ON A CLASS OF ELLIPTIC AND PARABOLIC EQUATIONS IN CONVEX DOMAINS WITHOUT BOUNDARY CONDITIONS

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Abstract. We consider the operator $\mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle$, where U is a convex real function defined in a convex open set $\mathcal{O} \subset \mathbb{R}^N$ and $\lim_{|x|\to\infty} U(x) = \lim_{|x|\to\partial\mathcal{O}} U(x) = +\infty$. We study the realization of $\mathcal{A}$ in the spaces $C_b(\overline{\mathcal{O}})$, $C_b(\mathcal{O})$ and $B_b(\mathcal{O})$, and prove several properties of the associated Markov semigroup. In contrast with the case of bounded coefficients, elliptic equations and parabolic Cauchy problems such as (1.3) and (1.4) below are uniquely solvable in reasonable classes of functions, without imposing any boundary condition. We prove that the associated semigroup coincides with the transition semigroup of a stochastic variational inequality on $C_b(\overline{\mathcal{O}})$.

1. Introduction

We consider the differential operator $\mathcal{A}$ defined by

$$\mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle \tag{1.1}$$

in a convex open set $\mathcal{O} \subset \mathbb{R}^N$, where $U : \mathcal{O} \mapsto \mathbb{R}$ is a convex function such that

$$\lim_{x\to\partial\mathcal{O},\, x\in\mathcal{O}} U(x) = +\infty, \quad \lim_{|x|\to+\infty,\, x\in\mathcal{O}} U(x) = +\infty. \tag{1.2}$$

No further regularity assumptions will be made, except in Proposition 3.5.

The aim of this note is the study of the realization of $\mathcal{A}$ and of the associated Markov semigroup $T(t)$ in spaces of bounded functions in $\mathcal{O}$. Therefore, for $\lambda > 0$ and φ Borel measurable and bounded we shall study the elliptic equation

$$\lambda f(x) - (\mathcal{A}f)(x) = \varphi(x), \quad x \in \mathcal{O}, \tag{1.3}$$

and the parabolic Cauchy problem

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x), & t > 0,\ x \in \mathcal{O}, \\ \\ u(0,x) = \varphi(x), & x \in \mathcal{O}. \end{cases} \tag{1.4}$$

Solutions to (1.3) and (1.4) are readily constructed by classical methods. See e.g. [2, 3]. Uniqueness of the solution is not obvious. If the gradient DU were bounded, problems (1.3) and (1.4) would have unique solutions in reasonable classes of functions satisfying some prescribed boundary condition (Dirichlet, Neumann, or Robin boundary conditions, see [13]). But since U blows up near the boundary $\partial\mathcal{O}$, also $|DU|$ does.

Appropriate settings for uniqueness are spaces of bounded functions with bounded first order space derivatives. More precisely, we denote respectively by $B_b(\mathcal{O})$, $C_b(\mathcal{O})$, $C_b(\overline{\mathcal{O}})$ the spaces of bounded and Borel measurable, continuous, continuous up to the boundary, real valued functions, and we set

$$D(A_\infty) = \{f \in C^1_b(\mathcal{O}) \textstyle\bigcap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O}) :\ \mathcal{A}f \in B_b(\mathcal{O})\},$$

$$D(A_{C_b}) = \{f \in C^1_b(\mathcal{O}) \textstyle\bigcap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O}) :\ \mathcal{A}f \in C_b(\mathcal{O})\},$$

$$D(A_C) = \{f \in C^1_b(\mathcal{O}) \textstyle\bigcap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O}) :\ \mathcal{A}f \in C_b(\overline{\mathcal{O}})\}.$$

Work supported by the MIUR research project PRIN 2006 "Operatori di Kolmogorov".

We shall prove that for each $\lambda > 0$ and for each $\varphi \in B_b(\mathcal{O})$ problem (1.3) has a unique solution $f \in D(A_\infty)$. Consequently, if $\varphi \in C_b(\mathcal{O})$ (resp. $\varphi \in C_b(\overline{\mathcal{O}})$) then problem (1.3) has a unique solution $f \in D(A_{C_b})$ (resp., $f \in D(A_C)$). Moreover, the operator

$$A_{C_b} : D(A_{C_b}) \mapsto C_b(\mathcal{O}), \quad Af = \mathcal{A}f$$

is dissipative, and so it is m-dissipative. By the Crandall–Liggett Theorem, its part in the closure $\overline{D(A_{C_b})}$ of $D(A_{C_b})$ in $C_b(\mathcal{O})$ is the infinitesimal generator of a strongly continuous semigroup $\{T_0(t) : t \geq 0\}$ in $\overline{D(A_{C_b})}$. However, $\overline{D(A_{C_b})}$ is strictly contained in $C_b(\mathcal{O})$, while we want to work in the whole space $C_b(\mathcal{O})$, and in $B_b(\mathcal{O})$. The same difficulty arises in the space $C_b(\overline{\mathcal{O}})$ if $\mathcal{O}$ is unbounded: $\overline{D(A_C)}$ is strictly contained in $C_b(\overline{\mathcal{O}})$.

Concerning problem (1.4), we shall prove that for any $\varphi \in B_b(\mathcal{O})$ there exists a unique solution of problem (1.4) in a large class of functions. Moreover the following assertions hold.

(i) Setting

$$T(t)\varphi := u(t), \quad t \geq 0,$$

where u is the solution of (1.4), $\{T(t) : t \geq 0\}$ is a Markov semigroup in the space $B_b(\mathcal{O})$.

(ii) $T(t)$ is an extension of $T_0(t)$. Moreover we have

$$R(\lambda, A_{B_b})\varphi(x) = \int_0^\infty e^{-\lambda t} T(t)\varphi(x)dt, \quad \lambda > 0, \ f \in B_b(\mathcal{O}), \ x \in \mathcal{O}.$$

(iii) $T(t)$ is irreducible, strong Feller and it has a unique invariant measure μ given by

$$\mu(dx) = \left(\int_{\mathcal{O}} e^{-2U(x)} dx \right)^{-1} e^{-2U(x)} dx. \tag{1.5}$$

Moreover,

$$\lim_{t \to +\infty} T(t)\varphi(x) = \int_{\mathcal{O}} \varphi(y)\mu(dy), \quad \forall\, x \in \mathcal{O}, \ \varphi \in B_b(\mathcal{O}). \tag{1.6}$$

(iv) $T(t)$ maps $B_b(\mathcal{O})$ into $C^1_b(\mathcal{O})$, and for each $t > 0$ we have

$$\|(T(t)\varphi)^2 + t|DT(t)\varphi|^2\|_\infty \leq \|\varphi\|^2_\infty, \quad \varphi \in B_b(\mathcal{O}). \tag{1.7}$$

The above mentioned "large class" for uniqueness is the set of functions $u : [0, +\infty) \times \mathcal{O} \mapsto \mathbb{R}$ such that $t \mapsto u(t, \cdot)$ belongs to $C([0, +\infty); L^2(\mathcal{O}, \mu))$ and $t \mapsto u_t(t, \cdot)$, $t \mapsto \mathcal{A}u(t, \cdot)$ belong to $C((0, +\infty); L^2(\mathcal{O}, \mu))$.

Indeed, the proofs of our statements rely heavily on the results of [8] where we showed that the realization A_2 of $\mathcal{A}$ in $L^2(\mathcal{O}, \mu)$, with domain

$$D(A_2) = \{u \in H^2(\mathcal{O}, \mu) : \mathcal{A}u \in L^2(\mathcal{O}, \mu)\} = \{u \in H^2(\mathcal{O}, \mu) : \langle DU, Du\rangle \in L^2(\mathcal{O}, \mu)\} \tag{1.8}$$

is a self-adjoint and dissipative operator, therefore it generates an analytic contraction semigroup e^{tA_2} in $L^2(\mathcal{O}, \mu)$. Hence, in the language of the theory of evolution equations in Banach spaces, we have uniqueness of the classical solution of the Cauchy problem $u' = A_2 u$, $t > 0$, $u(0) = \varphi$ in the space $L^2(\mathcal{O}, \mu)$.

