' \in (0, -\max\{\mathrm{Re}\,\lambda : \lambda \in \sigma^-(A)\})$ we have $(t,\xi) \to e^{\omega' t} w(t,\xi) \in C^{1+\theta/2,2+\theta}([0,\infty) \times \overline{\Omega})$. Conversely, if problem (4.1) has a forward solution w which satisfies (4.19) and $\|P^- w(0,\cdot)\|_{C^{2+\theta}(\overline{\Omega})} \leq r_1$, then $w(0,\cdot) \in$ graph ψ.

The graph of φ is called local unstable manifold. The graph of ψ is called local stable manifold.

In the case where the operator A is hyperbolic, i.e. (2.11) holds, both the local stable manifold and the local unstable manifold do exist, and Theorem 4.13 is a saddle point theorem.

What is missing up to now is a center manifold theory for problems with fully nonlinear boundary condition. Boundary conditions of the type $\mathcal{B}u = G(u)$ with nonlinear G may be treated, see [35]. But existence of a center manifold in the case where G depends nonlinearly on the gradient is still an open problem.

5 The fully nonlinear approach to free boundary problems

A class of free boundary problems of parabolic type may be reduced to abstract evolution equations of the type treated in section 2, or to evolution equations in Hölder spaces of the type treated in sections 3 and 4.

The prototypes of these problems are (1.2) and (1.3). The different physical nature of these problems is reflected in their different mathematical nature. (1.3) is a Stefan type problem, where the velocity of the free boundary is explicit, while in (1.2) it is implicit.

However, the initial step to reduce the free boundary problems to fixed boundary ones by natural changements of coordinates that involve one of the unknowns is the same for both models. After that, the procedures are different: we eliminate the free boundary and arrive at a final problem of the type (4.1) for (1.2), we eliminate u and we arrive at a final problem for the free boundary of the type (2.1) for (1.3).

(1.2) and (1.3) are the simplest significant examples of a wide class of parabolic free boundary problems, that can be studied with the same methods. We mention the papers [19, 20, 14, 15] for problems of the type (1.3), arising in several fields, such as flow of viscous fluids through porous media, the injection moulding process, diblock copolymer melts; and [6, 9, 7, 8, 10, 31] for problems of the type (1.2), arising in combustion theory.

5.1 Hele-Shaw models

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary, consisting of two disjoint nonempty parts J and Γ. For each t, the free boundary Γ_t will be sought as the range of an unknown function $s(t,\cdot)\in h^{2+\alpha}(\Gamma;\mathbb{R}^N)$ with small C^1 norm, in such a way that the mapping $\xi\mapsto\xi+s(t,\xi)\nu(\xi)$ is a $h^{2+\alpha}$ diffeomorphism between Γ and Γ_t, and Γ_t and J are disjoint. Here and in what follows we denote by $\nu=\nu(\xi)$ the exterior normal vector to $\partial\Omega$ at $\xi\in\partial\Omega$. Ω_t will be the open set diffeomorphic to Ω with boundary $J\cup\Gamma_t$.

Problem (1.3) is associated with an initial condition,

$$s(0,\xi)=s_0(\xi),\ \ \xi\in\Gamma. \tag{5.1}$$

Now we transform the free boundary problem (1.3) into a fixed boundary one.

For $a>0$ we define a map

$$X:\Gamma\times[-a,a]\to\mathbb{R}^N,\quad X(\xi',r)=\xi'+r\nu(\xi'). \tag{5.2}$$

If a is sufficiently small, then (5.2) is a diffeomorphism to a compact neighborhood $\mathcal{R}$ of Γ. In $\mathcal{R}$ every ξ can be written in a unique way as $\xi=X(\xi',r)$ with $\xi'\in\Gamma$ and $r\in[-a,a]$. So, $\xi'=\xi'(\xi)$ is the nearest point to ξ in Γ, and $r=r(\xi)$ is the signed distance from ξ to Γ.

We will look for Ω_t close to Ω in some time interval I in the sense that its free boundary Γ_t will be given by

$$\Gamma_t=\{x=\xi'+s(t,\xi')\nu(\xi'),\ \xi'\in\Gamma\}, \tag{5.3}$$

where $s:\Gamma\times I\to[-a,a]$ is a smooth function which is one of the unknowns of the problem. In other words, Γ_t is the zero level set of the function

$$\mathcal{R}\mapsto\mathbb{R},\ \ \xi\mapsto N(t,\xi)=r(\xi)-s(t,\xi'(\xi)).$$

It follows that the exterior normal vector at Γ_t is given by $\nu=DN/|DN|$, and the normal velocity V at $x\in\Gamma_t$ is

$$V(t,x)=-\frac{\partial/\partial t\,N(t,x)}{DN(t,x)}=\frac{\partial s(t,\xi'(x))/\partial t\,N(t,x)}{DN(t,x)}.$$

The equation $V=-\dfrac{\partial u}{\partial\nu}$ in (1.3) may be rewritten as

$$N_t(t,x)=\langle Du(t,x),DN(t,x)\rangle,\ \ t>0,\ x\in\Gamma_t.$$

It will be convenient to extend the vector field

$$\Phi(t,\xi)=s(t,\xi)\nu(\xi),\ \ \xi\in\Gamma, \tag{5.4}$$

to the whole of $\mathbb{R}^N$, by setting

$$\Phi(t,\xi) = \begin{cases} \alpha(r)s(t,\xi')\nu(\xi') & \text{if } \xi \in \mathcal{R}, \\ \\ 0 \quad \text{otherwise} , \end{cases} \tag{5.5}$$

where $r = r(\xi)$, $\xi' = \xi'(\xi)$, and $\alpha : \mathbb{R} \mapsto [0,1]$ is a smooth mollifier which is equal to 1 near 0 and has compact support in $(-a, a)$.

