On Fredholm properties of $\mathcal{L}u = u' - A(t)u$ for paths of sectorial operators

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Abstract

We consider a path of sectorial operators $t \mapsto A(t) \in C^\alpha(\mathbb{R}, L(D,X))$, $0 < \alpha < 1$, in general Banach space X, with common domain $D(A(t)) = D$, and with hyperbolic limits at $\pm\infty$. We prove that there exist exponential dichotomies in the halflines $(-\infty, -T]$ and $[T, +\infty)$ for large T, and we study the operator $(\mathcal{L}u)(t) = u'(t) - A(t)u(t)$ in the space $C^\alpha(\mathbb{R}, D) \cap C^{1+\alpha}(\mathbb{R}, X)$. In particular, we give sufficient conditions in order that $\mathcal{L}$ is a Fredholm operator. In this case the index of $\mathcal{L}$ is given by an explicit formula, which coincides to the well known spectral flow formula in finite dimension. Such sufficient conditions are satisfied, for instance, if the embedding $D \hookrightarrow X$ is compact.

1 Introduction

Let $\{A(t) : t \in \mathbb{R}\}$ be a family of sectorial operators in a general Banach space X, with common domain $D(A(t)) = D$. Under reasonable regularity assumptions (i.e. $t \mapsto A(t) \in C^\alpha(\mathbb{R}, L(D,X))$, with $\alpha > 0$), forward Cauchy problems such as

$$\begin{cases} u'(t) - A(t)u(t) = f(t), & a \le t \le b, \\ u(a) = x, \end{cases}$$

are well understood if $a < b \in \mathbb{R}$. See e.g. [16], [3, 4, 2], [11, ch. 6]. Problems in unbounded time intervals, including backward Cauchy problems and problems on the whole real line, are of different nature, and at present a satisfactory theory is available only in the case where the associated evolution operator $G(t,s)$ has an exponential dichotomy on $\mathbb{R}$. This has been worked out in the periodic case $A(t+T) = A(t)$, see [9, §7.2], [11, ch. 6], [8].

In this paper we consider the asymptotically autonomous case, when there exist the limits

$$\lim_{t \to +\infty} A(t) = A_{+\infty}, \quad \lim_{t \to -\infty} A(t) = A_{-\infty},$$

in $L(D,X)$, and such limits are hyperbolic, i.e.

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{+\infty}) \cap i\mathbb{R} = \emptyset.$$

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Then we study the operator $\mathcal{L}$ defined by

$$\begin{cases} \mathcal{L} : D(\mathcal{L}) = C^{1+\alpha}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, D) \mapsto C^\alpha(\mathbb{R}, X), \\ \\ (\mathcal{L}u)(t) = u'(t) - A(t)u(t), \quad t \in \mathbb{R}. \end{cases} \tag{1.1}$$

The operators $A_{+\infty}$ and $A_{-\infty}$, being limits of sectorial operators, are sectorial. Therefore the sets $\sigma(A_{+\infty}) \cap \{ \text{ Re}\,\lambda > 0\}$ and $\sigma(A_{-\infty}) \cap \{ \text{ Re}\,\lambda > 0\}$ are compact, and the relevant spectral projections $P_{+\infty}$, $P_{-\infty}$ are well defined.

If $X = D = \mathbb{R}^N$ it is well known that $\mathcal{L}$ is a Fredholm operator, with index equal to

$$ind\, \mathcal{L} = dim\ P_{-\infty}(\mathbb{R}^N) - \ dim\ P_{+\infty}(\mathbb{R}^N). \tag{1.2}$$

This number may be written also as

$$dim\ (I - P_{+\infty})(\mathbb{R}^N) - dim\ (I - P_{-\infty})(\mathbb{R}^N),$$

and as minus the spectral flow. The spectral flow is the sum of the algebraic multiplicities of the eigenvalues of $A(t)$ whose real part changes from negative to positive, minus the sum of the algebraic multiplicities of the eigenvalues whose real part changes from positive to negative, as t increases. See [13, 14] for precise definitions and [7] for some applications.

Here we generalize one of these formulae, showing (corollary 3.13) that if $P_{-\infty}(X)$ and $P_{+\infty}(X)$ are finite dimensional, then $\mathcal{L}$ is a Fredholm operator and its index is equal to

$$ind\, \mathcal{L} = dim\ P_{-\infty}(X) - \ dim\ P_{+\infty}(X). \tag{1.3}$$

In its turn, the assumption that $P_{-\infty}(X)$ and $P_{+\infty}(X)$ are finite dimensional is satisfied, for instance, if D is compactly embedded in X. Note that in general the dimensions of $(I - P_{-\infty})(X)$ and of $(I - P_{+\infty})(X)$ are infinite.

The literature on the subject deals mainly with the case where X is a Hilbert space, and $C^\alpha(\mathbb{R}, X)$, $C^{1+\alpha}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, D)$ are replaced respectively by $L^2(\mathbb{R}, X)$, $H^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, D)$. In the paper [14], Robbin and Salamon considered a C^1 path of self-adjoint operators, with common domain D compactly embedded in X. They proved that $\mathcal{L} : D(\mathcal{L}) = H^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, D) \mapsto L^2(\mathbb{R}, X)$ is a Fredholm operator with index given again by a generalization of the spectral flow formula.

More recently, Abbondandolo and Majer in [1] considered a path of bounded, not necessarily self-adjoint, operators in a Hilbert space, and studied $\mathcal{L}$ as an operator from $H^1(\mathbb{R}, X)$ to $L^2(\mathbb{R}, X)$. Among other results, they proved that if $(I - P_{+\infty})(X)$ and $(I - P_{-\infty})(X)$ are finite dimensional, then $\mathcal{L}$ is Fredholm with index equal the difference of their dimensions. They considered also situations in which the subspaces $P_{+\infty}(X)$, $(I - P_{+\infty})(X)$, $P_{-\infty}(X)$ and $(I - P_{-\infty})(X)$ are infinite dimensional and $\mathcal{L}$ comes out to be a Fredholm operator, and showed that in general the index of $\mathcal{L}$ does not depend only on the endpoints $A_{-\infty}$ and $A_{+\infty}$. We note that in their case, since the operators $A(t)$ are bounded, the choice of the spaces is not essential, and $H^1(\mathbb{R}, X)$, $L^2(\mathbb{R}, X)$ could be replaced as well by $C^1_b(\mathbb{R}, X)$, $C_b(\mathbb{R}, X)$ or by $C^{1+\alpha}(\mathbb{R}, X)$, $C^\alpha(\mathbb{R}, X)$, respectively.

From the point of view of Cauchy problems, our assumptions are halfway between the assumptions of [14] and those of [1]. Indeed, both forward and backward Cauchy problems are ill posed in general under the assumptions of [14]; in our case forward Cauchy problems are well posed but backward Cauchy problems are ill posed in general, and both forward and backward Cauchy problems are well posed under the assumptions of [1]. In terms of evolution operators, the evolution operator $G(t, s)$ does not exist under the assumptions of [14], it exists only for $t \geq s$ under our assumptions, and it exists for all t, $s \in \mathbb{R}$ under the assumptions of [1].

In the paper [14] the Hilbert space structure is essential. Some of the proofs of [1] may be extended without important differences to a Banach space setting, but other ones work only in Hilbert spaces.

Our main tool is the existence of exponential dichotomies in the halflines $(-\infty, -T]$ and $[T, +\infty)$ for T large enough. Indeed, both $A_{+\infty}$ and $A_{-\infty}$ are sectorial hyperbolic operators, so that their evolution operators $e^{(t-s)A_{+\infty}}$ and $e^{(t-s)A_{-\infty}}$ have constant exponential dichotomies on $\mathbb{R}$. For t large, $A(t)$ is a small perturbation of $A_{+\infty}$, so that $G(t,s)$, $t \geq s \geq T$, is a small perturbation of $e^{(t-s)A_{+\infty}}$; similarly, for t small $A(t)$ is a small perturbation of $A_{-\infty}$, so that $G(t,s)$, $s \leq t \leq -T$, is a small perturbation of $e^{(t-s)A_{-\infty}}$. Then we can prove existence of exponential dichotomies by perturbation arguments.

Exponential dichotomies for abstract parabolic evolution operators in general Banach spaces have been recently studied by Schnaubelt in [15]. He considers a family of operators $A(t)$, $t \geq 0$, with possibly nonconstant but dense domains, such that $A(t)$ goes to a hyperbolic operator $A_{+\infty}$ in a suitable sense as $t \to +\infty$, and he proves the existence of an exponential dichotomy in $[T, +\infty)$ for T large by a perturbation argument different from ours. His method may be extended to get an exponential dichotomy in a halfline $(-\infty, -T]$.