In the second part of this paper we study the connection of $T(t)$ with the stochastic variational inequality

$$\begin{cases} dX(t) + \partial U(X(t))dt \ni dW(t), \\ \\ X(0) = x \in \mathcal{O}, \end{cases} \tag{1.9}$$

where $\partial U(x) = \{x^* \in \mathbb{R}^N : U(y) \geq U(x) + \langle x^*, y - x\rangle \ \forall y \in \mathbb{R}^N\}$ denotes the subgradient of U at x, and $W(t)$ is a standard N-dimensional Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The theory of [5] implies that problem (1.9) has a unique solution (for the precise notion of solution see Sect. 3). Since the paper [5] treats a very general situation, it is very complicated; here we give a simpler proof of existence and uniqueness in our specific case, following the approach of [1].

Then we consider the corresponding transition semigroup,

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))],$$

and we prove that $P_t\varphi(x) = T(t)\varphi(x)$ for each $\varphi \in C_b(\overline{\mathcal{O}})$ and $x \in \mathcal{O}$.

2. Analytical results

2.1. **Preliminaries.** We quote some results from [8] about the realization of $\mathcal{A}$ in $L^2(\mathcal{O},\mu)$, where μ is the measure defined in (1.5). The domain $D(A_2)$ is defined in (1.8).

Theorem 2.1. *The resolvent set of A_2 contains $(0,+\infty)$. For every $\lambda > 0$ we have*

$$\left\{\begin{array}{ll} (i) & \|R(\lambda,A_2)f\|_{L^2(\mathcal{O},\mu)} \le \dfrac{1}{\lambda}\,\|f\|_{L^2(\mathcal{O},\mu)}, \\[2ex] (ii) & \|\,|DR(\lambda,A_2)f|\,\|_{L^2(\mathcal{O},\mu)} \le \sqrt{\dfrac{2}{\lambda}}\,\|f\|_{L^2(\mathcal{O},\mu)}, \\[2ex] (iii) & \|\,|D^2R(\lambda,A_2)f|\,\|_{L^2(\mathcal{O},\mu)} \le 2\sqrt{2}\,\|f\|_{L^2(\mathcal{O},\mu)}. \end{array}\right. \tag{2.1}$$

Moreover, A_2 is symmetric, $R(\lambda,A_2)$ preserves positivity, and $R(\lambda,A_2)\mathbb{1} = \mathbb{1}/\lambda$.

Therefore, A_2 is a dissipative self-adjoint operator in the Hilbert space $L^2(\mathcal{O},\mu)$, that generates a contraction, analytic semigroup e^{tA_2}. Additional properties are stated in the following proposition.

Proposition 2.2. *The following statements hold.*

(i) *For each $u \in D(A_2)$ we have*

$$\int_{\mathcal{O}} A_2u(x)\,\mu(dx) = 0.$$

Consequently, for each $f \in L^2(\mathcal{O},\mu)$ and $t > 0$ we have

$$\int_{\mathcal{O}} e^{tA_2}f(x)\,\mu(dx) = \int_{\mathcal{O}} f(x)\,\mu(dx).$$

(ii) *The space $H^1(\mathcal{O},\mu)$ is the domain of $(-A_2)^{1/2}$.*

(iii) *Every function in $D(A_2)$ is the limit in $H^2(\mathcal{O},\mu)$ of a sequence of functions in $H^2(\mathcal{O},\mu)$ that are restrictions to $\mathcal{O}$ of functions belonging to $C^2_b(\mathbb{R}^N)$.*

(iv) *e^{tA_2} is a symmetric Markov semigroup in $L^2(\mathcal{O},\mu)$, in the sense of* [9, §1.3, 1.4].

(v) *The kernel of A_2 consists of constant functions.*

(vi) *For all $f \in L^2(\mathcal{O},\mu)$ we have*

$$\lim_{t\to+\infty} e^{tA_2}f = \int_{\mathcal{O}} f(y)\mu(dy) \quad \text{in } L^2(\mathcal{O},\mu).$$

By the general theory of semigroups, e^{tA_2} may be extended in a standard way to a contraction semigroup e^{tA_p} in $L^p(\mathcal{O},\mu)$ for each $p \in [1,+\infty]$. See [9].

2.2. **Further properties of e^{tA_2}.** We begin with local smoothing properties.

Proposition 2.3. *e^{tA_2} maps $L^2(\mathcal{O},\mu)$ into $C(\mathcal{O})$ (in fact, into $C^{1+\alpha}_{loc}(\mathcal{O})$ for any $\alpha \in (0,1)$) for each $t > 0$.*

Proof. The proof follows from standard interior regularity properties of solutions to parabolic equations with locally bounded and measurable coefficients. We write a sketch below, but it is really standard.

Let $t_0 > 0$, $x_0 \in \mathcal{O}$. Let $r > 0$ be such that the closed ball $B = B(x_0,r)$ is contained in $\mathcal{O}$, and let $\theta \in C^\infty_0((0,+\infty)\times B)$ be a cut off function, such that $\theta \equiv 1$ in $[t_0/2, 3t_0/2] \times$

$B(x_0, r/2)$. The function $u(t,x) = \theta(t,x)e^{tA_2}\varphi(x)$ satisfies

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x) + f(t,x), \quad t \geq 0,\ x \in B, \\ \\ u(0,x) = 0, \quad x \in B, \\ \\ u(t,x) = 0, \quad t \geq 0,\ x \in \partial B, \end{cases}$$

with

$$f(t,\cdot) = e^{tA_2}\varphi\left(\theta_t - \frac{1}{2}\Delta\theta + \langle D\theta, DU\rangle\right)\right) - \langle D\theta, De^{tA}\varphi\rangle.$$

Since e^{tA_2} is an analytic semigroup in $L^2(\mathcal{O},\mu)$ and $DU \in (L^\infty(B))^N$, $t \mapsto De^{tA_2}\varphi(t,\cdot) \in C^\infty((0,+\infty); (H^1(B))^N)$, then $t \mapsto f(t,\cdot)$ belongs to $C_0^\infty([0,+\infty); L^{2^*}(B))$, where 2^* is the Sobolev exponent $2N/(N-2)$ if $N > 2$, it is any number larger than 2 if $N \leq 2$. Setting the problem in the space $L^{2^*}(B)$, it follows that $u \in C^\infty([0,+\infty); W^{2,2^*}(B))$. If $2^* > N$ then $Du \in C^\infty([0,+\infty); (C(B))^N)$. If $2^* \leq N$ we iterate this procedure and in a finite number of steps we arrive at $Du \in C^\infty([0,+\infty); (C(B))^N)$, for any dimension N. So, $t \mapsto f(t,\cdot)$ is smooth with values in $L^\infty(B)$. Then u is smooth with values in the domain of the realization of $\mathcal{A}$ with Dirichlet boundary condition in $L^\infty(B)$, which is contained in $C^{1+\alpha}(B)$ for each $\alpha \in (0,1)$. $\square$

e^{tA_2} is a semigroup in $L^2(\mathcal{O},\mu)$, which is a space of equivalence classes of functions. Let us denote by $[\cdot]$ the equivalence classes of functions in $L^2(\mathcal{O},\mu)$. The statement of Proposition 2.3 has to be understood as: for each $[\varphi] \in L^2(\mathcal{O},\mu)$, in the equivalence class $[e^{tA_2}[\varphi]]$ there is a unique continuous (in fact, $C^{1+\alpha}_{loc}$) function.