The extension Φ is used now to transform (1.3) to a problem on the fixed domain Ω. We define the coordinate transformation

$$x = \Theta(t,\xi) = \xi + \Phi(t,\xi). \tag{5.6}$$

Note that $\Theta(t,\cdot)$ differs from the identity only in a small neighborhood of Γ, and it maps Ω onto Ω_t.

Denoting by $\widetilde{u}$ the unknown u in the new variables, i.e. $\widetilde{u}(t,\xi) = u(t,\xi + \Phi(t,\xi))$, the couple (s,u) satisfies (1.3)-(5.1) if and only if $(s,\widetilde{u})$ satisfies

$$\begin{cases} \mathcal{A}\widetilde{u} = 0, & t > 0,\ \xi \in \overline{\Omega}, \\ \\ \widetilde{u} = 0, \quad s_t + \mathcal{B}\widetilde{u} = 0, & t > 0,\ \xi \in \Gamma, \\ \\ \dfrac{\partial}{\partial \nu}\widetilde{u} = b, & t > 0,\ \xi \in J, \\ \\ s(0,\cdot) = s_0, & \xi \in \Gamma, \end{cases} \tag{5.7}$$

where $\mathcal{A}$ is the Laplacian expressed in the new variables, i.e.,, setting $\Xi(x) = \Theta(t,\cdot)^{-1}(x)$,

$$\mathcal{A} = \sum_{i,j,h=1}^{N} \frac{\partial \Xi_j}{\partial x_i}(\Theta(t,\xi)) \frac{\partial \Xi_h}{\partial x_i}(\Theta(t,\xi)) D_{jh} + \sum_{i,j=1}^{N} \frac{\partial^2 \Xi_j}{\partial x_i^2}(\Theta(t,\xi)) D_j, \tag{5.8}$$

and $\mathcal{B}$ is the normal derivative expressed in the new variables. Since

$$n(t,\xi + \Phi) = \frac{(I + {}^tD\Phi)^{-1}\nu(\xi)}{|(I + {}^tD\Phi)^{-1}\nu(\xi)|}$$

and

$$Du(t,\xi + \Phi) = (I + {}^tD\Phi)^{-1} D\widetilde{u}(t,\xi),$$

we get

$$\mathcal{B}v = \frac{\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1}Dv \rangle}{|(I + {}^tD\Phi)^{-1}\nu|}. \tag{5.9}$$

Note that $\mathcal{A} = \mathcal{A}(s)$, $\mathcal{B} = \mathcal{B}(s)$ depend on s through Φ.

Now we are able to decouple the system (5.7), expressing $\widetilde{u}$ in terms of s.

If the function b is smooth enough ($b \in h^{1+\alpha}(\Gamma)$), for each $\sigma \in h^{2+\alpha}(\Gamma)$ with small C^1 norm there is a unique $v \in h^{2+\alpha}(\overline{\Omega})$ such that

$$\begin{cases} \mathcal{A}(\sigma)v = 0, & \xi \in \overline{\Omega}, \\ v(\xi) = 0, & \xi \in \Gamma, \\ \dfrac{\partial}{\partial \nu} v = b, & \xi \in J. \end{cases} \tag{5.10}$$

This is because $\mathcal{A}(\sigma)$ is a second order elliptic operator with h^α coefficients and without zero order terms.

We denote by $\mathcal{F}$ the function

$$\mathcal{F}(\sigma) = \mathcal{B}(\sigma)v \tag{5.11}$$

where v is the solution to (5.10), and we rewrite (5.7) as a final problem for s,

$$\begin{cases} s_t(t,\xi) + \mathcal{F}(s(t,\cdot))(t,\xi) = 0, & t \geq 0,\ \xi \in \Gamma, \\ s(0,\xi) = s_0(\xi), & \xi \in \Gamma. \end{cases} \tag{5.12}$$

This problem will be seen as an evolution equation in the space $E_0 = h^{1+\alpha}(\Gamma)$, for which the assumptions of Theorem 2.7 are satisfied. Indeed, the following statements have been proved in [18].