Once we have the powerful tool of exponential dichotomies at our disposal, we can characterize the bounded solutions to forward Cauchy problems,

$$\begin{cases} u'(t) - A(t)u(t) = f(t), \;\; t > s, \\ \\ u(s) = x, \end{cases}$$

for $s \geq T$, and the bounded solutions to backward Cauchy problems,

$$\begin{cases} u'(t) - A(t)u(t) = g(t), \;\; t < s, \\ \\ u(s) = x, \end{cases}$$

for $s \leq -T$, for continuous bounded f and g. By "solution" we mean, as usual, "mild solution", see e.g. [11, ch. 6]. In fact, here we meet one of the typical difficulties of abstract parabolic equations: if f is continuous in some interval $[a,b]$, in general there is no $u \in C^1([a,b], X) \cap C([a,b], D)$ such that $u' - A(\cdot)u = f$, even in the autonomous case $A(t) \equiv A$. On the contrary, if $f \in C^\alpha([a,b], X)$ then the equation $u' - A(\cdot)u = f$ has classical solutions in $[a,b]$, and the solutions belong to $C^{1+\alpha}([a,b], X) \cap C^\alpha([a,b], D)$ provided necessary compatibility conditions at $t = a$ hold. See next theorem 3.4.

This is why we work in Hölder spaces rather than in spaces of continuous functions. Another possible choice could be $L^p(\mathbb{R}, D) \cap W^{1,p}(\mathbb{R}, X)$ with $1 < p < \infty$, but in such a case we need further suitable assumptions on X and on the operators $A(t)$.

The paper is structured as follows. In section 2 we show that $G(t,s)$ has exponential dichotomies in $(-\infty, -T]$ and in $[T, +\infty)$ for T large, and we use them for the study of forward and backward Cauchy problems in halflines. In section 3 we define the stable and unstable linear spaces of the associated system $u'(t) = A(t)u(t)$ and we investigate the connections between these spaces and the Fredholm properties of $\mathcal{L}$, extending to our situation some results of [1]. In section 4 we give some applications to paths of elliptic operators in different Banach spaces.

2 Exponential dichotomies

We begin this section with some standard notation.

Let X be a Banach space with norm $\|\cdot\|$, and let $I \subset \mathbb{R}$ be any interval. If $0 < \alpha < 1$, $C^\alpha(I, X)$ is the space of all the bounded and α-uniformly Hölder continuous functions from I to X, and $C^{1+\alpha}(I, X)$ is the space of all the differentiable functions from I to X with derivative in $C^\alpha(I, X)$. $B(I, X)$ is the space of the bounded functions from I to X. Such spaces are endowed with the natural norms

$$\|u\|_{B(I,X)} = \|u\|_\infty = \sup_{t \in I} \|u(t)\|,$$

$$\|u\|_{C^\alpha(I,X)} = \|u\|_\infty + [u]_{C^\alpha} = \|u\|_\infty + \sup_{t>s,\, t,s\in I} \frac{\|u(t)-u(s)\|}{(t-s)^\alpha},$$

$$\|u\|_{C^{1+\alpha}(I,X)} = \|u\|_\infty + \|u'\|_\infty + [u']_{C^\alpha}.$$

Let D be another Banach space, continuously embedded in X, not necessarily dense in X. In this section we shall establish exponential dichotomies in suitable halflines $(-\infty,-T]$, $[T,+\infty)$ for the evolution operator associated to a path of sectorial[(1)] operators. Precisely, we assume that

$$\begin{cases} (i) \quad \exists \alpha\in(0,1):\; t\mapsto A(t)\in C^\alpha(\mathbb{R}, L(D,X)), \\ \\ (ii) \quad \forall t\in\mathbb{R},\; A(t): D(A(t)) = D \to X \text{ is a sectorial operator.} \end{cases} \tag{2.1}$$

By the equality $D(A(t)) = D$ we mean that the graph norm of $A(t)$ is equivalent to the norm of D.

In any case (see [16] if D is dense, [11, ch. 6] if D is not dense) there exists an associated evolution operator $G(t,s)$. For all the properties of $G(t,s)$ we refer to [11, ch. 6].

We recall that a (not necessarily strongly continuous) evolution operator $G(t,s)$ in a Banach space X, is a family of bounded linear operators $G(t,s)$, $t\ge s\in\mathbb{R}$, such that

$$G(t,t) = I, \quad G(t,s)G(s,r) = G(t,r), \quad t\ge s\ge r.$$

$G(t,s)$ is associated to the family $\{A(t)\}$ if for each $x\in X$ the function $t\mapsto G(t,s)x$ is in $C^1((s,+\infty),X)\cap C((s,+\infty),D)$ and

$$\frac{d}{dt}G(t,s)x = A(t)G(t,s)x, \quad x\in X,\; t>s.$$

An evolution operator $G(t,s)$ is said to have an exponential dichotomy with exponent $\beta>0$ and bound $N>0$ in an interval $I\subset\mathbb{R}$ if there exists a family of projections $P(t)\in L(X)$, $t\in I$, such that

a) $G(t,s)P(s) = P(t)G(t,s), \quad \forall s,t\in I,\; s\le t,$

b) $G(t,s): P(s)(X)\to P(t)(X)$ is invertible with inverse $\widetilde{G}(s,t), \quad \forall s,t\in I,\; s\le t,$

c) $\|G(t,s)(I-P(s))\| \le Ne^{-\beta(t-s)} \quad \forall s,t\in I,\; s\le t,$

d) $\|\widetilde{G}(s,t)P(t)\| \le Ne^{-\beta(t-s)} \quad \forall s,t\in I,\; s\le t.$

Exponential dichotomies for evolution operators associated to families of sectorial operators satisfying the assumption (2.1) are easily seen to enjoy some regularity properties.

Lemma 2.1 *Let $\{A(t):\; t\in\mathbb{R}\}$ satisfy (2.1), and assume that the associated evolution operator $G(t,s)$ has an exponential dichotomy in I.*

Then the function $s\mapsto P(s)x$ is continuous in I if $x\in\overline{D}$, it is locally θ-Hölder continuous if $x\in(X,D)_{\theta,\infty}$, it is locally Lipschitz continuous if $x\in D$.

For each $x\in X$, the function $s\mapsto G(t,s)(I-P(s))x$ is locally α-Hölder continuous in $\{s\in I:\; s<t\}$, and the function $s\mapsto \widetilde{G}(t,s)P(s)x$ is locally α-Hölder continuous in $\{s\in I:\; s>t\}$.

[1]We recall that a linear operator $A: D(A)\subseteq X$ is said to be sectorial if the resolvent set of A contains a sector $S=\{\lambda\in\mathbb{C}: \lambda\ne\omega,\; |\arg(\lambda-\omega)|<\theta\}$ with $\omega\in\mathbb{R}$, $\theta>\pi/2$, and there is $M>0$ such that $\|(\lambda-\omega)R(\lambda,A)\|_{L(X)}\le M$ for each $\lambda\in S$.

Proof — The first statement follows from the argument used in [9, p. 227] and the regularity properties of $G(t,\cdot)x$ stated in corollary 6.1.10 of [11].

From [11, cor. 6.1.12] it follows that for each $x \in X$ and $t \in I$, the function $s \mapsto G(t,s)x$ is locally α-Hölder continuous in $\{s \in I: \ s < t\}$. Writing $G(t,s+h)P(s+h)x - G(t,s)P(s)x = P(t)(G(t,s+h)x - G(t,s)x)$ for $s,\ s+h \in I,\ s,\ s+h < t$, we see that also $s \mapsto G(t,s)(I-P(s))x$ is locally α-Hölder continuous in $\{s \in I: \ s < t\}$.

Fix now $t \in I$ and $T > 0$ such that $t+T \in I$. For $s,\ s+h \in (t, t+T]$ we have $\widetilde{G}(t,s+h)P(s+h)x - \widetilde{G}(t,s)P(s)x = \widetilde{G}(t,t+T)[G(t+T,s+h)P(s+h)x - G(t+T,s)P(s)x]$ and from the local Hölder continuity of $G(t+T,\cdot)P(\cdot)x$ the last statement follows. ∎

The main assumption of this paper is that there exist the limits (in $L(D,X)$)

$$\lim_{t\to+\infty} A(t) = A_{+\infty}, \quad \lim_{t\to-\infty} A(t) = A_{-\infty}, \tag{2.2}$$

and that they are hyperbolic, i.e.

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{-\infty}) \cap i\mathbb{R} = \emptyset. \tag{2.3}$$

To construct exponential dichotomies we shall argue by perturbation, using as a main tool the following theorem from Henry's book ([9, thm. 7.6.10]). Although it is stated for strongly continuous evolution operators, its proof relies on discrete dichotomies and it is independent of the density of D.

Theorem 2.2 *Suppose that an evolution operator $T(t,s)$ has an exponential dichotomy on $\mathbb{R}$ with exponent β and bound M, and moreover*

$$\sup_{0\le t-s\le 1} \|T(t,s)\|_{L(X)} < \infty.$$

If $0 < \beta_1 < \beta$ and $M_1 > M$, there exists $K > 0$ (depending only on β, β_1, M, M_1, and on $\sup_{0\le t-s\le 1}\|T(t,s)\|_{L(X)}$) such that any evolution operator $S(t,s)$ satisfying

$$\|T(t,s) - S(t,s)\|_{L(X)} \le K \quad \textit{whenever} \quad 0 \le t-s \le 1,$$

has an exponential dichotomy on $\mathbb{R}$ with exponent β_1 and bound M_1.