With this specification, we have the next corollary.

Corollary 2.4. *e^{tA_2} is a strong Feller semigroup (i.e., $e^{tA_2}\varphi$ is continuous for each $\varphi \in B_b(\mathcal{O})$ and $t > 0$).*

Another feature of e^{tA_2} is the following.

Corollary 2.5. *e^{tA_2} is irreducible.*

Proof. Let φ be the characteristic function of a ball $B \subset\subset \mathcal{O}$. We have to show that $e^{tA_2}\varphi > 0$ for each $t > 0$. This follows from local Harnack inequalities for parabolic operators with locally bounded and measurable coefficients. $\square$

In the papers [6, 8] we constructed the resolvent $R(\lambda, A)$ for positive λ by approximation with problems in the whole $\mathbb{R}^N$. But it is useful to approach $R(\lambda, A)$ by resolvents of operators defined in bounded regular sets $\mathcal{O}_n \subset \mathcal{O}$. We use the following lemma.

Lemma 2.6. *There exists a nested sequence of convex bounded open sets $\mathcal{O}_n$ with smooth boundary, whose union is $\mathcal{O}$.*

Proof. For each $n \in \mathbb{N}$, $n > \min U$, the set $K_n := \{x \in \mathcal{O} : U(x) \leq n\}$ is non empty and compact, therefore its distance from $\partial\mathcal{O}$ is positive.

We construct a convex open set $\mathcal{O}_n$ with smooth boundary, such that $K_{n-1} \subset \mathcal{O}_n \subset K_{n+2}$. The approximations U_ε obtained by convolution with smooth mollifiers are well defined in K_{n+2} for ε small enough, and converge uniformly to U over K_{n+2} as $\varepsilon \to 0$. Since U is convex, the functions U_ε are convex too.

Let $\varepsilon = \varepsilon(n)$ be so small that $\|U_\varepsilon - U\|_{L^\infty(K_{n+2})} \leq 1/2$. Then $U_\varepsilon \leq U + 1/2 \leq n - 1/2$ over K_{n-1}, $U_\varepsilon \geq U - 1/2 \geq n + 1/2$ over $K_{n+2} \setminus K_{n+1}$, so that the level line $U_\varepsilon = n$ is contained in the interior of $K_{n+2} \setminus K_{n-1}$. The gradient of U_ε does not vanish at any point of the level line, because it vanishes only at minimum points, and $U_\varepsilon(x) \leq n - 1/2$ in ∂K_{n-1}, so that in the level line there are no minimum points.

Therefore we can define $\mathcal{O}_n$ as $\mathcal{O}_n = \{x \in \mathring{K}_{n+2} : U_\varepsilon(x) < n\}$. $\square$

Since $\mathcal{O}_n$ is smooth and bounded, the general theory of PDE's yields that the realization A_n of $\mathcal{A}$ with Neumann boundary conditions in $L^2(\mathcal{O}_n, dx) = L^2(\mathcal{O}_n, \mu)$ generates an analytic semigroup $T_n(t)$. In the paper [7] we proved estimates quite similar to (2.1) for the operators A_n. In particular,

$$\begin{aligned}\|R(1,A_n)\varphi\|_{L^2(\mathcal{O}_n,\mu)} &\leq \|\varphi\|_{L^2(\mathcal{O}_n,\mu)},\\ \|\,|DR(1,A_n)\varphi|\,\|_{L^2(\mathcal{O}_n,\mu)} &\leq \sqrt{2}\|\varphi\|_{L^2(\mathcal{O}_n,\mu)},\\ \|\,|D^2R(1,A_n)\varphi|\,\|_{L^2(\mathcal{O}_n,\mu)} &\leq 2\sqrt{2}\|\varphi\|_{L^2(\mathcal{O}_n,\mu)},\end{aligned}$$

for each $\varphi \in L^2(\mathcal{O}_n;\mu)$. Note that the constants are universal, i.e. they do not depend on n. This implies that the domains of A_n are uniformly embedded in the spaces $H^2(\mathcal{O}_n,\mu)$, i.e. there is $C>0$ independent of n such that

$$\|u\|_{H^2(\mathcal{O}_n,\mu)} \leq C(\|u\|_{L^2(\mathcal{O}_n,\mu)} + \|A_n u\|_{H^2(\mathcal{O}_n,\mu)}), \quad u \in D(A_n). \tag{2.2}$$

Although in this paper we deal with real valued functions, in the next proposition we need to consider complex valued functions to use the standard representation formula of analytic semigroups through Dunford integrals. So, we consider the spaces $L^2(\mathcal{O},\mu,\mathbb{C})$, $L^2(\mathcal{O}_n,\mu,\mathbb{C})$, and the complexifications of the operators A, A_n (still denoted by A, A_n).

Proposition 2.7. *For each $\lambda \notin (-\infty,0]$ and each $\varphi \in L^2(\mathcal{O},\mu;\mathbb{C})$, $R(\lambda,A)\varphi$ is the a.e. pointwise limit of $R(\lambda,A_n)\varphi_{|\mathcal{O}_n}$. For each $t>0$, $T(t)\varphi$ is the a.e. pointwise limit of $T_n(t))\varphi_{|\mathcal{O}_n}$.*

Proof. Since A and each A_n are self-adjoint and dissipative, their spectra are contained in $(-\infty,0]$ and the norms of their resolvent operators, in the spaces $L^2(\mathcal{O},\mu;\mathbb{C})$ and $L^2(\mathcal{O}_n,\mu;\mathbb{C})$ respectively, do not exceed $(|\lambda|\cos(\theta/2))^{-1}$ for each $\lambda \notin (-\infty,0]$, with $\theta = \arg(\lambda)$.

Fix $\lambda \notin (-\infty,0]$. For each $n\in\mathbb{N}$, the function $u_n := R(\lambda,A_n)\varphi_{|\mathcal{O}_n}$ is well defined in $\mathcal{O}_k$ for $n\geq k$, and since $A_nR(\lambda,A_n) = \lambda R(\lambda,A_n) - I$, by the obvious extension of (2.2) to complex valued functions the sequence $(u_n)_{n\geq k}$ is bounded in $H^2(\mathcal{O}_k,\mu;\mathbb{C})$ by a constant independent of n. By the usual diagonal procedure, we can find a subsequence u_{n_h} that converges weakly in each $H^2(\mathcal{O}_k,\mu;\mathbb{C})$ and pointwise a.e. in $\mathcal{O}_k$ to a function $u \in H^2(\mathcal{O}_k,\mu;\mathbb{C})$, for every $k\in\mathbb{N}$. Since $\|u\|_{H^2(\mathcal{O}_k,\mu;\mathbb{C})}$ is bounded by a constant independent of k, then $u\in H^2(\mathcal{O},\mu;\mathbb{C})$. Moreover, the weak convergence implies that $\lambda u - \mathcal{A}u = \varphi$ in $\mathcal{O}_k$. Since $\mathcal{O}$ is the union of the sets $\mathcal{O}_k$, then u satisfies $\lambda u - \mathcal{A}u = \varphi$ in $\mathcal{O}$. It follows that $u = R(\lambda,A)\varphi$. Since for any other converging subsequence the limit has to be $(R(\lambda,A)\varphi)_{|\mathcal{O}_k}$, then the sequence $(u_n)_{n\geq k}$ (and not only a subsequence) converges weakly in $H^2(\mathcal{O}_k,\mu;\mathbb{C})$ and pointwise a.e. in $\mathcal{O}_k$ to $R(\lambda,A)\varphi$, and this proves the first part of the statement.