Theorem 5.1 *If $a > 0$ is small enough, for each $\beta \in (0,1)$ the function $\mathcal{F} : \mathcal{V}_\beta := \{s \in h^{2+\beta}(\Gamma) : \|h\|_{C^1(\Gamma)} < a\} \mapsto h^{1+\beta}(\Gamma)$, is smooth.*

Assume that $b(x) \geq 0$ for each $x \in J$, and that $b \not\equiv 0$. Then for each $s \in \mathcal{V}_\beta$, the operator $A_\beta = -\mathcal{F}'(s) : h^{2+\beta}(\Gamma) \mapsto h^{1+\beta}(\Gamma)$ is a sectorial operator in $h^{1+\beta}(\Gamma)$.

Theorem 5.1 allows to apply the theory of section 2.

We recall that the little-Hölder spaces $h^\beta(\Gamma)$ are stable by continuous interpolation, in the sense that for nonintegers $\beta_1 < \beta_2$ we have

$$(h^{\beta_1}(\Gamma), h^{\beta_2}(\Gamma))_\theta = h^{\beta_1 + \theta(\beta_2 - \beta_1)}(\Gamma)$$

for each $\theta \in (0,1)$ such that $\beta_1 + \theta(\beta_2 - \beta_1)$ is not integer. In particular, for $0 < \beta < \alpha < 1$ we have

$$h^{1+\alpha}(\Gamma) = (h^{1+\beta}(\Gamma), h^{2+\beta}(\Gamma))_\theta \tag{5.13}$$

with $\theta = \alpha - \beta$.

We fix $0 < \beta < \alpha < 1$ and we set $E_0 = h^{1+\alpha}(\Gamma)$, $E_1 = h^{2+\alpha}(\Gamma)$. By Theorem 5.1, the function $F = -\mathcal{F}$ is defined and smooth in an open set $\mathcal{O} = \mathcal{V}_\alpha \subset E_1$ with values in E_0, for each $s \in \mathcal{O}$, $F'(s) = A_\alpha$ is sectorial, and it may be seen as the part in E_0 of the sectorial operator $A_\beta : h^{2+\beta}(\Gamma) \mapsto h^{1+\beta}(\Gamma)$. By (5.13) we have $D_{A_\beta}(\theta) = h^{1+\alpha}(\Gamma) = E_0$, and by Theorem 5.1 we have $D_{A_\beta}(\theta+1) = E_1$, provided $\theta = \alpha - \beta$. Therefore, assumption (H3) is satisfied and Theorem 2.7 is applicable. We arrive at a final wellposedness theorem, see [18].

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with boundary $J \cup \Gamma$, $J \neq \varnothing$ interior to Γ and with positive distance from Γ. Let $b \in h^{2+\alpha}(J)$ $(0 < \alpha < 1)$ be such that $b(x) \geq 0$ for each $x \in J$, and $b \not\equiv 0$.*

Then for each $s_0 \in h^{2+\alpha}(\Gamma)$ with small C^1 norm there are $T > 0$ and a classical solution (s,u) of (1.3)-(5.1) in $[0,T]$. The function s is such that $t \mapsto s(t,\cdot)$ is in $C([0,T]; h^{2+\alpha}(\Gamma)) \cap C^1([0,T]; h^{1+\alpha}(\Gamma))$, and $\|s(t,\cdot)\|_{C^1(\Gamma)} \leq a$. Denoting by Ω_t the bounded open set with boundary $J \cup \mathrm{Range}\, s(t,\cdot)$, u is continuous in $\{(t,x) : 0 \leq t \leq T, x \in \overline{\Omega_t}\}$ and $u(t,\cdot)$ belongs to $h^{2+\alpha}(\overline{\Omega_t})$ for each $t \in [0,T]$. The couple (s,u) is the unique solution to (1.3)-(5.1) enjoying these regularity properties.

Problem (1.3) is the simplest example of a class of free boundary problems that can be treated similarly. Among them we quote one- and two-phase Hele-Shaw models with surface tension, also called Mullins-Sekerka models. See [18, 19, 20, 15, 14].

Let us describe the two-phase problem. Ω is again a bounded open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For $t \geq 0$, Γ_t is a compact connected hypersurface which is the boundary of an open set $\Omega_t \subset \Omega$. The normal velocity of Γ_t and its mean curvature are denoted by $V(t,\cdot)$ and $\kappa(t,\cdot)$, respectively. Again, V is taken to be positive for expanding hypersurfaces, and moreover κ is positive for uniformly convex hypersurfaces. $\Omega_1(t)$ and $\Omega_2(t)$ are the open subset of Ω separated by Γ_t, $\Omega_1(t)$ being the interior region; $n(t,\cdot)$ is the unit exterior normal vector to $\partial\Omega_1(t)$. We consider the system

$$
\begin{cases}
\Delta u = 0, & t > 0,\ x \in \Omega_1(t) \cup \Omega_2(t), \\
u = \kappa,\ \ V = -\left[\dfrac{\partial u}{\partial n}\right], & t > 0,\ x \in \Gamma_t, \\
\dfrac{\partial u}{\partial n} = 0, & t > 0,\ x \in \partial\Omega,
\end{cases}
\tag{5.14}
$$

where the brackets denote the jump across the free boundary: $[\partial u/\partial n] = \partial u_1/\partial n - \partial u_2/\partial n$, u_j being the restrictions of u to $\Omega_j(t)$. As before, the unknowns are the free boundary Γ_t for $t > 0$ and the function $u(t, x)$ for $t > 0$, $x \in \overline{\Omega}$, while the initial hypersurface Γ_0 is given.