The idea of the proof is simple. Since the operators $A_{+\infty}$ and $A_{-\infty}$ are limits of sectorial operators, they are sectorial. The evolution operators $T_+(t,s) := e^{(t-s)A_{+\infty}}$ and $T_-(t,s) := e^{(t-s)A_{-\infty}}$ have (constant) exponential dichotomies on $\mathbb{R}$, because of assumption (2.3). More precisely, $e^{(t-s)A_{\pm\infty}}$ has the constant exponential dichotomy $P(s) := P_{\pm\infty}$, where $P_{\pm\infty}$ is the spectral projection associated to the spectral set $\sigma_+(\pm\infty) := \{z \in \sigma(A_{\pm\infty}) : \operatorname{Re} z > 0\}$, i.e.

$$P_{\pm\infty} = \frac{1}{2\pi i}\int_\gamma R(\lambda, A_{\pm\infty})d\lambda, \tag{2.4}$$

γ being any counterclockwise oriented, regular in stretches, closed curve around $\sigma_+(\pm\infty)$, with index 1 with respect to $\sigma_+(\pm\infty)$ and with range in $\{\operatorname{Re}\lambda > 0\}$.

Since $A(t) \to A_{+\infty}$ as $t \to \infty$, and $A(t) \to A_{-\infty}$ as $t \to -\infty$, $G(t,s)$ is close to $T_+(t,s)$ for $t \ge s \ge T$, and it is close to $T_-(t,s)$ for $s \le t \le -T$ if T is large enough. Then we modify $G(t,s)$ for $s < T$, in such a way that the modified evolution operator $\widetilde{G}(t,s)$ is still close to $T_+(t,s)$ for all values of s and $t \in [s, s+1]$, and we apply theorem 2.2 to get an exponential dichotomy in $[T, +\infty)$. We argue similarly to get an exponential dichotomy in $(-\infty, -T]$.

Theorem 2.3 *Let $t \mapsto A(t)$ be a path of sectorial operators satisfying (2.1), (2.2), and (2.3). Then there is $T \geq 0$ such that the corresponding evolution operator $G(t,s)$ has exponential dichotomies on $[T,+\infty)$ and on $(-\infty,-T]$.*

Proof — To begin with, we prove the existence of an exponential dichotomy in $[T,+\infty)$ for T large.

Fix $T \geq 0$, and let $\psi \in C^\infty(\mathbb{R})$ be such that $\|\psi\|_\infty \leq 1$ and $\psi(t) = 1$ for $t \in (-\infty, T-1]$, $\psi(t) = 0$ for $t \in [T,+\infty)$. Define the family of operators

$$\widetilde{A}(t) := \psi(t)A_{+\infty} + (1-\psi(t))A(t), \quad t \in \mathbb{R}.$$

If T is large enough, for each $t \in \mathbb{R}$ $\widetilde{A}(t)$ is a small perturbation of $A_{+\infty}$ in $L(D,X)$, and therefore it is a sectorial operator. Moreover, $t \mapsto \widetilde{A}(t)$ is in $C^\alpha(\mathbb{R}, L(D,X))$. Therefore there exists an evolution operator $G_{\widetilde{A}}(t,s)$ such that

$$G_{\widetilde{A}}(t,s) = \begin{cases} e^{(t-s)A_{+\infty}}, & s \leq t \leq T-1, \\ G(t,s), & t \geq s \geq T. \end{cases}$$

To use theorem 2.2, with $T(t,s) = e^{(t-s)A_{+\infty}}$ and $S(t,s) = G_{\widetilde{A}}(t,s)$, we need to know that $\|G_{\widetilde{A}}(t,s) - e^{(t-s)A_{+\infty}}\|_{L(X)}$ is small for $s \leq t \leq s+1$.

Let $0 < \varepsilon < (\|A_{+\infty}^{-1}\|_{L(X,D)})^{-1}$, and let T be such that such that $\|A_{+\infty} - A(t)\|_{L(D,X)} \leq \varepsilon$, for every $t \geq T-1$. Then $\widetilde{A}(t)$ is invertible for every t, and there exist three constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2\pi)$, $M > 0$, such that the resolvent sets of $\widetilde{A}(t)$ contain a common sector $\Sigma = \{z \in \mathbb{C} :\ z \neq \omega,\ |\arg\,(z-\omega)| < \theta\}$, and $\|(\lambda - \omega)R(\lambda, \widetilde{A}(t))\|_{L(X)} \leq M$ for $\lambda \in \Sigma$, for all $t \in \mathbb{R} \cup \{-\infty,+\infty\}$. Moreover the graph norms of $D(\widetilde{A}(t))$, $t \in \mathbb{R} \cup \{-\infty,+\infty\}$, are uniformly equivalent to the D-norm, i.e. there exists $c > 0$ such that $c^{-1}\|x\|_D \leq \|\widetilde{A}(t)x\| + \|x\|_D \leq c\|x\|_D$ for every $x \in D$ and $t \in [-\infty,+\infty]$.

For $s \leq t \leq s+1$ we have

$$\begin{gathered} \|G_{\widetilde{A}}(t,s)x - e^{(t-s)A_{+\infty}}x\|_{L(X)} \leq \\ \leq \|G_{\widetilde{A}}(t,s)x - e^{(t-s)\widetilde{A}(s)}x\|_{L(X)} + \|e^{(t-s)\widetilde{A}(s)}x - e^{(t-s)A_{+\infty}}x\|_{L(X)}. \end{gathered} \tag{2.5}$$

To estimate the first addendum we recall the construction of the evolution operator $G_{\widetilde{A}}(t,s)$ of [11, ch. 6]. We have $G_{\widetilde{A}}(t,s)x - e^{(t-s)\widetilde{A}(s)}x = W(t,s)x$, where $W(t,s)x$ is the solution of the Cauchy problem

$$\begin{cases} w'(t) = \widetilde{A}(t)w(t) + (\widetilde{A}(t) - \widetilde{A}(s))e^{(t-s)\widetilde{A}(s)}x, \quad t > s, \\ \\ w(s) = 0. \end{cases}$$

Since

$$\|\widetilde{A}(t) - \widetilde{A}(s)\|_{L(D,X)} \leq \begin{cases} [A]_{C^\alpha(\mathbb{R},L(D,X))}(t-s)^\alpha, \\ 2\varepsilon, \end{cases}$$

for $t \geq s$, then

$$\|\widetilde{A}(t) - \widetilde{A}(s)\|_{L(D,X)} \leq (2\varepsilon)^{\frac{1}{2}}([A]_{C^\alpha(\mathbb{R},L(D,X))}(t-s)^\alpha)^{\frac{1}{2}}, \qquad t \geq s,$$

so that

$$\|\widetilde{A}\|_{C^{\alpha/2}(\mathbb{R},L(D,X))} \leq K_1\varepsilon^{\frac{1}{2}}.$$

By theorem 6.1.4 of [11], replacing α by $\alpha/2$ we obtain that there exists a positive number K_2, depending only on the above constants ω, θ, M, c, such that

$$\|W(\cdot,s)x\|_{C^{\alpha/2}([s,s+1],X)} \leq K_2\varepsilon^{1/2}\|x\|, \quad s \in \mathbb{R}.$$

It follows that

$$\|G_{\widetilde{A}}(t,s) - e^{(t-s)\widetilde{A}(s)}\|_{L(X)} \le K_2 \varepsilon^{1/2}, \quad s \le t \le s+1. \tag{2.6}$$

The estimate of the second addendum in (2.5) is standard. Indeed, setting $B(s) := \widetilde{A}(s) - 2\omega I$, $B_{+\infty} := A_{+\infty} - 2\omega I$, we have

$$\|e^{\sigma B(s)} - e^{\sigma B_{+\infty}}\|_{L(X)} = \left\| \int_\gamma e^{\lambda\sigma}(R(\lambda, B(s)) - R(\lambda, B_{+\infty}))d\lambda \right\| =$$

$$= \left\| \int_{\widetilde{\gamma}} \left(R\left(\frac{z}{\sigma}, B(s)\right) - R\left(\frac{z}{\sigma}, B_{+\infty}\right)\right) e^z \frac{dz}{\sigma} \right\|,$$

where $\gamma := \{r\exp(-i\theta),\ r \ge 0\} \cup \{r\exp(i\theta),\ r \ge 0\}$ and $\widetilde{\gamma} = \sigma\gamma$ are oriented counterclockwise. Since the resolvent is an analytic function and γ is homotopic to $\widetilde{\gamma}$, we may replace $\widetilde{\gamma}$ by γ in the last integral, and we estimate it by

$$\left\| \int_\gamma \left(R\left(\frac{z}{\sigma}, B(s)\right) - R\left(\frac{z}{\sigma}, B_{+\infty}\right)\right) e^z \frac{dz}{\sigma} \right\| \le \int_\gamma \left\| R\left(\frac{z}{\sigma}, B(s)\right) - R\left(\frac{z}{\sigma}, B_{+\infty}\right)\right\| |e^z| \frac{dz}{\sigma} \le$$

$$\le \int_\gamma \frac{|e^z|}{\sigma} \frac{M\sigma}{|z+\sigma\omega|} K_3 \|B(s) - B_{+\infty}\|_{L(D,X)} dz \le K_4 \varepsilon,$$

where $K_3 := \sup\{\|R(\lambda, B(s))\|_{L(X,D)} :\ |\arg\lambda| = \theta\}$. It follows

$$\|e^{\sigma\widetilde{A}(s)}x - e^{\sigma A_{+\infty}}x\|_{L(X)} = e^{2\omega\sigma}\|e^{\sigma B(s)} - e^{\sigma B_{+\infty}}\|_{L(X)} \le e^{2\omega} K_4 \varepsilon$$

for each $\sigma \in [0,1]$. Taking into account (2.6) and (2.5), we get

$$\|G_{\widetilde{A}}(t,s) - e^{(t-s)A_{+\infty}}\|_{L(X)} \le K_2 \varepsilon^{1/2} + K_4 e^{2\omega}\varepsilon, \quad s \le t \le s+1$$

so that if T is large enough then $\|G_{\widetilde{A}}(t,s) - e^{(t-s)A_{+\infty}}\|_{L(X)}$ is small, and $G_{\widetilde{A}}(t,s)$ has an exponential dichotomy in $\mathbb{R}$ thanks to theorem 2.2. But $G_{\widetilde{A}}(t,s)$ coincides with $G(t,s)$ for $t \ge s \ge T$, and therefore $G(t,s)$ has an exponential dichotomy in $[T,+\infty)$.