Representing $T(t)$ and $T_n(t)$ by Dunford integrals over the same contours, we get pointwise convergence of $T_n(t))\varphi_{|\mathcal{O}_n}$ to $T(t)\varphi$. □

2.3. **Definition and properties of $T(t)$.** We set

$$T(t)\varphi = e^{tA_2}\varphi, \quad t>0,\ \varphi\in B_b(\mathcal{O}),$$

or, to be more fastidious: $T(t)\varphi$ is the unique continuous function in the equivalence class of $e^{tA_2}\varphi$.

Since the density e^{-2U} of μ is positive in $\mathcal{O}$, then $L^\infty(\mathcal{O},\mu) = L^\infty(\mathcal{O},dx)$. Therefore, $T(t)\varphi = e^{tA_2}\varphi \in L^\infty(\mathcal{O},dx)$ for each $\varphi\in C_b(\mathcal{O})$. So, $T(t)$ is a contraction semigroup in $B_b(\mathcal{O})$ and in $C_b(\mathcal{O})$. Since it is the restriction of e^{tA_2} to $B_b(\mathcal{O})$, it inherits several properties of e^{tA_2}. In particular, it is irreducible and strong Feller.

Proposition 2.8. *$T(t)$ maps $C_b(\mathcal{O})$ into $C^1_b(\mathcal{O})$, and for each $t>0$ we have*

$$\|(T(t)\varphi)^2 + t|DT(t)\varphi|^2\|_\infty \leq \|\varphi\|^2_\infty, \quad \varphi\in C_b(\mathcal{O}). \tag{2.3}$$

Proof. The idea is to approach $T(t)\varphi$ by the solutions of Cauchy-Neumann problems in bounded convex open sets $\mathcal{O}_n$ with smooth boundary, for which we get an estimate similar to (2.3) in the usual way.

We consider the $\mathcal{O}_n$'s constructed in Lemma 2.6.

Fix $p > N$. From the general theory of PDE's we know that the realization A_n of $\mathcal{A}$ in $L^p(\mathcal{O}_n)$ with Neumann boundary condition generates an analytic semigroup $T_n(t)$.

Let $u(t,x) := T_n(t)\varphi_{|\mathcal{O}_n}(x)$ be the solution of the problem

$$\begin{cases} D_t u = \mathcal{A}u = \frac{1}{2}\Delta u - \langle DU, Du\rangle, & t>0,\ x\in\mathcal{O}_n, \\ \\ \dfrac{\partial u}{\partial \nu} = 0, \quad t>0,\ x\in\partial\mathcal{O}_n, \\ \\ u(0,x) = \varphi(x), \quad x\in\mathcal{O}_n. \end{cases} \tag{2.4}$$

The usual procedure to get estimate (2.3) for u needs C^2 coefficients, so we approach again U by the functions U_ε used in Lemma 2.6, and consider the solutions u_ε of

$$\begin{cases} D_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon - \langle DU_\varepsilon, Du_\varepsilon\rangle, & t>0,\ x\in\mathcal{O}_n, \\ \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0, \quad t>0,\ x\in\partial\mathcal{O}_n, \\ \\ u_\varepsilon(0,x) = \varphi(x), \quad x\in\mathcal{O}_n. \end{cases} \tag{2.5}$$

The procedure of the paper [2] (i.e., using the maximum principle in the equation satisfied by $v(t,x) := |(u_\varepsilon(t,x))^2 + t|Du_\varepsilon(t,x)|^2|)$ gives

$$|(u_\varepsilon(t,x))^2 + t|Du_\varepsilon(t,x)|^2| \le \|\varphi\|_\infty^2, \quad t>0,\ x\in\overline{\mathcal{O}_n}. \tag{2.6}$$

The difference $v_\varepsilon := u - u_\varepsilon$ satisfies

$$\begin{cases} D_t v_\varepsilon = \frac{1}{2}\Delta v_\varepsilon - \langle DU, Dv_\varepsilon\rangle + \langle DU_\varepsilon - DU, Du_\varepsilon\rangle, & t>0,\ x\in\mathcal{O}_n, \\ \\ \dfrac{\partial v_\varepsilon}{\partial \nu} = 0, \quad t>0,\ x\in\partial\mathcal{O}_n, \\ \\ v_\varepsilon(0,x) = 0, \quad x\in\mathcal{O}_n, \end{cases}$$

so that

$$(u-u_\varepsilon)(t,\cdot) = \int_0^t T_n(t-s)g_\varepsilon(s)\,ds$$

where

$$g_\varepsilon(s) = \langle DU_\varepsilon - DU, Du_\varepsilon(s,\cdot)\rangle.$$

Since $DU \in (L^\infty(\mathcal{O}_n))^N$, then $DU_\varepsilon - DU$ goes to 0 in $(L^p(\mathcal{O}_n))^N$. Since $\|Du_\varepsilon(s,\cdot)\|_\infty \le C/\sqrt{s}$ by (2.6), then $\|s^{1/2}g_\varepsilon(s)\|_{L^p}$ goes to 0 as $\varepsilon\to 0$. It follows that $(u-u_\varepsilon)(t,\cdot)$ goes to 0 in $D_{A_n}(1/2,p)$ and hence in $W^{1,p}(\mathcal{O}_n)$. Since $p>N$ (the space dimension), $(u-u_\varepsilon)(t,\cdot)$ goes to 0 in $L^\infty(\mathcal{O}_n)$. The convergence in the sup norm is enough to bound the Lipschitz constant of $u(t,\cdot)$. Therefore,

$$\|(T_n(t)\varphi_{|\mathcal{O}_n})^2 + t|DT_n(t)\varphi_{|\mathcal{O}_n}|^2\|_\infty \le \|\varphi\|_\infty^2. \tag{2.7}$$

On the other hand,for each $t>0$ $T(t)\varphi$ is the a.e. pointwise limit of $T_n(t)\varphi_{|\mathcal{O}_n}$ by proposition 2.7. By estimates (2.7) and the Arzelà-Ascoli theorem, for each compact set $K\subset\Omega$ a subsequence converges uniformly to $T(t)\varphi$ on K. Therefore, for each $x\in K$,

$$|(T(t)\varphi)^2(x) + t|DT(t)\varphi|^2(x)| \le \|\varphi\|_\infty^2,$$

and the statement follows. □

Corollary 2.9. *$T(t)$ maps $B_b(\mathcal{O})$ into $C_b^1(\mathcal{O})$, and for each $t>0$ estimate* (2.3) *holds for each $\varphi\in B_b(\mathcal{O})$.*

Proof. Let $t>0$, $\varepsilon\in(0,t)$ and $\varphi\in B_b(\mathcal{O})$. We know that $T(\varepsilon)\varphi\in C_b(\mathcal{O})$, and (2.3) applied with φ replaced by $T(\varepsilon)\varphi$ and t replaced by $t-\varepsilon$ gives

$$\|(T(t)\varphi)^2 + (t-\varepsilon)|DT(t)\varphi|^2\|_\infty \le \|T(\varepsilon)\varphi\|_\infty^2 \le \|\varphi\|_\infty^2.$$

Letting $\varepsilon\to 0$ yields the statement. □

As a corollary of the gradient estimate we obtain a nice convergence result as $t \to \infty$.

Corollary 2.10. *For each $\varphi \in B_b(\mathcal{O})$ we have*

$$\lim_{t\to+\infty} T(t)\varphi(x) = m := \int_{\mathcal{O}} \varphi(y)\mu(dy), \quad \forall\, x \in \mathcal{O},$$

and the convergence is uniform on each compact subset $K \subset\subset \mathcal{O}$.