The procedure described for the Hele-Shaw flow gives local existence and uniqueness of a regular solution to (5.14) for each regular (precisely: $h^{2+\alpha}$) initial hypersurface Γ_0.

Fixed any smooth hypersurface $\Gamma \subset \Omega$, for initial data close to Γ we look for Γ_t as the graph of an unknown function $s(t,\cdot)$ defined on Γ, and we proceed as before. We eliminate the unknown u, expressing it in terms of s. More precisely, $u(t,\cdot)$ has to be the solution v to

$$
\begin{cases}
\mathcal{A}(\sigma)v = 0, & \xi \in \Omega_1 \cup \Omega_2, \\
v(\xi) = K(\sigma), & \xi \in \Gamma, \\
\dfrac{\partial}{\partial \nu} v = 0, & \xi \in \partial\Omega,
\end{cases}
\tag{5.15}
$$

with $\sigma = s(t,\cdot)$, the fixed open sets Ω_1 and Ω_2 are the images $\Omega_1(t)$ and $\Omega_2(t)$ under the change of coordinates, and $K(\sigma)$ is the transformed mean curvature operator after the change of coordinates. The jump condition $V = -[\partial u/\partial n]$ is transformed to $s_t + \mathcal{B}(s)v = 0$, where now $\mathcal{B}(s)$ is the transformed jump of the normal derivative across Γ_t. So we arrive at a final equation for s of the type (5.12), where now $\mathcal{F}$ is a third order nonlocal operator with quasilinear structure. This is because the curvature depends on s through its derivatives up to the second order, in a quasilinear way since it depends linearly on the second order derivatives of s. The character of (5.12) is still parabolic and $\mathcal{F}$ is still smooth; more precisely it was proved in [19] that if $a > 0$ is small enough, for each $\beta \in (0,1)$ the function $\mathcal{F} : \mathcal{U}_\beta := \{s \in h^{3+\beta}(\Gamma) :\ \|h\|_{C^1(\Gamma)} < a\} \mapsto h^\beta(\Gamma)$, is C^∞, and the operator $A_\beta = -\mathcal{F}'(s) : h^{3+\beta}(\Gamma) \mapsto h^\beta(\Gamma)$ is a sectorial operator in $h^{1+\beta}(\Gamma)$.

So, as far as local existence, uniqueness and regularity are concerned, we are not forced to use Theorem 2.7 but we can use other theories of abstract quasilinear evolution equations in Banach spaces, that allow less regular initial data. In the paper [19] the theory developed in [2, 3] was used to arrive at a final result of existence and uniqueness of a solution to (5.12), $s \in C^\infty((0,T] \times \Gamma)$, such that $t \mapsto s(t,\cdot) \in C([0,T]; h^{2+\beta}(\Gamma)) \cap C((0,T]; h^{3+\beta}(\Gamma))$, for each initial datum $s_0 \in h^{2+\beta+\varepsilon}(\Gamma)$, $\varepsilon > 0$, with small C^1 norm. The number $T > 0$ depends on s_0. If in addition $s_0 \in h^{3+\beta}(\Gamma)$, then $t \mapsto s(t,\cdot) \in C([0,T]; h^{3+\beta}(\Gamma))$. (The last statement comes by applying Theorem 2.7 with $E_0 = h^\beta(\Gamma)$, $E_1 = h^{3+\beta}(\Gamma)$.)

Coming back to (5.14) we get a unique regular local solution (Γ_t, u).

It is easy to see that (5.14) admits spheres as stationary solutions, and it is of interest to study their stability. Fixed any sphere $\mathcal{S} \subset \Omega$, we need to know some spectral properties of the linearized operator $A = -\mathcal{F}'(0) : h^{3+\beta}(\mathcal{S}) \mapsto h^\beta(\mathcal{S})$. It was proved in [20] that the spectrum of A consists of a sequence of negative eigenvalues plus the semisimple isolated eigenvalue 0, with multiplicity $N+1$. Therefore, this is a critical case of stability. The center manifold theory of section 2, with $E_0 = h^\beta(\Gamma)$, $E_1 = h^{3+\beta}(\Gamma)$, may be applied, and it gives the existence of a $(N+1)$-dimensional locally invariant manifold $\mathcal{M} \subset h^{3+\beta}(\mathcal{S})$ which attracts all the small orbits. Going further in the analysis, in [20] it was proved that the center manifold itself consists of spheres, and for each small initial datum the solution exists in the large and converges to one of these spheres exponentially fast as $t \to \infty$. The convergence is in the $h^{3+\beta}(\mathcal{S})$ norm, even for initial data in $h^{2+\beta+\varepsilon}(\mathcal{S})$. See [20] for the details.