The proof that $G(t,s)$ has an exponential dichotomy in $(-\infty,-T]$ for large T is similar; of course the operators $\widetilde{A}(t)$ have to be defined now by

$$\widetilde{A}(t) := \varphi(t)A_{-\infty} + (1-\varphi(t))A(t), \quad t \in \mathbb{R},$$

where $\varphi \in C^\infty(\mathbb{R})$, $\|\varphi\|_\infty \le 1$ and $\varphi(t) = 1$ for $t \in [-T+1,+\infty)$, $\varphi(t) = 0$ for $t \in (-\infty,-T]$. ∎

Once one knows that there exist exponential dichotomies in suitable halflines, several results for forward and backward Cauchy problems follow. We quote below some of the results of [11, ch. 6] that will be used later. In fact, such results were proved under the periodicity assumption $A(t+T) = A(t)$, but they rely uniquement on the existence of exponential dichotomies, as it is easy to check.

Let $\{P(s) :\ s \in (-\infty,-T] \cup [T,+\infty)\}$ be any exponential dichotomy for $G(t,s)$ in the halflines $I = (-\infty,-T]$, $I = [T,+\infty)$.

Theorem 2.4 *Let $t_0 \ge T$, let $f \in C^\alpha([t_0,+\infty),X)$, and $x \in X$. Then the solution u of*

$$u'(t) = A(t)u(t) + f(t), \quad t > t_0; \quad u(t_0) = x, \tag{2.7}$$

is bounded in $[t_0,+\infty)$ if and only if

$$P(t_0)x = -\int_{t_0}^\infty \widetilde{G}(t_0,s)P(s)f(s)ds, \tag{2.8}$$

in which case u is given by

$$u(t) = G(t,t_0)(I-P(t_0))x + \int_{t_0}^{t} G(t,s)(I-P(s))f(s)ds - \int_{t}^{\infty} \widetilde{G}(t,s)P(s)f(s)ds. \quad (2.9)$$

If in addition $x \in D$ *and* $A(t_0)x + f(t_0) \in (X,D)_{\alpha,\infty}$*, then* $u \in C^{1+\alpha}([t_0,+\infty),X) \cap C^{\alpha}([t_0,+\infty),D)$*, and* $u' \in B([t_0,+\infty),(X,D)_{\alpha,\infty})$*. There is* $C>0$ *such that*

$$\begin{aligned} &\|u\|_{C^{1+\alpha}([t_0,+\infty),X)} + \|u\|_{C^{\alpha}([t_0,+\infty),D)} + \|u'\|_{B([t_0,+\infty),(X,D)_{\alpha,\infty})} \\ &\le C(\|x\|_D + \|A(t_0)x + f(t_0)\|_{(X,D)_{\alpha,\infty}} + \|f\|_{C^{\alpha}([t_0,+\infty),X)}). \end{aligned} \quad (2.10)$$

Theorem 2.5 *Let* $t_0 \le -T$*,* $y \in X$*,* $g \in C^{\alpha}((-\infty,t_0];X)$*. Then problem*

$$v'(t) = A(t)v(t) + g(t), \;\; t < t_0; \;\;\; v(t_0) = y, \quad (2.11)$$

has a bounded solution v in $(-\infty,t_0]$ *if and only if*

$$(I-P(t_0))y = \int_{-\infty}^{t_0} G(t_0,s)(I-P(s))g(s)ds, \quad (2.12)$$

in which case v is given by

$$v(t) = G(t,t_0)P(t_0)y + \int_{t_0}^{t} \widetilde{G}(t,s)P(s)g(s)ds + \int_{-\infty}^{t} G(t,s)(I-P(s))g(s)ds, \quad (2.13)$$

and belongs to $C^{1+\alpha}((-\infty,t_0],X) \cap C^{\alpha}((-\infty,t_0]),D)$*, while* v' *is bounded with values in* $(X,D)_{\alpha,\infty}$*. There exists* $C>0$ *such that*

$$\begin{aligned} &\|v\|_{C^{1+\alpha}((-\infty,t_0],X)} + \|v\|_{C^{\alpha}((-\infty,t_0],D)} + \|v'\|_{B((-\infty,t_0],(X,D)_{\alpha,\infty})} \\ &\le C(\|y\| + \|g\|_{C^{\alpha}((-\infty,t_0],X)}). \end{aligned} \quad (2.14)$$

As a particular case of theorems 2.4 and 2.5 we may consider the autonomous case, when $A(t)$ is a constant hyperbolic operator. In fact in next proposition 2.7 we shall use the result of theorem 2.5 with $A(t)$ replaced by $A_{-\infty}$. In this case we have a trivial exponential dichotomy on the whole $\mathbb{R}$, with $P(t) = P_{-\infty}$ for each t, and the proofs (see [11, thms. 4.4.3, 4.4.6]) are simpler.

Proposition 2.6 *We have*

$$dim\ P(t)(X) = dim\ P_{+\infty}(X), \;\; t \ge T,$$
$$dim\ P(t)(X) = dim\ P_{-\infty}(X), \;\; t \le -T.$$

Proof — Since $\widetilde{G}(t,s)$ is an isomorphism between $P(s)(X)$ and $P(t)(X)$ for $s \ge t \ge T$, and between $P(s)(X)$ and $P(t)(X)$ for $t \le s \le -T$, then the dimensions of $P(t)(X)$ for $t \ge T$ are constant, and the dimensions of $P(t)(X)$ for $t \le -T$ are constant.

Set $X_0 := \overline{D}$. The parts of the operators $G(t,s)$ in X_0 still have exponential dichotomies in $[T,+\infty)$ and in $(-\infty,T]$, given by the parts of $P(s)$ in X_0. By [15, thm. 3.3], dim $P(T)(X_0) =$ dim $P_{+\infty}(X_0)$. Since $P(T)(D) = P(T)(X)$ and $P_{+\infty}(D) = P_{+\infty}(X)$, then $P(T)(X_0) = P(T)(X)$ and $P_{+\infty}(X_0) = P_{+\infty}(X)$. Therefore, dim $P_+(t)(X) =$ dim $P_{+\infty}(X)$, for every $t \ge T$.

A similar argument yields the second part of the statement. ∎

Now we give a better insight on the structure of the spaces $P(t)(X)$, seen as graphs of linear bounded operators over $P_{-\infty}(X)$ for $t \le -T$. This extends to our situation the analogous results of [1]. The fact that $P(t)(X)$ is a graph over $P_{-\infty}(X)$ gives also an alternative proof of the second part of proposition 2.6.

Proposition 2.7 *If T is big enough, for each $t_0 < -T$ the space $P(t_0)(X)$ is the graph of a linear bounded operator $\Gamma_- : P_{-\infty}(X) \mapsto (I - P_{-\infty})(X)$.*

Proof — Let $t_0 \leq -T$ and let $u_0 \in P(t_0)(X)$. By theorem 2.5 there exists a unique bounded backward solution u of the Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t), & t \leq t_0, \\ u(t_0) = u_0, \end{cases} \tag{2.15}$$

and $u \in C^\alpha((-\infty, t_0], X)$. We rewrite the differential equation as a perturbation of $u' = A_{-\infty}u$, i.e. $u'(t) = A_{-\infty}u(t) + f(u)(t)$, with $f(u)(t) := A(t)u(t) - A_{-\infty}u(t)$. Applying theorem 2.5 with $A(t)$ replaced by $A_{-\infty}$, $P(t)$ replaced by $P_{-\infty}$, it follows that

$$u(t) = e^{(t-t_0)A_{-\infty}}x + \int_{t_0}^{t} e^{(t-s)A_{-\infty}} P_{-\infty} f(u)(s)ds + \int_{-\infty}^{t} e^{(t-s)A_{-\infty}}(I - P_{-\infty})f(u)(s)ds,$$

for some $x \in P_{-\infty}(X)$. Therefore, we fix $x \in P_{-\infty}(X)$ and we consider the operator Λ_x defined on $C^{\alpha/2}((-\infty, -t_0], D)$ by

$$(\Lambda_x u)(t) := e^{(t-t_0)A_{-\infty}}x + \int_{t_0}^{t} e^{(t-s)A_{-\infty}} P_{-\infty} f(u)(s)ds + \int_{-\infty}^{t} e^{(t-s)A_{-\infty}}(I - P_{-\infty})f(u)(s)ds.$$