Proof. Since $\{T(t)\varphi :\ t \geq 1\}$ is bounded in $C^1_b(\mathcal{O})$, a sequence $(T(t_n)\varphi)$, with $\lim_{n\to\infty} t_n = +\infty$, converges pointwise in $\mathcal{O}$ and uniformly on each compact subset of $\mathcal{O}$. The limit is the mean value m, because $T(t)\varphi$ goes to m in $L^2(\mathcal{O},\mu)$ as $t \to +\infty$. In fact, for every sequence $t_n \to +\infty$ a subsequence $T(t_{k(n)})\varphi$ converges pointwise in $\mathcal{O}$, and this implies that $\lim_{t\to+\infty} T(t)\varphi(x) = m$ for each $x \in \mathcal{O}$. The convergence is uniform on each compact set because $\lim_{t\to+\infty} \|\,|DT(t)\varphi|\,\|_\infty = 0$ by Proposition 2.8. □

Another corollary is the following representation formula for $T(t)$.

Lemma 2.11. *For each $t > 0$, $x \in \mathcal{O}$ there exists a probability measure $p_{t,x}$ in $\mathcal{O}$ such that*

$$(T(t)\varphi)(x) = \int_{\mathcal{O}} \varphi(y)p_{t,x}(dy),\ \varphi \in C_b(\mathcal{O}), \tag{2.8}$$

Proof. Fix $t > 0$ and $x \in \mathcal{O}$. Denoting by $C_c(\mathcal{O})$ the space of real valued continuous functions with compact support in $\mathcal{O}$, the functional $C_c(\mathcal{O}) \mapsto \mathbb{R}$, $\varphi \mapsto (T(t)\varphi)(x)$ is linear and positive. Therefore (see e.g. [12, Thm. 2.14]) there exists a unique Borel measure $p_{t,x}$ on $\mathcal{O}$ such that (2.8) holds for each $\varphi \in C_c(\mathcal{O})$.

Let us show that $p_{t,x}(\mathcal{O})$ is finite. If $\{\varphi_n : n \in \mathbb{N}\}$ is an increasing sequence of functions in $C_c(\mathcal{O})$ that converges pointwise to $\mathbb{1}$, for each $n \in \mathbb{N}$ we have

$$\int_{\mathcal{O}} \varphi_n p_{t,x}(dy) = T(t)\varphi_n(x) \leq T(t)\mathbb{1}(x) = 1,$$

and the left hand side converges to $p_{t,x}(\mathcal{O})$, by the Monotone Convergence Theorem.

Let now $\varphi \in C_b(\mathcal{O})$, and let $\{\varphi_n : n \in \mathbb{N}\}$ be a sequence of functions in $C_c(\mathcal{O})$ that converges pointwise to φ, and such that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$. The functions $T(t)\varphi_n$ are equibounded and also equi-continuous, since $\|\,|DT(t)\varphi_n|\,\|_\infty \leq \|\varphi_n\|_\infty/\sqrt{t}$ by Proposition 2.8. Consequently a subsequence $T(t)\varphi_{\alpha(n)}$ converges pointwise in $\mathcal{O}$. The limit is $T(t)\varphi$, because $T(t)\varphi_n$ goes to $T(t)\varphi$ in $L^2(\mathcal{O},\mu)$. So, for each $x \in \mathcal{O}$ we have

$$(T(t)\varphi)(x) = \lim_{n\to\infty} (T(t)\varphi_{\alpha(n)})(x) = \lim_{n\to\infty} \int_{\mathcal{O}} \varphi_{\alpha(n)}(y)p_{t,x}(dy) = \int_{\mathcal{O}} \varphi(y)p_{t,x}(dy)$$

by dominated convergence. Therefore, the representation formula (2.8) holds for all continuous and bounded functions.

As a last step, from $T(t)\mathbb{1} = \mathbb{1}$ we get $p_{t,x}(\mathcal{O}) = 1$. □

Now we can define the realizations of $\mathcal{A}$ in $B_b(\mathcal{O})$, in $C_b(\mathcal{O})$ and in $C_b(\overline{\mathcal{O}})$ through their resolvent operators. Namely, we define A_∞ as the unique linear operator in $B_b(\mathcal{O})$ whose resolvent for $\lambda > 0$ is given by

$$R(\lambda, A_\infty)\varphi(x) = \int_0^\infty e^{-\lambda t}(T(t)\varphi)(x)dt, \quad \varphi \in B_b(\mathcal{O}). \tag{2.9}$$

In other words, $R(\lambda, A_\infty)$ is the restriction of $R(\lambda, A_2)$ to $B_b(\mathcal{O})$, that maps $B_b(\mathcal{O})$ and its subspaces $C_b(\mathcal{O})$ and $C_b(\overline{\mathcal{O}})$ into $C_b(\overline{\mathcal{O}})$. Note that, since $T(t)$ is a contraction semigroup, then $\|R(\lambda, A_\infty)\| \leq 1/\lambda$ for each $\lambda > 0$ so that A_∞ is a m-dissipative operator.

Proposition 2.12. $D(A_\infty) = \{u \in C^1_b(\mathcal{O}) \bigcap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O}) :\ \mathcal{A}u \in B_b(\mathcal{O})\}$.

Proof. Let $u \in D(A_\infty)$. Then $u = R(\lambda, A_\infty)\varphi$ for some $\lambda > 0$ and $\varphi \in B_b(\mathcal{O})$, so that $u \in H^2(\mathcal{O},\mu) \cap B_b(\mathcal{O})$, and $A_\infty u = \mathcal{A}u \in B_b(\mathcal{O})$ since $A_\infty u = \lambda u + \varphi$. Moreover, the representation formula (2.9) and Corollary 2.9 yield $u \in C^1_b(\mathcal{O})$.

To prove that $u \in W^{2,p}_{loc}(\mathcal{O})$ for each $p \geq 1$ we use the same procedure of Proposition 2.3, adapted to the elliptic case which makes it easier. We rewrite it for convenience.

Let $x_0 \in \mathcal{O}$. Let $r > 0$ be such that the closed ball $B = B(x_0, r)$ is contained in $\mathcal{O}$, and let $\theta \in C_0^\infty(B)$ be a cut off function, such that $\theta \equiv 1$ in $B(x_0, r/2)$. The function $v(x) = \theta(x)u(x)$ belongs to $D(A_\infty)$ and satisfies

$$\begin{cases} \lambda v(x) - \mathcal{A}v(x) = f(x), \quad x \in B, \\ \\ v(x) = 0, \quad x \in \partial B, \end{cases}$$

with

$$f(x) = (\mathcal{A}\theta)(x)u(x) - \langle D\theta(x), Du(x)\rangle.$$

Since the coefficients of $\mathcal{A}$ are in $L^\infty(B)$ and $u \in C^1(B)$, then $f \in L^\infty(B)$ and by the general theory of elliptic PDE's, $v \in W^{2,p}(B)$ for every $p \in [1, +\infty)$. Since v coincides with u in $B(x_0, r/2)$, then $u \in W^{2,p}(B(x_0, r/2))$ for every $p \in [1, +\infty)$.