5.2 Models from combustion theory

Perhaps surprisingly, wellposedness of the Cauchy problem for (1.2) is still an open problem in dimension $N \geq 2$. The Cauchy problem consists of (1.2) supplemented by an initial condition,

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}_0, \tag{5.16}$$

where Ω_0 is a given open set in $\mathbb{R}^N$. There are results of existence of weak solutions without uniqueness ([11]), of existence and uniqueness of regular classical solutions with loss of regularity with respect to the inital data ([6]), of local wellposedness for initial data close to special solutions such as travelling waves or selfsimilar solutions ([7]) and for special geometries ([4, 21, 29]). But none of them can be considered a standard wellposedness result.

We describe here the approach leading to the fully nonlinear evolution equations discussed in section 3. The change of coordinats used here is the same of the previous subsection, taking as reference set Ω the initial set Ω_0. Let us assume that Ω_0 is a nonempty open set in $\mathbb{R}^N$ with $C^{3+\alpha}$ boundary Γ. The boundary Γ_t of Ω_t is sought again as the range of an unknown function $s(t,\cdot):\Gamma\mapsto\mathbb{R}^N$. The coordinate transformation (5.6) transforms Ω_t into the fixed domain Ω_0 and it leads to a Cauchy problem for the couple $(s,\widetilde{u})$ where $\widetilde{u}(t,\xi)=u(t,x)$ is again the function u in the new coordinates:

$$\begin{cases} \widetilde{u}_t - \langle D\widetilde{u}, (I+{}^tD\Phi)^{-1}\Phi_t\rangle = \mathcal{A}(s)\widetilde{u}, & t>0,\ x\in\overline{\Omega}_0, \\ \widetilde{u}=0,\ \ \mathcal{B}(s)\widetilde{u}=-1, & t>0,\ x\in\Gamma_0, \\ s(0,\cdot)=0,\ \ \widetilde{u}(0,\cdot)=u_0, & x\in\Gamma_0, \\ \widetilde{u}(0,\cdot)=u_0, & x\in\overline{\Omega}_0, \end{cases} \tag{5.17}$$

where, as before, $\mathcal{A}(s)$ is the Laplacian in the new coordinates, and $\mathcal{B}(s)$ is the normal derivative in the new coordinates, see formulas (5.8), (5.9).

System (5.17) still has to be decoupled. Instead of proceeding like in problem (1.3), we introduce a new unknown w by splitting $\widetilde{u}$ as

$$\widetilde{u}(t,\xi)=u_0(\xi)+\langle Du_0(\xi),\Phi(t,\xi)\rangle+w(t,\xi). \tag{5.18}$$

At $t=0$ we have $\widetilde{u}(0,\xi)=u_0(\xi)$, $\Phi(0,\cdot)\equiv 0$, so that

$$w(0,\xi)=0,\ \ \xi\in\overline{\Omega}_0. \tag{5.19}$$

(5.18) allows to get s in terms of w thanks to the boundary condition $u=0$ at Γ_t, which gives

$$s(t,\xi)=w(t,\xi),\ \ t\geq 0,\ \xi\in\Gamma_0, \tag{5.20}$$

so that

$$\Phi(t,\xi)=w(t,\xi')\widetilde{\nu}(\xi),\ \ t\geq 0,\ \xi\in\overline{\Omega}_0, \tag{5.21}$$

where $\widetilde{\nu}(\xi)$ is the extension of the normal vector field in formula (5.5): $\widetilde{\nu}(\xi)=\alpha(r)\nu(\xi')$ if $\xi\in\mathcal{R}$, $\widetilde{\nu}(\xi)=0$ otherwise. Replacing (5.21) in (5.17) we get

$$w_t=\mathcal{F}_1(\xi,w,Dw,D^2w)+\mathcal{F}_2(\xi,w,Dw)s_t,\ \ t\geq 0,\ \xi\in\overline{\Omega}_0, \tag{5.22}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ are obtained respectively from

$$\mathcal{A}(s)(u_0 + \langle Du_0, \Phi\rangle + w) \tag{5.23}$$

and from

$$-\langle Du_0 - (I + D\Phi)^{-1}D(u_0 + \langle Du_0, \Phi\rangle + w), \widetilde{\nu}\rangle, \tag{5.24}$$

replacing $\Phi = w(t, \xi')\widetilde{\nu}(\xi)$.

Equation (5.22) still contains s_t, that we eliminate using again the identity $s = w$ at the boundary which gives $s_t = w_t$. Replacing in (5.22) for $\xi \in \Gamma_0$ we get

$$s_t(1 - \mathcal{F}_2(\xi, w, Dw)) = \mathcal{F}_1(\xi, w, Dw, D^2w), \;\; t \geq 0, \; \xi \in \Gamma_0.$$