We claim that Λ_x is a $1/2$-contraction if T is big enough. Indeed, let $\varepsilon > 0$, and let T be such that $\|A_{-\infty} - A(t)\|_{L(D,X)} \leq \varepsilon$, for every $t \leq -T$. The same argument used in the proof of theorem 2.3 yields $\|A(\cdot) - A_{-\infty}\|_{C^{\alpha/2}((-\infty,-T],L(D,X))} \leq K(\varepsilon + \varepsilon^{1/2})$, and this implies $\|f(u_1) - f(u_2)\|_{C^{\alpha/2}((-\infty,t_0],X)} \leq C(\varepsilon + \varepsilon^{1/2})\|u_1 - u_2\|_{C^{\alpha/2}((-\infty,t_0],D)}$ for each couple of functions u_1, u_2 in $C^{\alpha/2}((-\infty, t_0], D)$. Estimate (2.14) with α replaced by $\alpha/2$ implies now that Λ_x is a $1/2$-contraction if ε is small enough. In this case, there is a unique fixed point of Λ_x, which is the unique solution to (2.15) with final datum $u_0 = x + \int_{-\infty}^{t_0} e^{(t_0-s)A_{-\infty}}(I - P_{-\infty})f(u)(s)ds$.

Defining

$$\Gamma_- x := (I - P_{-\infty})u(t_0) = \int_{-\infty}^{t_0} e^{(t_0-s)A_{-\infty}}(I - P_{-\infty})f(u)(s)ds,$$

where u is the unique fixed point of Λ_x, we obtain that $W^u(t_0)$ is the graph of the operator Γ_-, and the statement follows. ■

Arguing as in proposition 2.7 and using [11, cor. 4.3.6, thm. 4.4.3] instead of theorem 2.4, one can prove that the spaces $(I - P(t))(D)$ are graphs of linear bounded operators $\Gamma_+ : (I - P_{+\infty})(D) \mapsto P_{+\infty}(D)$ for $t \geq T$. It follows that the dimension of $(I - P(t))(D)$ is equal to the dimension of $(I - P_{+\infty})(D)$, but this is not of much interest for us because in the most important applications such dimension is $+\infty$.

3 Properties of the operator $\mathcal{L}$

Throughout this section $\{A(t) : t \in \mathbb{R}\}$ is a family of operators satisfying assumptions (2.1), (2.2) and (2.3), $G(t, s)$ is the associated evolution operator, and $\mathcal{L}$ is the operator defined in (1.1).

It is convenient to introduce the notion of stable and unstable subspaces, as in [1].

Definition 3.1 *Let $t_0 \in \mathbb{R}$. We define the stable space at t_0 by*

$$W^s(t_0) := \{x \in X : \lim_{t \to +\infty} G(t, t_0)x = 0\},$$

and the unstable space at t_0 by

$$W^u(t_0) := \{ \quad x \in X \ : \ \exists\, u \in C((-\infty, t_0]; D) \cap C^1((-\infty, t_0]; X)$$

$$\text{such that } u'(t) = A(t)u(t),\ u(t_0) = x,\ \lim_{t\to -\infty} u(t) = 0\}.$$

Some easy properties of $W^s(t_0)$ and $W^u(t_0)$ follow.

Proposition 3.2 *The following statements hold true:*

(i) for each $t_0 \geq T$, $W^s(t_0) = (I - P(t_0))(X)$; for each $t_0 \leq -T$, $W^u(t_0) = P(t_0)(X)$;

(ii) for each $t_0 \geq T$, $W^s(t_0) = \{x \in X \ : \ t \to G(t, t_0)x \in B([t_0, +\infty), X)\}$;

(iii) for each $t_0 \leq -T$,

$$W^u(t_0) = \{ \quad x \in D \ : \ \exists\, u \in C((-\infty, t_0]; D) \cap C^1((-\infty, t_0]; X) \cap B((-\infty, t_0]; X)$$

$$\text{such that } u'(t) = A(t)u(t),\ u(t_0) = x\};$$

(iv) for each $t,\ t_0 \in \mathbb{R}$ with $t \geq t_0$, $G(t, t_0)W^s(t_0) \subseteq W^s(t)$;

(v) for each $t,\ t_0 \in \mathbb{R}$ with $t \geq t_0$, $G(t, t_0)W^u(t_0) = W^u(t)$;

(vi) for each $t_0 \in \mathbb{R}$, $W^s(t_0)$ is closed.

Proof — Statements (i), (ii), (iii) follow from theorems 2.4 and 2.5, taking $f \equiv 0$ and $g \equiv 0$. Statement (iv) is an obvious consequence of the semigroup law $G(s, t_0)x = G(s, t)G(t, t_0)x$ for $s > t > t_0$, $x \in X$.

Let us prove (v). Let $x \in W^u(t_0)$, and let $u \in C((-\infty, t_0]; D) \cap C^1((-\infty, t_0]; X)$ be such that $u(t_0) = x$ and $\lim_{t\to -\infty} u(t) = 0$. Therefore, $x \in D$ and $u'(t_0) = A(t_0)x \in \overline{D}$, which implies that $s \mapsto G(s, t_0)x$ is in $C([t_0, t]; D) \cap C^1([t_0, t]; X)$ (see [11, cor. 6.1.9]), and the function v defined by $v(s) = u(s)$ for $s \leq t_0$, $v(s) = G(s, t_0)x$ for $t_0 < s \leq t$ belongs to $C((-\infty, t]; D) \cap C^1((-\infty, t]; X)$ and goes to 0 as s goes to $-\infty$. Therefore, $v(t) = G(t, t_0)x \in W^u(t)$. This shows the inclusion $\subseteq$. The other inclusion is obvious.

Let us prove (vi). If $t_0 \geq T$ (risp. $t_0 \leq -T$) then $W^s(t_0)$ (risp. $W^u(t_0)$) is the range of the projection $I - P(t_0)$ (risp. $P(t_0)$) and then it is obviously closed. If $t_0 < T$, let $\{x_n\} \subseteq W^s(t_0)$ be such that $x_n \to x$ as $n \to +\infty$. Then, for each $t > t_0$, $G(t, t_0)x_n \to G(t, t_0)x$ as $n \to +\infty$, and by statement (iv), $G(t, t_0)x_n \in W^s(t)$. Taking $t = T$ and recalling that $W^s(T) = (I - P(T))(X)$ is closed, we get $G(T, t_0)x \in W^s(T)$. Therefore, $G(t, t_0)x = G(t, T)G(T, t_0)x$ goes to 0 as t goes to $+\infty$. This means that $x \in W^s(t_0)$, so that $W^s(t_0)$ is closed. ∎

To study the operator $\mathcal{L}$ it is useful to introduce the realizations of the operator $u \mapsto u' - A(\cdot)u$ in Hölder spaces on halflines.

Definition 3.3

$$\begin{cases} \mathcal{L}^+ : D(\mathcal{L}^+) = C^{1+\alpha}([T, +\infty), X) \cap C^\alpha([T, +\infty), D) \longrightarrow C^\alpha([T, +\infty), X), \\ \\ (\mathcal{L}^+ u)(t) = u'(t) - A(t)u(t), \quad t \geq T; \end{cases} \tag{3.1}$$

$$\begin{cases} \mathcal{L}^- : D(\mathcal{L}^-) = C^{1+\alpha}((-\infty, T], X) \cap C^\alpha((-\infty, T], D) \longrightarrow C^\alpha((-\infty, T], X), \\ \\ (\mathcal{L}^- u)(t) = u'(t) - A(t)u(t), \quad t \leq T. \end{cases} \tag{3.2}$$

Using theorems 2.4 and 2.5 it is possible to write down right inverses for $\mathcal{L}^+$ and $\mathcal{L}^-$:

$$
\begin{aligned}
(R^+h)(t) \;&=\; -\int_t^{+\infty} \widetilde{G}(t,s)P(s)h(s)ds + \int_T^t G(t,s)(I-P(s))h(s)ds \\
&\quad -G(t,T)A(T)^{-1}(I-P(T))h(T), \quad t \geq T, \;\; h \in C^\alpha([T,+\infty),X),
\end{aligned}
\tag{3.3}
$$

$$
(R^-h)(t) = \begin{cases}
\displaystyle\int_{-\infty}^t G(t,s)(I-P(s))h(s)ds + \int_{-T}^t \widetilde{G}(t,s)P(s)h(s)ds, & t \leq -T, \\[2ex]
\displaystyle\int_{-\infty}^{-T} G(t,s)(I-P(s))h(s)ds + \int_{-T}^t G(t,s)h(s)ds, & -T \leq t \leq T, \\[2ex]
h \in C^\alpha((-\infty,T],X).
\end{cases}
\tag{3.4}
$$

To prove that in fact R^+ and R^- are right inverses of $\mathcal{L}^+$ and $\mathcal{L}^-$, we shall need a theorem due to Sinestrari about maximal Hölder regularity for forward Cauchy problems in bounded intervals, whose proof may be found in [11, ch. 6].