Let now $u \in C_b^1(\mathcal{O}) \cap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O})$ be such that $\mathcal{A}u \in C_b(\mathcal{O})$. To prove that $u \in D(A_\infty)$, fix $\lambda > 0$ and set $\lambda u - \mathcal{A}u = f$. The function $w = R(\lambda, A_\infty)f$ is in $D(A_\infty)$ by definition, and we have to show that it coincides with u. By the first part of the proof, we know that it belongs to $C_b^1(\mathcal{O}) \cap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O})$, moreover it satisfies $\lambda w - \mathcal{A}w = f$. Therefore the difference $v : u - w$ belongs to $C_b^1(\mathcal{O}) \cap_{p\geq 1} W^{2,p}_{loc}(\mathcal{O})$ and satisfies $\lambda v - \mathcal{A}v = 0$. Multiplying by v and integrating over the open sets $\mathcal{O}_k$ given by Lemma 2.6, we get

$$\lambda \|v\|^2_{L^2(\mathcal{O}_k,\mu)} + \frac{1}{2}\| \, |Dv| \, \|^2_{L^2(\mathcal{O}_k,\mu)} = \frac{1}{2}\int_{\partial\mathcal{O}_k} \frac{\partial v}{\partial n} v e^{-2U} d\sigma.$$

The integral in the right hand side does not exceed $c_k \| \, |Dv| \, \|_\infty \|v\|_\infty$, where

$$c_k = \int_{\partial\mathcal{O}_k} e^{-2U} d\sigma.$$

goes to 0 as $k \to \infty$ because $e^{-2U(x)} \leq e^{-2(k-1)}$ for each $x \in \partial\mathcal{O}_k$, while the surface measure of $\partial\mathcal{O}_k$ is either bounded (if $\mathcal{O}$ is bounded) or grows at most polynomially with k, if $\mathcal{O}$ is unbounded. This is because $\mathcal{O}_k$ is a convex set contained in $\{x \in \mathcal{O} : U(x) \leq k+2\}$ which in its turn is contained in the ball $B(0, (k+2+b)/a)$ where $a > 0$, $b \in \mathbb{R}$ are such that $U(x) \geq a|x| - b$ for every $x \in \mathcal{O}$. Therefore, the surface measure of $\partial\mathcal{O}_k$ does not exceed the surface measure of the ball $B(0, (k+2+b)/a)$.

Letting $k \to \infty$ we obtain $\lambda \|v\|^2_{L^2(\mathcal{O},\mu)} = 0$ so that $v = 0$. □

2.4. **Moreau-Yosida approximations.** We extend U to the whole $\mathbb{R}^N$ setting $U(x) = +\infty$ for $x \in \mathbb{R}^N \setminus \mathcal{O}$. We introduce the Moreau-Yosida approximations of the extension (still denoted by U),

$$U_\alpha(x) = \inf\left\{U(y) + \frac{1}{2\alpha}|x-y|^2 : \; y \in \mathcal{O}\right\}, \;\; x \in \mathbb{R}^N, \; \alpha > 0. \tag{2.10}$$

As well known (see e.g. [4]) they enjoy the following properties:

(i) $U_\alpha(x) \leq U(x)$, $U_\alpha(x) \uparrow U(x)$ as $\alpha \to 0$, for each $x \in \mathbb{R}^N$;
(ii) $U_\alpha \in C^1(\mathbb{R}^N)$ and DU_α is Lipschitz continuous for each $\alpha > 0$;
(iii) $DU_\alpha(x) \to D_0U(x)$ and $|DU_\alpha(x)| \uparrow |D_0U(x)|$ as $\alpha \to 0$, for each $x \in \mathrm{dom}\,\partial U = \mathcal{O}$.

Here we denote by $D_0U(x)$ the element of minimal norm in $\partial U(x)$.

We consider the operators in the whole $\mathbb{R}^N$ defined by

$$A_\alpha : D(A_\alpha) := H^2(\mathbb{R}^N, \mu_\alpha) \mapsto L^2(\mathbb{R}^N, \mu_\alpha), \;\; A_\alpha = \frac{1}{2}\Delta - \langle DU_\alpha, D\cdot \rangle,$$

$$\mu_\alpha(dx) = \left(\int_{\mathbb{R}^N} e^{-2U_\alpha(y)} dy\right)^{-1} e^{-2U_\alpha(x)} dx.$$

Each of them is the (self-adjoint) infinitesimal generator of a contraction semigroup $T_\alpha(t)$ in $L^2(\mathbb{R}^N, \mu_\alpha)$, and we have

$$(T_\alpha(t)f)(x) = \mathbb{E}(f(X_\alpha(t,x)))$$

for each $f \in C_b(\mathbb{R}^N)$, where $X_\alpha(t,x)$ is the unique solution to

$$\left\{\begin{array}{l} dX(t,x) = -DU_\alpha(X(t,x))dt + dW(t), \quad t \geq 0, \\ \\ X(0,x) = x, \end{array}\right. \tag{2.11}$$

and $W(t)$ is a standard N-dimensional Brownian motion.

Moreover, we have ([2])

$$\| \, |DT_\alpha(t)f| \, \|_\infty \leq \frac{1}{\sqrt{t}} \|f\|_\infty, \quad f \in C_b(\mathbb{R}^N), \; t > 0, \tag{2.12}$$

and for $\lambda > 0$ we have ([7])

$$\left\{\begin{array}{ll} (i) & \|R(\lambda, A_\alpha)f\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leq \dfrac{1}{\lambda} \, \|f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}, \\ \\ (ii) & \| \, |DR(\lambda, A_\alpha)f| \, \|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leq \dfrac{2}{\sqrt{\lambda}} \, \|f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}, \\ \\ (iii) & \| \, |D^2R(\lambda, A_\alpha)f| \, \|_{L^2(\mathbb{R}^N,\mu_\alpha)} \leq 2\sqrt{2} \, \|f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}. \end{array}\right. \tag{2.13}$$

Estimates (2.13) yield estimates on $T_\alpha(t)$.

Corollary 2.13. *There is $C > 0$ such that*

$$\|tA_\alpha T_\alpha(t)\|_{L(L^2(\mathbb{R}^N,\mu_\alpha))} + \|T_\alpha(t)\|_{L(H^2(\mathbb{R}^N,\mu_\alpha))} \leq C, \quad \alpha > 0, \; t > 0. \tag{2.14}$$

Proof. For each self-adjoint dissipative operator A in a Hilbert space H, the resolvent set of A contains $\mathbb{C} \setminus (-\infty, 0]$ and we have

$$\|R(\lambda, A)\|_{L(H)} \leq \frac{1}{|\lambda| \cos(\theta/2)}, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

with $\theta = \arg \lambda$. Denoting by $S(t)$ the analytic semigroup generated by A and representing it by the usual Dunford integral, we get

$$\|tS(t)\|_{L(H,D(A))} + \|S(t)\|_{L(D(A))} \leq C_1, \quad t > 0, \tag{2.15}$$

with C_1 independent of A.

Let us take $H = L^2(\mathbb{R}^N, \mu_\alpha)$ and $A = A_\alpha$. By estimates (2.13), the graph norm of A_α is equivalent to the norm of $H^2(\mathbb{R}^N, \mu_\alpha)$, with equivalence constants independent of α. Therefore, (2.15) yields (2.14). $\square$

Theorem 2.14. *Let $f \in C_b(\overline{\mathcal{O}})$, and let $\widetilde{f}$ be any extension of f belonging to $C_b(\mathbb{R}^N)$. Then we have*

$$\lim_{\alpha \to 0} T_\alpha(t)\widetilde{f}(x) = T(t)f(x), \quad t > 0, \; x \in \overline{\mathcal{O}}.$$

Proof. The extension $\widetilde{f}$ belongs to $L^2(\mathbb{R}^N, \mu_\alpha)$ for each $\alpha > 0$. By estimates (2.14), $(t,x) \mapsto u_\alpha(t,x) := T_\alpha(t)\widetilde{f}(x)$ is bounded in $L^2((\varepsilon, +\infty); H^2(\mathbb{R}^N, \mu_\alpha)) \cap H^1((\varepsilon, +\infty); L^2(\mathbb{R}^N, \mu_\alpha))$ (because $d/dt \, u_\alpha = A_\alpha u_\alpha$) for each $\varepsilon > 0$, and by estimate (2.12) it is bounded in $C_b((\varepsilon, +\infty); C^1_b(\mathbb{R}^N))$ for each $\varepsilon > 0$.