At $t = 0$ we have $w \equiv 0$, and $\mathcal{F}_2$ vanishes at $(\xi, 0, 0)$, so that, at least for t small, $\mathcal{F}_2(\cdot, w(t, \cdot), Dw(t, \cdot))$ is different from 1 and we get s_t in terms of w,

$$s_t = \mathcal{F}_3(\xi, w, Dw, D^2w) = \frac{\mathcal{F}_1(\xi, w, Dw, D^2w)}{1 - \mathcal{F}_2(\xi, w, Dw)}, \;\; t \geq 0, \; \xi \in \Gamma_0, \tag{5.25}$$

which, replaced in (5.22), gives the final equation for w,

$$w_t = \mathcal{F}(w)(\xi), \;\; t \geq 0, \; \xi \in \overline{\Omega}_0, \tag{5.26}$$

where

$$\mathcal{F}(w)(\xi) = \mathcal{F}_1(\xi, w, Dw, D^2w) + \mathcal{F}_2(\xi, w, Dw)\mathcal{F}_3(\xi, w, Dw, D^2w). \tag{5.27}$$

Note that $\mathcal{F}(0)(\xi) = \Delta u_0(\xi)$, and $\mathcal{F}(w)(\xi) = \Delta w + \Delta u_0(\xi)$ if ξ is far from the boundary Γ_0.

The function $\mathcal{F}(v)$ is defined for $v \in C^2(\overline{\Omega}_0)$ with small C^1 norm; precisely, it is defined for $v \in C^2(\overline{\Omega}_0)$ such that $1 - \mathcal{F}_2(\cdot, v(\cdot), Dv(\cdot)) \neq 0$. From formulas (5.20), (5.23), (5.24), (5.25) we see that $\mathcal{F}(v)(\xi)$ depends smoothly on v, Dv, D^2v and their traces at the boundary; therefore the function $v \mapsto \mathcal{F}(v)$ is continuously differentiable from $\mathcal{O} = \{v \in C^2(\overline{\Omega}_0) : \|v\|_{C^1(\overline{\Omega}_0)} \leq r\}$ to $C(\overline{\Omega}_0)$, and from $\mathcal{O}_\alpha = \{v \in C^{2+\alpha}(\overline{\Omega}_0) : \|v\|_{C^1(\overline{\Omega}_0)} \leq r\}$ to $C^\alpha(\overline{\Omega}_0)$ if r is small.

The boundary condition for w comes from the boundary condition $\partial u/\partial n = -1$ in (1.2). Using (5.9) we get

$$\langle (I + {}^tD\Phi)^{-1}\nu, (I + {}^tD\Phi)^{-1}D(u_0 + \langle Du_0, \Phi\rangle + w\rangle + |(I + {}^tD\Phi)^{-1}\nu| = 0, \tag{5.28}$$

which gives

$$\mathcal{G}(\xi, w(t,\xi), Dw(t,\xi)) = 0, \;\; t \geq 0, \; \xi \in \Gamma_0, \tag{5.29}$$

when we replace $\Phi = w(t,\xi')\widetilde{\nu}(\xi)$ in (5.28). The function $\mathcal{G}(\xi, u, p)$ is smooth with respect to u and p_i, $i = 1, \ldots, N$, and its derivatives are continuous in (ξ, u, p) and $C^{1/2+\alpha/2}$ in ξ. It follows that $v \mapsto \mathcal{G}(\cdot, v, Dv)$ is smooth from a neighborhood of 0 in $C^1(\overline{\Omega}_0)$ to $C(\Gamma_0)$, and from a neighborhood of 0 in $C^{1+\alpha}(\overline{\Omega}_0)$ to $C^\alpha(\Gamma_0)$. Moreover $\mathcal{G}(\xi, 0, 0) = 0$.

Concerning the linear parts of $\mathcal{F}$ and $\mathcal{G}$ near 0, the following lemma was proved in [6].

Lemma 5.3 *$\mathcal{F}_v(0)$ is the sum of the Laplacian plus a nonlocal differential operator of order* 1. *Moreover,*

$$\mathcal{G}_v(0)w = \mathcal{B}w := \frac{\partial w}{\partial \nu} + \frac{\partial^2 u_0}{\partial \nu^2} w.$$

The final problem for the only unknown w may be rewritten in the form discussed in section 3, as

$$\begin{cases} w_t = \mathcal{A}w + F(w), \;\; t \geq 0, \; \xi \in \overline{\Omega}_0, \\ \\ \mathcal{B}w = G(w), \;\; t \geq 0, \; \xi \in \Gamma_0, \\ \\ w(0, \cdot) = 0, \;\; \xi \in \overline{\Omega}_0. \end{cases} \tag{5.30}$$

The difference between (5.30) and (1.1) is that the linear operator $\mathcal{A}$ and the nonlinearities F, G contain nonlocal terms. The nonlocal part of $\mathcal{A}$ concerns only first order derivatives, so that it may be considered as a non important perturbation. F depends nonlocally also on the second order derivatives of w, but it is (at least) quadratic near $w = 0$. The arguments used in the proof of Theorem 3.2 work also for (5.30), and give a local existence and uniqueness result for w. Precisely, there is $R_0 > 0$ such that for every $R \geq R_0$ and for every sufficiently small $T > 0$ problem (5.30) has a unique solution in the ball $B(0, R) \subset C^{1+\alpha/2, 2+\alpha}([0,T] \times \overline{\Omega}_0)$. For the details of the proof see [6].