Theorem 3.4 *Let $a < b \in \mathbb{R}$, and let $f \in C^\alpha([a,b],X)$, $x \in D$ be such that $A(a)x + f(a) \in (X,D)_{\alpha,\infty}$. Then the solution u to problem*

$$
\begin{cases}
u'(t) = A(t)u(t), & a \leq t \leq b, \\[1ex]
u(a) = x,
\end{cases}
$$

belongs to $C^{1+\alpha}([a,b],X) \cap C^\alpha([a,b],D)$, and there is $C > 0$, independent of f and x, such that

$$
\|u\|_{C^{1+\alpha}([a,b],X)} + \|u\|_{C^\alpha([a,b],D)} \leq C(\|f\|_{C^\alpha([a,b],X)} + \|x\|_D + \|A(a)x + f(a)\|_{(X,D)_{\alpha,\infty}}).
\tag{3.5}
$$

Proposition 3.5 *The following statements hold.*

(i) R^+ is a bounded operator from $C^\alpha([T,+\infty),X)$ to $D(\mathcal{L}^+)$, and we have $\mathcal{L}^+R^+h = h$ for each $h \in C^\alpha([T,+\infty),X)$.

(ii) R^- is a bounded operator from $C^\alpha((-\infty,T],X)$ to $D(\mathcal{L}^-)$, and we have $\mathcal{L}^-R^-h = h$ for each $h \in C^\alpha((-\infty,T],X)$.

Proof — (i) Let $h \in C^\alpha([T,+\infty),X)$. Setting $x = -A(T)^{-1}(I-P(T))h(T)$, we have $A(T)x + h(T) = P(T)h(T) \in D \subseteq (X,D)_{\alpha,\infty}$. By theorem 2.4, $R^+h \in D(\mathcal{L}^+)$ and $(R^+h)'(t) = A(t)R^+h(t) + h(t)$ for $t \geq T$, i.e. $\mathcal{L}^+R^+h = h$. From estimate (2.10) it follows that $R^+ \in L(C^\alpha([T,+\infty),X); D(\mathcal{L}^+))$.
(ii) Let now $h \in C^\alpha((-\infty,T],X)$. By theorem 2.4, the restriction of R^-h to $(-\infty,-T]$ is in $C^{1+\alpha}((-\infty,-T],X) \cap C^\alpha((-\infty,-T],D)$, its norm does not exceed $C\|h\|_{C^\alpha((-\infty,-T],X)}$, and $(R^-h)'(t) = A(t)R^-h(t) + h(t)$ for $t \leq -T$. Moreover,

$$
(R^-h)'(-T) = A(-T)(R^-h)(-T) + h(-T) \in (X,D)_{\alpha,\infty},
$$

$$
\|(R^-h)'(-T)\|_{(X,D)_{\alpha,\infty}} \leq C\|h\|_{C^\alpha((-\infty,-T],X)}.
$$

Theorem 3.4 implies now that the restriction of R^-h to $[-T,T]$ is in $C^{1+\alpha}([-T,T],X) \cap C^\alpha([-T,T],D)$, that its norm does not exceed

$$
C(\|h\|_{C^\alpha([-T,T],X)} + \|(R^-h)(-T)\|_D + \|A(-T)(R^-h)(-T) + h(-T)\|_{(X,D)_{\alpha,\infty}}),
$$

and that $(R^-h)'(t) = A(t)R^-h(t)+h(t)$ for $-T \le t \le T$. Patching together the restrictions of R^-h to $(-\infty, -T]$ and to $[-T, T]$ the statement follows. ∎

Now we prove a trace lemma that will be a key tool in the proofs of next theorems.

Lemma 3.6 *For every $w_0 \in P(T)(X)$ there exists $h_0 \in D(\mathcal{L})$ such that $(R^+h_0)(T) = w_0$, $(R^-h_0)(T) = 0$ and $\|h_0\|_{D(\mathcal{L})} \le K\|w_0\|$, where K is a nonnegative constant independent of w_0.*

Proof — Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that

$$\|\varphi\|_\infty \le 1, \quad \varphi(t) = 0 \ \ \forall t \le T, \quad \int_T^{+\infty} \varphi(s)ds = -1,$$

and set

$$h_0(t) := \varphi(t)G(t,T)w_0, \quad t \ge T, \quad h_0(t) := 0, \quad t \le T.$$

Then there exists a nonnegative constant K such that $\|h_0\|_{D(\mathcal{L})} \le K\|w_0\|$. Moreover $R^+h_0(T) = w_0$, $R^-h_0(T) = 0$ and the statement follows. ∎

Now we can state a characterization of the kernel and of the range of $\mathcal{L}$ in terms of the stable and the unstable subspaces at T.

Proposition 3.7 *(i) Ker $\mathcal{L}^+ = \{h \in C^\alpha([T,+\infty), D) \cap C^{1+\alpha}([T,+\infty), X) \ : \ h'(t) = A(t)h(t),\ h(T) \in W^s(T)\}$,*

(ii) Ker $\mathcal{L}^- = \{h \in C^\alpha((-\infty,T], D) \cap C^{1+\alpha}((-\infty,T], X) \ : \ h'(t) = A(t)h(t),\ h(T) \in W^u(T)\}$,

(iii) Ker $\mathcal{L} = \{h \in C^\alpha(\mathbb{R}, D) \cap C^{1+\alpha}(\mathbb{R}, X) \ : \ h'(t) = A(t)h(t),\ h(T) \in W^s(T) \cap W^u(T)\}$,

(iv) Range $\mathcal{L} = \{h \in C^\alpha(\mathbb{R}, X) \ : \ R^+h(T) - R^-h(T) \in W^s(T) + W^u(T)\}$,

(v) $\overline{\text{Range } \mathcal{L}} = \{h \in C^\alpha(\mathbb{R}, X) \ : \ R^+h(T) - R^-h(T) \in \overline{W^s(T) + W^u(T)}\}$.

Proof — Statements (i), (ii) and (iii) follow from proposition 3.2.

Let us prove statement (iv). Let $h \in$ Range $\mathcal{L}$, then the restrictions of h to $[T, +\infty)$ and to $(-\infty, T]$ belong to the ranges of $\mathcal{L}^+$ and of $\mathcal{L}^-$, respectively. If u is any solution to $\mathcal{L}u = h$ then

$$u(t) = \begin{cases} (R^+h)(t) + v_+(t), \ t \ge T, \\ \\ (R^-h)(t) + v_-(t), \ t \le T, \end{cases}$$

for some $v_+ \in$ Ker $\mathcal{L}^+$, $v_- \in$ Ker $\mathcal{L}^-$. In particular, $u(T) = R^+h(T) + v_+(T) = v_-(T) + R^-h(T)$ and $v_-(T) \in W^u(T)$, $v_+(T) \in W^s(T)$. Then $R^+h(T) - R^-h(T) = v_+(T) - v_-(T) \in W^s(T) + W^u(T)$.

Let now $h \in C^\alpha(\mathbb{R}, X)$ be such that $R^+h(T) - R^-h(T) = v_s + v_u \in W^s(T) + W^u(T)$. Let $u_0 := R^+h(T) - v_s = R^-h(T) + v_u$, and define the function

$$u(t) := \begin{cases} a(t) := G(t,T)(u_0 - R^+h(T)) + R^+h(t), \ \ t \ge T, \\ \\ b(t) := \widetilde{u}(t) + R^-h(t), \ \ t \le T, \end{cases}$$

where $\widetilde{u}(t)$ is any backward solution of $v'(t) = A(t)v(t)$, $t \le T$, with final datum $\widetilde{u}(T) = v_u$, and going to 0 as t goes to $-\infty$. Note that, since $(R^+h)(T)$, $(R^-h)(T)$ are in D, and $A(T)(R^+h)(T)$, $A(T)(R^-h)(T)$ are in $(X,D)_{\alpha,\infty}$, then also $v_s \in D$, $A(T)v_s \in (X,D)_{\alpha,\infty}$. Therefore, the function a is in $C^\alpha([T,+\infty), D) \cap C^{1+\alpha}([T,+\infty), X)$. Moreover $b \in$

$C^\alpha((-\infty,T],D)\cap C^{1+\alpha}((-\infty,T],X)$ and $a(T)=u_0=b(T)$, $a'(T)=A(T)u_0=b'(T)$. Therefore $u\in D(\mathcal{L})$, and $u'(t)-A(t)u(t)=h(t)$ for each $t\in\mathbb{R}$. It follows that $h\in$ Range $\mathcal{L}$.