Since $U_\alpha(x)$ goes to $U(x)$ monotonically as $\alpha \to 0$, then $e^{-2U_\alpha(x)}$ goes to $e^{-2U(x)}$ monotonically, and $(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx)^{-1}$ goes to $(\int_{\mathcal{O}} e^{-2U(x)}dx)^{-1}$. Therefore for each $\psi \in L^2(\mathbb{R}^N, \mu_{\alpha_0})$ for some α_0, the restriction $\psi_{|\mathcal{O}}$ belongs to $L^2(\mathcal{O}, \mu)$ and $\|\psi\|_{L^2(\mathcal{O},\mu_\alpha)}$ goes to $\|\psi\|_{L^2(\mathcal{O},\mu)}$ as $\alpha \to 0$.

It follows that the restrictions of u_α to $(\varepsilon, +\infty) \times \mathcal{O}$ are bounded in $L^2((\varepsilon, +\infty); H^2(\mathcal{O}, \mu)) \cap H^1((\varepsilon, +\infty); L^2(\mathcal{O}, \mu))$ by a constant independent of α.

A sequence (u_{α_n}) converges weakly in $L^2((\varepsilon, +\infty); H^2(\mathcal{O}, \mu)) \cap H^1((\varepsilon, +\infty); L^2(\mathcal{O}, \mu))$ to a function $v \in L^2((\varepsilon, +\infty); H^2(\mathcal{O}, \mu)) \cap H^1((\varepsilon, +\infty); L^2(\mathcal{O}, \mu))$, for each $\varepsilon > 0$. Moreover, for each $t > 0$ and for each compact set $K \subset \mathbb{R}^N$ the convergence is uniform on $K \cap \overline{\mathcal{O}}$. Since

$$\frac{d}{dt} \, u_{\alpha_n} - \frac{1}{2}\Delta_x u_{\alpha_n} + \langle DU_{\alpha_n}, D_x u_{\alpha_n} \rangle = 0,$$

for each $w \in C_c^\infty((0,+\infty)\times\mathcal{O})$ we have

$$\int_0^\infty \int_{\mathcal{O}} (d/dt\, u_{\alpha_n} - \frac{1}{2}\Delta_x u_{\alpha_n} + \langle DU_{\alpha_n}, D_x u_{\alpha_n}\rangle) w\, dt\, \mu(dx) = 0$$

and letting $n\to\infty$ (recalling that DU_{α_n} goes to DU in $L^2(K,dx)$ for each compact set $K\subset\mathcal{O}$), we get

$$\int_0^\infty \int_{\mathcal{O}} (u_t - \frac{1}{2}\Delta_x u + \langle DU, D_x u\rangle) w\, dt\, \mu(dx) = 0$$

and since w is arbitrary,

$$u_t(t,x) - \frac{1}{2}\Delta_x u(t,x) + \langle DU(x), D_x u(t,x)\rangle = 0, \quad t>0,\ x\in\mathcal{O}\, a.e.$$

If we prove that u is continuous in $[0,+\infty)$ with values in $L^2(\mathcal{O},\mu)$ and $u(0)=f$ we are done: indeed, in this case $u\in C([0,+\infty);L^2(\mathcal{O},\mu))\cap L^2((\varepsilon,+\infty);H^2(\mathcal{O},\mu))\cap H^1((\varepsilon,+\infty);L^2(\mathcal{O},\mu))$ for each $\varepsilon>0$, it satisfies $u_t - A_2 u = 0$, $u(0)=f$, and it follows that $u(t) = e^{tA_2}f$. By uniqueness we obtain that $T_\alpha(t)f(x)$ converges to $e^{tA_2}f(x) = T(t)f(x)$ (not only a sequence $T_{\alpha_n}(t)f(x)$).

To prove that u is continuous, the first step is boundedness. Set

$$M_\alpha := \left(\int_{\mathbb{R}^N} e^{-2U_\alpha(x)}dx\right)^{1/2},\ \alpha>0, \quad M_0 := \left(\int_{\mathbb{R}^N} e^{-2U(x)}dx\right)^{1/2}.$$

Since M_α is increasing, then

$$M_0 \le M_\alpha \le M_1, \quad 0<\alpha<1.$$

Since $H^1((\varepsilon,+\infty);L^2(\mathcal{O},\mu))$ is continuously embedded in $C_b((\varepsilon,+\infty);L^2(\mathcal{O},\mu))$, then for each $t>0$ the sequence $u_{\alpha_n}(t)$ converges weakly to $u(t)$ in $L^2(\mathcal{O},\mu)$. It follows that for each $t>0$

$$\begin{aligned} \|u(t)\|_{L^2(\mathcal{O},\mu)} &\le \limsup_{n\to\infty}\|u_{\alpha_n}(t)\|_{L^2(\mathcal{O},\mu)} \le \limsup_{n\to\infty}\frac{M_{\alpha_n}}{M_0}\|u_{\alpha_n}(t)\|_{L^2(\mathcal{O},\mu_{\alpha_n})} \\ &\le \limsup_{n\to\infty}\frac{M_{\alpha_n}}{M_0}\|f\|_{L^2(\mathcal{O},\mu_{\alpha_n})} = \|f\|_{L^2(\mathcal{O},\mu)}. \end{aligned} \tag{2.16}$$

As a second step, we prove that the functions u_α, $\alpha\in(0,1]$, are equi-uniformly continuous from $[a,+\infty)$ to $L^2(\mathcal{O},\mu)$ for each $a>0$.

For $t>s>0$ we have

$$\begin{aligned} &\|u_{\alpha_n}(t)-u_{\alpha_n}(s)\|_{L^2(\mathcal{O},\mu)} \le \frac{M_{\alpha_n}}{M_0}\|u_{\alpha_n}(t)-u_{\alpha_n}(s)\|_{L^2(\mathcal{O},\mu_{\alpha_n})} \\ &\le \frac{M_{\alpha_n}}{M_0}\int_s^t \|A_{\alpha_n}T_{\alpha_n}(r)f\|_{L^2(\mathbb{R}^N,\mu_{\alpha_n})}dr \le \frac{CM_1|t-s|}{M_0 s}\|f\|_{L^2(\mathcal{O},\mu_{\alpha_n})}, \end{aligned} \tag{2.17}$$

where C is the constant in (2.14).

Equi-continuity up to $t=0$ is a bit more delicate; in fact we prove a weaker estimate that however implies that u is continuous. Precisely, we prove that for each $\varepsilon>0$ there are $\alpha_0>0$, $t_0>0$ such that

$$\|u_\alpha(t)-f\|_{L^2(\mathcal{O},\mu)} \le \varepsilon, \quad 0<\alpha\le\alpha_0,\ 0<t\le t_0. \tag{2.18}$$

We prove (2.18) in two steps: first, for $f\in C_c^\infty(\mathcal{O})$, and then for any $f\in C_b(\overline{\mathcal{O}})$.