Note that we cannot use Theorem 4.1 to get local existence for w because $F(0) \neq 0$.

Now we come back to the original problem (1.2). Recalling that $s(t,\xi) = w(t,\xi)$ for each $t \in [0,T]$, $\xi \in \partial\Omega$, we can define Γ_t. Of course s has the same regularity of w, i.e. it is in $C^{1+\alpha/2, 2+\alpha}([0,T] \times \Gamma_0)$. Then we define $\widetilde{u}$ through

(5.18), where Φ is given by (5.5). Again, $\widetilde{u}$ has the same regularity of w. As a last step we define u through the change of coordinates, $u(t,x) = \widetilde{u}(t,\xi)$ where $x = \xi + \Phi(t,\xi)$. This leads to loss of regularity: starting with initial data in $C^{3+\alpha}$ we get a local solution with $C^{2+\alpha}$ space regularity. The final result (see [6]) is the following.

Theorem 5.4 *Let $\Omega_0 \subset \mathbb{R}^N$ be a nonempty bounded open set with $C^{3+\alpha}$ boundary Γ_0, and let $u_0 \in C^{3+\alpha}(\overline{\Omega}_0)$ satisfy the compatibility conditions $u_0 = 0$, $\partial u_0/\partial n = -1$ at Γ_0. Then there is $T > 0$ such that problem (1.2) has a solution (Ω_t, u) such that the $(N+1)$-dimensional hypersurface $\mathcal{S} = \{(t,x) : 0 \le t \le T,\ x \in \Gamma_t\}$ and each $\Gamma_t = \partial\Omega_t$ are of class $C^{1+\alpha/2,2+\alpha}$, and the function $u : \{(t,x);\ 0 \le t \le \delta,\ x \in \overline{\Omega}_t\} \mapsto \mathbb{R}$ is of class $C^{1+\alpha/2,2+\alpha}$.*

If in addition Γ_0 and u_0 are in $C^{4+\alpha}$, and the further compatibility condition $\mathcal{B}(\Delta u_0) = 0$ at Γ_0 holds, then $\mathcal{S}$ and each Γ_t are of class $C^{3/2+\alpha/2,3+\alpha}$, and the function u is of class $C^{3/2+\alpha/2,3+\alpha}$. Moreover, the couple (Ω_t, u) is the unique solution with such regularity properties.

If the initial data are close to the initial datum (Ω, U) of a given regular solution, the same method gives existence and uniqueness of a local classical solution without loss of regularity. Moreover we can go further in the investigation of the stability properties of the established solution.

The free boundary problem is transformed into a fixed boundary problem in Ω by the changement of coordinates (5.6), so that the unknown s is the signed distance from $\Gamma = \partial\Omega$. The splitting (5.18) is replaced by

$$\widetilde{u}(t,\xi) = U(\xi) + \langle DU(\xi), \Phi(t,\xi)\rangle + w(t,\xi), \ \ \xi \in \Omega. \tag{5.31}$$

This gives again $s(t,\xi) = w(t,\xi)$ for $\xi \in \Gamma$. The final problem for w has initial datum $w_0 = \widetilde{u}_0 - U - \langle DU, \Phi(0,\xi)\rangle$ which does not vanish, but it is small.

Let us show how to apply the results of section 4. The simplest situation would be to have initial data close to stationary solutions, but (1.2) has no bounded stationary solutions.

So we consider the case where (Ω_0, u_0) is close to the initial datum of a self-similar solution. Existence and properties of self-similar solutions have been discussed in [11]; they are solutions of the type (Ω_t, u) where

$$u(t,x) = (T-t)^\alpha f(|x|/(T-t)^\beta), \ \ \Omega_t = \{|x| < r(T-t)^\beta\}, \ \ 0 \le t < T,$$

with T, α, β, $r > 0$.

It is easy to see that there exist self-similar solutions only for $\alpha = \beta = 1/2$, and that the function $g(x) = f(|x|)$ has to be an eigenfunction of an Ornstein-Uhlenbeck type operator in the ball $B(0,r)$,

$$\Delta g - \frac{1}{2}\langle x, Dg(x)\rangle + \frac{1}{2}g = 0,$$

with two boundary conditions, $g = 0$, $\partial g/\partial\nu = -1$. In other words, f has to solve

$$\begin{cases} f''(\eta) + \frac{N-1}{\eta}f'(\eta) + \frac{1}{2}f(\eta) = \frac{1}{2}\eta f'(\eta) \quad \text{for } 0 < \eta \leq r, \\ \\ f'(0) = f(r) = 0 \ , \ f'(r) = -1, \end{cases} \tag{5.32}$$

which looks overdetermined, but it is not, because also r is an unknown. In [11] it is proved that there exist a unique $r > 0$ and a unique C^2 function $f : [0,r] \mapsto \mathbb{R}$ satisfying (5.32) and such that $f(\eta) > 0$ for $0 \leq \eta < r$. Moreover f is analytic.