To prove statement (v) we have only to check the inclusion $\supseteq$ since the other one is obvious. It is possible to follow an argument already used in [1]. Let $h\in C^\alpha(\mathbb{R},X)$ be such that $R^+h(T)-R^-h(T)\in\overline{W^s(T)+W^u(T)}$, and let $\varepsilon>0$. Since $R^+h(T)-R^-h(T)\in\overline{W^s(T)+W^u(T)}$ there exist $v_0=v_0(\varepsilon)\in W^s(T)+W^u(T)$ and $w=w(\varepsilon)$, $\|w\|\le\varepsilon$, such that $R^+h(T)-R^-h(T)=v_0+w$. Let $w_0=P(T)w$, then by lemma 3.6 there exists $h_0\in C^{1+\alpha}(\mathbb{R},X)\cap C^\alpha(\mathbb{R},D)$ such that $(R^+h_0)(T)=w_0$, $(R^-h_0)(T)=0$ and $\|h_0\|_{C^\alpha(\mathbb{R},X)}\le K\|w_0\|$. We set $h_\varepsilon:=h-h_0$. $h_\varepsilon\in$ Range $\mathcal{L}$, indeed

$$\begin{aligned} R^+h_\varepsilon(T)-R^-h_\varepsilon(T) &= R^+h(T)-R^-h(T)-(R^+h_0(T)-R^-h_0(T)) \\ &= v_0+w-w_0=v_0+(I-P(T))w \\ &\in W^s(T)+W^u(T). \end{aligned}$$

Moreover $\|h_\varepsilon-h\|_{C^\alpha(\mathbb{R},X)}=\|h_0\|_{C^\alpha(\mathbb{R},X)}\le K\|w_0\|=K\|P(T)\|_{L(X)}\varepsilon$. Then $h_\varepsilon\to h$ as $\varepsilon\to0^+$ and the statement follows. ∎

We recall the definitions of semi-Fredholm and Fredholm operators, and of semi-Fredholm and Fredholm couples of subspaces.

Definition 3.8 *Let E and F be two Banach spaces. We say that a closed linear operator $T:D(T)\subseteq E\to F$ is a semi-Fredholm operator if* Range T *is closed and if at least one of the dimensions dim* Ker $T<\infty$, *codim* Range T, *is finite. If both dimensions are finite we say that T is a Fredholm operator. If T is a semi-Fredholm operator the index of T is defined as*

$$ind\,T:=dim\,\text{Ker}\,T-codim\,\text{Range}\,T.$$

Definition 3.9 *Let V and W be two subspaces of a Banach space E. We say that (V,W) is a semi-Fredholm couple if $V+W$ is closed and if at least one of the dimensions dim $(V\cap W)<\infty$, codim $(V+W)$, is finite. If both dimensions are finite we say that (V,W) is a Fredohlm couple. If (V,W) is a semi-Fredholm couple the index of (V,W) is defined as*

$$ind\,(V,W):=dim\,(V\cap W)-codim\,(V+W).$$

Now we are able to describe the properties of $\mathcal{L}$ in terms of properties of the subspaces $W^s(T)$ and $W^u(T)$.

Theorem 3.10 *(i) Range $\mathcal{L}$ is closed if and only if $W^s(T)+W^u(T)$ is closed.*

(ii) $\mathcal{L}$ is onto if and only if $W^s(T)+W^u(T)=X$.

(iii) If $\mathcal{L}$ is one to one then $W^s(T)\cap W^u(T)=\{0\}$. Moreover if $G(T,-T)_{|P(-T)(X)}$ is one to one the converse is also true.

(iv) If $\mathcal{L}$ is invertible then $W^s(T)\oplus W^u(T)=X$. Moreover if $G(T,-T)_{|P(-T)(X)}$ is one to one the converse is also true.

(v) If $\mathcal{L}$ is a semi-Fredholm operator then $(W^s(T),W^u(T))$ is a semi-Fredholm couple and

$$ind\,(W^s(T),W^u(T))\le ind\,\mathcal{L}.$$

If in addition the kernel of $G(T,-T)_{|P(-T)(X)}$ is finite dimensional, then

$$ind\,(W^s(T),W^u(T))=ind\,\mathcal{L}-dim\,\text{Ker}\,G(T,-T)_{|P(-T)(X)}. \tag{3.6}$$

Conversely, if $(W^s(T), W^u(T))$ *is a semi-Fredholm couple and the kernel of* $G(T, -T)_{|P(-T)(X)}$ *is finite dimensional, then* $\mathcal{L}$ *is a semi-Fredholm operator and (3.6) holds.*

Proof — The "if" part of statement (i) follows immediately from (iv) and (v) of proposition 3.7. Now suppose that Range $\mathcal{L}$ is closed and let $v_0 \in \overline{W^s(T) + W^u(T)}$. We shall show that $w_0 := P(T)v_0 \in W^s(T) + W^u(T)$ so that $v_0 = (I - P(T))v_0 + P(T)v_0 \in W^s(T) + W^u(T)$. By lemma 3.6 there exists $h \in D(\mathcal{L})$ such that $R^+h(T) = w_0$, $R^-h(T) = 0$. Since $R^+h(T) - R^-h(T) = w_0 \in \overline{W^s(T) + W^u(T)}$, then by proposition 3.7 (v) $h \in \overline{\text{Range}\,\mathcal{L}} = \text{Range}\,\mathcal{L}$. Therefore there exists $u \in D(\mathcal{L})$ such that $\mathcal{L}u = h$. By the construction of h we get that $u_{|(-\infty,T]} \in \text{Ker}\,\mathcal{L}^-$ and $u_{|[T,+\infty)} - R^+h \in \text{Ker}\,\mathcal{L}^+$. Then, by proposition 3.7 (ii) and (i), $u(T) \in W^u(T)$ and $u(T) - R^+h(T) \in W^s(T)$. Hence $w_0 = R^+h(T) = (R^+h(T) - u(T)) + u(T) \in W^s(T) + W^u(T)$.

The "if" part of statement (ii) is an easy consequence of statement (iv) of proposition 3.7. To prove the converse, suppose that $\mathcal{L}$ is onto, let $v_0 \in X$ and argue like in statement (i).

Let us prove statement (iii). Let $\mathcal{L}$ be one to one, and let $x \in W^s(T) \cap W^u(T)$. Then we can find a function $v \in D(\mathcal{L})$ such that $v(T) = x$ and $v'(t) = A(t)v(t)$, $t \in \mathbb{R}$. Therefore $v \in \text{Ker}\,\mathcal{L} = \{0\}$, hence $v \equiv 0$, thus $x = v(T) = 0$.

Conversely, suppose that $G(T, -T)_{|P(-T)(X)}$ is one to one and $W^s(T) \cap W^u(T) = \{0\}$. Let $u \in \text{Ker}\,\mathcal{L}$, then $u(T) \in W^s(T) \cap W^u(T) = \{0\}$ and $x := u(-T) \in W^u(-T) = P(-T)(X)$. Therefore $u(t) = \widetilde{G}(t, -T)x$ for $t \leq -T$ and $0 = u(T) = G(T, -T)x$. Then $G(T, -T)x = 0$ so that $x = 0$ and $u \equiv 0$.

The proof of statement (iv) follows immediately from (ii) and (iii).

To prove (v), first of all we recall that Range $\mathcal{L}$ is closed if and only if $W^s(T) + W^u(T)$ is closed. In this case we proceed in three steps.

Step 1: *codim* Range $\mathcal{L} =$ *codim* $(W^s(T) + W^u(T))$.
Indeed, if we define

$$J : \frac{C^\alpha(\mathbb{R}, X)}{\textit{Range}\ \mathcal{L}} \longrightarrow \frac{X}{W^s(T) + W^u(T)}, \quad J[h] := [R^+h(T) - R^-h(T)],$$

it is easy to prove that J is an isomorphism. Indeed, J is one to one thanks to statement (iv) of proposition 3.7, and J is onto because by lemma 3.6 for every $x \in X$ there is $h \in C^\alpha(\mathbb{R}, X)$ such that $R^+h(T) - R^-h(T) = P(T)x$, and $P(T)x$ is equivalent to x in the quotient space $X/(W^s(T) + W^u(T))$.

Step 2: *dim* Ker $\mathcal{L} =$ *dim* $(W^s(T) \cap W^u(T)) +$ *dim* Ker $G(T, -T)_{|P(-T)(X)}$.
Define $\Lambda := \{u \in \text{Ker}(\mathcal{L}) : \ u(t) = 0 \ \forall t \geq T\}$. Then $u \in \Lambda$ iff $u(-T) \in P(-T)(X)$, $G(T, -T)u(-T) = 0$, and $u(t) = \widetilde{G}(t, -T)u(-T)$ for $t \leq -T$, $u(t) = G(t, -T)u(-T)$ for $t \geq -T$. Therefore, the dimension of Λ is equal to the dimension of Ker $G(T, -T)_{|P(-T)(X)}$.
The mapping

$$E : \frac{\text{Ker}\,\mathcal{L}}{\Lambda} \longrightarrow W^s(T) \cap W^u(T), \quad E[u] := u(T)$$

is well defined and bijective, and the statement follows.

Step 3: conclusion.
Steps (i) and (ii) imply that if $\mathcal{L}$ is a semi-Fredholm operator then either the codimension of $(W^s(T) + W^u(T))$ or both dimensions of $W^s(T) \cap W^u(T)$ and of Ker $G(T, -T)_{|P(-T)(X)}$ are finite. It follows that $(W^s(T), W^u(T))$ is a semi-Fredholm couple with index $\leq$ *ind* $\mathcal{L}$; if in addition the kernel of $G(T, -T)_{|P(-T)(X)}$ is finite dimensional then (3.6) holds.

Conversely, if Ker $G(T, -T)_{|P(-T)(X)}$ is finite dimensional and $(W^s(T), W^u(T))$ is a semi-Fredholm couple then $\mathcal{L}$ is a semi-Fredholm operator with index satisfying (3.6). ∎

Remark 3.11 In general, a parabolic evolution operator $G(t,s)$ is not one to one. See e.g. [12]. However, several sufficient conditions for backward uniqueness are known. We mention here the papers [5, 6] for abstract evolution operators in Hilbert spaces, and [17] for evolution operators associated to specific parabolic partial differential operators.

To prove next corollary 3.13, which is one of the main results of the paper, we shall use the following simple lemma.

Lemma 3.12 *Let X be a topological vector space and let V, W be two subspaces of X with dim $V < \infty$. Then*

i) if W is closed then $V + W$ is closed,

ii) if codim $W < \infty$ then dim V − codim W = dim $(V \cap W)$ − codim $(V + W)$.

Corollary 3.13 *If dim $P_{+\infty}(X) < \infty$ and dim $P_{-\infty}(X) < \infty$ then $\mathcal{L}$ is Fredholm with index*

$$ind\, \mathcal{L} = dim\, P_{-\infty}(X) - dim\, P_{+\infty}(X).$$

Proof — Since $\dim P(-T)(X) = \dim P_{-\infty}(X) < \infty$ then $\dim \operatorname{Ker} G(T,-T)_{|P(-T)(X)} < \infty$. Hence we have only to show that $(W^s(T), W^u(T))$ is a Fredholm pair. Since $W^u(T) = G(T,-T)W^u(-T) = G(T,-T)P(-T)(X)$ we have

$$dim\, W^u(T) = dim\, P(-T)(X) - dim\, \operatorname{Ker} G(T,-T)_{|P(-T)(X)} =$$

$$= dim\, P_{-\infty}(X) - dim\, \operatorname{Ker} G(T,-T)_{|P(-T)(X)} < \infty.$$

Moreover $\operatorname{codim} W^s(T) = \dim P(T)(X) = \dim P_{+\infty}(X) < \infty$. Using lemma 3.12 with $W := W^s(T)$ and $V := W^u(T)$ we get the statement. ∎

Corollary 3.14 *If the embedding $D \hookrightarrow X$ is compact then $\mathcal{L}$ is a Fredholm operator.*

Proof — Since the embedding is compact then $R(\lambda, A_{+\infty})$, $R(\lambda, A_{-\infty})$ are compact operators, and (see e.g. [10, p. 187]) the spectra of $A_{+\infty}$ and of $A_{-\infty}$ consist of isolated eigenvalues with finite algebraic multiplicity. Since the dimension of $P_{+\infty}(X)$ (respectively, $P_{-\infty}(X)$) is the sum of the dimensions of the generalized eigenspaces of $A_{+\infty}$ (respectively, $A_{-\infty}$) corresponding to eigenvalues with positive real part, then they are finite, and the statement follows. ∎

4 Examples

If the dimension of X is finite we have immediately that $\mathcal{L}$ is a Fredholm operator. But in infinite dimensions this is not true in general, as the following example (taken from [1]) shows.

Example 4.1 *Let V and W be two closed subspaces of a Hilbert space H. Then there exists an asymptotically hyperbolic path of operators $t \mapsto A(t) \in C^\infty(\mathbb{R}, L(H))$, such that each $A(t)$ is self-adjoint and*

$$W^s(T) = V, \qquad W^u(T) = W.$$

Since, in this case, every $A(t)$ is a bounded operator, then $G(t,s)$ is well defined and invertible for each t and s, so that *dim* Ker $G(T,-T)_{|P(-T)(X)} = 0$. This example and theorem 3.10 provide us an easy way to build examples of families of operators satisfying our assumptions such that $\mathcal{L}$ has closed or not closed range, it is invertible or not invertible, it is or it is not a Fredholm operator.

Another example in [1] shows that, if $\mathcal{L}$ is a Fredholm operator, its index could be every $m \in \mathbb{Z}$ and it does not depend only on the endpoints $A_{\pm\infty}$.

Example 4.2 *For each $k \in \mathbb{Z}$ there exists a family $\{A_k(t)\}$ of asymptotically hyperbolic paths of bounded operators in a Hilbert space having identical end-points at $\pm\infty$, such that $\mathcal{L}_{A_k}$ is a Fredholm operator of index k.*

Note that in the case of example 4.2 either *dim* $P_{-\infty}(X)$ or *dim* $P_{+\infty}(X)$ has to be infinite, otherwise corollary 3.13 would imply that the index of $\mathcal{L}_{A_k}$ is independent of k.

Now let us see an example where we can apply our results.

Example 4.3 Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^2 boundary, and let a_{ij}, b_i, $c : \mathbb{R} \times \overline{\Omega} \mapsto \mathbb{R}$ be continuous functions satisfying a Hölder condition in time with $0 < \alpha < 1$,

$$|f(t,x) - f(s,x)| \leq C|t-s|^{\alpha}, \;\; t,\, s \in \mathbb{R}, \;\; x \in \Omega,$$

for $f = a_{ij}$, b_i, c, and satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \mu|\xi|^2, \qquad t \in \mathbb{R},\; x \in \Omega.$$

We denote by $\nu(x)$ the exterior normal vector to $\partial\Omega$ at $x \in \partial\Omega$. We consider the realizations of the differential operators

$$\mathcal{A}(t,x,D) := \sum_{i,j=1}^{n} a_{ij}(x,t)D_{ij} + \sum_{i=1}^{n} b_i(x,t)D_i + c(x,t)I\;, \quad x \in \Omega,\; t \in \mathbb{R},$$

with Neumann boundary condition in three different Banach spaces:

$X_1 = L^p(\Omega)$, $1 < p < \infty$, with common domain

$$D_1 := \left\{ u \in W^{2,p}(\Omega) \; : \; \frac{\partial u}{\partial \nu} = 0 \right\}, \quad (A_1(t)u)(x) := \mathcal{A}(t,x,D)u, \;\; u \in D_1;$$

$X_2 = C(\overline{\Omega})$, in the case $a_{ij}(x,t) = \varphi_{ij}(x)\psi(t,x)$, so that we have the common domain

$$D_2 := \left\{ u \in \bigcap_{p>1} W^{2,p}(\Omega) \; : \; \sum_{i,j=1}^{n} \varphi_{ij}(x) D_{ij}u(x) \in C(\overline{\Omega}),\; \frac{\partial u}{\partial \nu} = 0 \right\},$$

$$(A_2(t)u)(x) := \mathcal{A}(t,x,D)u, \;\; u \in D_2;$$

$X_3 = C^{\theta}(\overline{\Omega})$, provided the boundary of Ω is $C^{2+\theta}$ and the coefficients a_{ij}, b_i, c are also uniformly θ-Hölder continuous with respect to the space variables; in this case the common domain is

$$D_3 := \left\{ u \in C^{2+\theta}(\overline{\Omega}) \; : \; \frac{\partial u}{\partial \nu} = 0 \right\}, \quad (A_3(t)u)(x) := \mathcal{A}(t,x,D)u, \;\; u \in D_3.$$

The operators $A_j(t)$, $j = 1,2,3$, are sectorial for every $t \in \mathbb{R}$. In addition, $t \mapsto A_j(t) \in C^{\alpha}(\mathbb{R}, L(D_j, X_j))$ for $j = 1,2$, and if the coefficients are more regular (for instance, if they are in $C^{2\alpha,2\theta}(\mathbb{R} \times \overline{\Omega})$ in the case α, $\theta \leq 1/2$) this is true also for $j = 3$. We need that there exist $\lim_{t\to+\infty} A_j(t) =: A_{j+}$ and $\lim_{t\to-\infty} A_j(t) =: A_{j-}$, $j = 1,2,3$ in $L(D_j, X_j)$. This is true if there exist $\lim_{t\to\pm\infty} a_{ij}(t) =: a_{ij}^{\pm}$, $\lim_{t\to\pm\infty} b_i(t) =: b_i^{\pm}$ and $\lim_{t\to\pm\infty} c(t) =: c^{\pm}$ in $L^\infty(\Omega)$ for $j = 1,2$ and in $C^\theta(\overline{\Omega})$ for $j = 3$. Moreover we need that the limits, A_{j+} and A_{j-}, $j = 1,2,3$, are hyperbolic. Under our assumptions, the spectra of A_{1+} and of A_{1-} are independent of p, and they do coincide with the spectra of A_{j+} and of A_{j-}, $j = 2,3$, respectively. Therefore, A_{j+} and A_{j-}, $j = 1,2,3$, are hyperbolic provided A_{1+} and of A_{1-} are hyperbolic for $p = 2$.

Under these assumptions we can use corollary 3.14 since the embeddings $D_j \hookrightarrow X_j$, $j = 1,2,3$, are compact. Hence we obtain that $\mathcal{L}_j$, $j = 1,2,3$, are Fredholm operators with index equal to the difference between the sum of the algebraic dimensions of the eigenspaces of A_{j-} corresponding to eigenvalues with positive real part and the sum of the algebraic dimensions of the eigenspaces of A_{j+} corresponding to eigenvalues with positive real part.

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