It is convenient now to transform the problem to selfsimilar variables

$$\hat{x} = \frac{x}{(T-t)^{\frac{1}{2}}}, \ \ \hat{t} = -\log(T-t), \tag{5.33}$$

and to set

$$\hat{u}(\hat{x},\hat{t}) = \frac{u(x,t)}{(T-t)^{\frac{1}{2}}}, \ \ \hat{\Omega}_{\hat{t}} = \{\hat{x} : x \in \Omega_t\}. \tag{5.34}$$

Omitting the hats, we arrive at

$$\begin{cases} u_t = \Delta u - \dfrac{1}{2}\langle x, Du\rangle + \dfrac{1}{2}u, \quad t > 0, \ x \in \Omega_t, \\ \\ u = 0, \quad \dfrac{\partial u}{\partial n} = 1, \quad t > 0, \ x \in \partial\Omega_t. \end{cases} \tag{5.35}$$

The selfsimilar solution is transformed by (5.33)–(5.34) into a stationary solution

$$U(x) = f(|x|), \Omega = \{x \in \mathbb{R}^N : |x| < r\}, \tag{5.36}$$

of (5.35). From now on we proceed as before: we change variables through the isomorphism (5.6), we set $\widetilde{u}(t,\xi) = u(t,x) - U(x)$, we define w by the splitting (5.19) and we arrive at a final equation for w in the fixed domain $\Omega = B(0, r)$,

$$\begin{cases} w_t = \Delta w - \frac{1}{2}\langle \xi, Dw\rangle + \frac{w}{2} + \phi(w, Dw, D^2 w), \quad t \geq 0, \ \xi \in \overline{\Omega}, \\ \frac{\partial w}{\partial \nu} + \left(\frac{N-1}{r} - \frac{r}{2}\right) w = \psi(w, Dw), \quad t \geq 0, \ \xi \in \partial\Omega, \\ w(0, \xi) = w_0(\xi), \quad \xi \in \overline{\Omega}, \end{cases} \tag{5.37}$$

where ϕ and ψ are smooth and quadratic near $w = 0$.

This problem fits into the theory discussed in section 4. Theorem 4.1 implies that for every $T > 0$ and $\alpha \in (0, 1)$ there are R, $\rho > 0$ such that (5.37) has a solution $w \in C^{1+\theta/2, 2+\theta}([0, T] \times \overline{\Omega})$ provided $\|w_0\|_{C^{2+\theta}(\overline{\Omega})} \leq \rho$ and $\partial w_0/\partial \nu + ((N-1)/r) - (r/2))w_0 = \psi(w_0, Dw_0)$. Moreover u is the unique solution in $B(0, R) \subset C^{1+\theta/2, 2+\theta}([0, T] \times \overline{\Omega})$.

To go on in the analysis, we define the operator A by

$$D(A) = \{v \in \cap_{p>1} W^{2,p}(\Omega) : \ \Delta v \in C(\overline{\Omega}), \ \mathcal{B}v = 0 \ \text{ on } \partial\Omega\}, \quad Av = \mathcal{A}v,$$

where

$$\mathcal{A}v = \Delta v - \frac{1}{2}\langle x, Dv\rangle + \frac{1}{2}v,$$
$$\mathcal{B}v = \frac{\partial v}{\partial \nu} + \left(\frac{N-1}{r} - \frac{r}{2}\right)v.$$

It has been proved in [7] that the spectrum of A consists of the semisimple eigenvalues 1, 1/2 plus a sequence of negative eigenvalues; moreover, the eigenspace with eigenvalue 1 is one-dimensional, the eigenspace wih eigenvalue 1/2 has dimension N.

The principle of linearized stability as stated in Theorem 4.6(ii) shows that the null solution of (5.37) is unstable in $C^{2+\theta}(\overline{\Omega})$, and therefore the selfsimilar solution of the original problem (1.2) is unstable. This is not surprising, because the original problem is invariant under translations in x and t; if we apply a small shift to (5.36), we obtain another selfsimilar solution which is transformed by (5.33)–(5.34) into a solution which starts close to (5.36) but moves far from it. Therefore, the local unstable manifold of (5.36) given by Theorem 4.13(i) must contain the images under (5.33) of shifts in space and time of (5.36), that are given by

$$\sqrt{1 + \epsilon_2 e^t}\, U\left(\frac{x - \epsilon_1 e^{\frac{1}{2}t}}{\sqrt{\epsilon_2 e^t + 1}}\right), \tag{5.38}$$

with $\epsilon_1 \in \mathbb{R}^N$ and $\epsilon_2 \in \mathbb{R}$. Since the local unstable manifold has to be $(N+1)$-dimensional, then it consists only of the images of (5.38) under the transformation (5.33). However, all the orbits in the unstable manifold have the same selfsimilar profile, so that the equilibrium (5.36) looks stable even if it unstable. Roughly speaking, the profile itself is stable.

The above discussion is taken from [7]. A study of the stability of the (planar) travelling wave solutions to (1.2) is in the paper [8]. Stability questions for more complicated multidimensional free boundary equations and systems arising in combustion theory have been studied by these methods in the papers [9, 10, 31] and in the papers quoted therein.

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