If $f \in C_c^\infty(\mathcal{O})$, equi-continuity at $s = 0$ can be proved as at any s. Indeed, since $f \in D(A_\alpha)$ for each $\alpha \in (0,1]$, for $t > 0$ we have

$$\|u_\alpha(t) - f\|_{L^2(\mathcal{O},\mu)} \le \frac{M_\alpha}{M_0}\|u_\alpha(t) - f\|_{L^2(\mathcal{O},\mu_\alpha)}$$

$$\le \frac{M_\alpha}{M_0}\int_0^t \|T_\alpha(r)A_\alpha f\|_{L^2(\mathbb{R}^N,\mu_\alpha)}dr \le t\,\frac{M_\alpha}{M_0}\|A_\alpha f\|_{L^2(\mathcal{O},\mu_\alpha)} \tag{2.19}$$

$$\le t\,\frac{M_1}{M_0}\left(\frac{1}{2}\|\Delta f\|_\infty + \|\,|DU|\,\|_{L^\infty(\mathrm{supp}\,f)}\|\,|Df|\,\|_\infty\right)|\mathrm{supp}\,f|^{1/2},$$

where $|\cdot|$ denotes the Lebesgue measure.

If f is just continuous and bounded, it belongs to $L^2(\mathcal{O},\mu_\alpha)$ for each α. Taking $\alpha = 1$, there is a sequence $(f_k)_{k\in\mathbb{N}}$ of smooth compactly supported functions that converge to f in $L^2(\mathcal{O},\mu_1)$ and hence in $L^2(\mathcal{O},\mu_\alpha)$ for every $\alpha \in (0,1)$, and also in $L^2(\mathcal{O},\mu)$. We have, for $0 < \alpha \le 1$,

$$\|u_\alpha(t)-f\|_{L^2(\mathcal{O},\mu)} \le \|T_\alpha(t)(\widetilde{f}-f_k)\|_{L^2(\mathcal{O},\mu)}+\|T_\alpha(t)f_k-f_k\|_{L^2(\mathcal{O},\mu)}+\|f_k-f\|_{L^2(\mathcal{O},\mu)}, \tag{2.20}$$

where

$$\|T_\alpha(t)(\widetilde{f} - f_k)\|_{L^2(\mathcal{O},\mu)} \le \frac{M_\alpha}{M_0}\,\|T_\alpha(t)(\widetilde{f} - f_k)\|_{L^2(\mathcal{O},\mu_\alpha)} \le \frac{M_\alpha}{M_0}\,\|T_\alpha(t)(\widetilde{f} - f_k)\|_{L^2(\mathbb{R}^N,\mu_\alpha)}$$

$$\le \frac{M_\alpha}{M_0}\,\|\widetilde{f} - f_k\|_{L^2(\mathbb{R}^N,\mu_\alpha)} \le \frac{M_\alpha}{M_0}\,(\|\widetilde{f}\|_{L^2(\mathbb{R}^N\setminus\mathcal{O},\mu_\alpha)} + \|\widetilde{f} - f_k\|_{L^2(\mathcal{O},\mu_\alpha)})$$

$$\le \frac{M_1}{M_0}\,(\|\widetilde{f}\|_\infty\,(\mu_\alpha(\mathbb{R}^N\setminus\mathcal{O}))^{1/2} + \|f - f_k\|_{L^2(\mathcal{O},\mu_1)}).$$

Similarly, $\|f_k - f\|_{L^2(\mathcal{O},\mu)} \le M_1/M_0\|f - f_k\|_{L^2(\mathcal{O},\mu_1)}$.

Given any $\varepsilon > 0$, let us fix k large enough such that

$$2\frac{M_1}{M_0}\,\|f - f_k\|_{L^2(\mathcal{O},\mu_1)} \le \frac{\varepsilon}{3}.$$

Letting α_0 be so small that

$$\frac{M_1}{M_0}\,\|\widetilde{f}\|_\infty\,(\mu_\alpha(\mathbb{R}^N\setminus\mathcal{O}))^{1/2} \le \frac{\varepsilon}{3}, \quad 0 < \alpha \le \alpha_0,$$

and using estimate (2.19) for f_k, we obtain (2.18). It follows that for each $\varepsilon > 0$ there is $t_0 > 0$ such that

$$\|u_\alpha(t) - u_\alpha(s)\|_{L^2(\mathcal{O},\mu)} \le 2\varepsilon, \quad 0 < \alpha \le \alpha_0,\ 0 < t,\ s \le t_0. \tag{2.21}$$

Together with (2.17), this estimate implies that u is uniformly continuous with values in $L^2(\mathcal{O},\mu)$; indeed for $t \ge s \ge 0$ we have

$$\|u(t) - u(s)\|^2_{L^2(\mathcal{O},\mu)} = \lim_{n\to\infty}\langle u_{\alpha_n}(t) - u_{\alpha_n}(s), u(t) - u(s)\rangle_{L^2(\mathcal{O},\mu)}$$

$$\le \limsup_{n\to\infty}\|u_{\alpha_n}(t) - u_{\alpha_n}(s)\|_{L^2(\mathcal{O},\mu)}\,2\|f\|_{L^2(\mathcal{O},\mu)},$$

by (2.16), and using (2.17) and (2.21) we are done. □

3. A stochastic differential inclusion

In this section we consider the stochastic differential inclusion

$$\begin{cases} dX(t) + \partial U(X(t))dt \ni dW(t), \\[2mm] X(0) = x \in \mathcal{O}, \end{cases} \tag{3.1}$$

where $W(t)$ is an $\mathbb{R}^N$-valued Brownian motion in a probability space $(\Omega,\mathcal{F},\mathbb{P})$, with continuous trajectories.

Definition 3.1. *Let $T>0$. A pair (X,η) is called a solution of* (3.1) *in* $[0,T]$ *if*

(i) $X:[0,T]\times\Omega\to\mathbb{R}^N$ *is a continuous (that is such that $X(\cdot,\omega)\in C([0,T];\mathbb{R}^N)$ for almost all $\omega\in\Omega$) adapted process and $X(s,\omega)\in\overline{\mathcal{O}}$, $\mathbb{P}$-a.s. for each $s\in[0,T]$;*

(ii) $\eta:[0,T]\times\Omega\to\mathbb{R}^N$ *is an adapted process such that $\eta(\cdot,\omega)\in BV([0,T];\mathbb{R}^N)$ for almost all $\omega\in\Omega$ and*

$$X(t)+\eta(t)=x+W(t),\quad t\in[0,T].$$

(iii) *For each $v\in C([0,T];\mathcal{O})$ we have*

$$\int_0^T\langle d\eta(s)-D_0U(v(s))ds,X(s)-v(s)\rangle\geq 0,\tag{3.2}$$

where $d\eta$ is the measure defined by the Stieltjes integral

$$d\eta([a,b])=\eta(b)-\eta(a),\quad 0\leq a\leq b\leq T.$$

Any solution enjoys further properties, stated in the next proposition.

Proposition 3.2. *Let (X,η) be a solution of* (3.1) *in* $[0,T]$. *Then for each $v\in C([0,T];\mathcal{O})$ we have*

$$\int_0^T\langle d\eta(s),X(s)-v(s)\rangle\geq\int_0^T(U(X(s))-U(v(s)))ds,\quad \mathbb{P}-a.s.\tag{3.3}$$

It follows that $s\mapsto U(X(s))$ belongs to $L^1(0,T)$ and $X(s)\in\mathcal{O}$ for almost all $s\in(0,T)$, $\mathbb{P}$-a.s.

Proof. Fix ω such that $X(\cdot,\omega)$ is continuous. For $0<\varepsilon<1$ set

$$v_\varepsilon(s)=\varepsilon v(s)+(1-\varepsilon)X(s),\quad s\in[0,T].$$

Then $v_\varepsilon(s)\in\mathcal{O}$ for each $s\in[0,T]$. Using (3.2) with $v=v_\varepsilon$ we get

$$\int_0^T\langle d\eta(s),X(s)-v_\varepsilon(s)\rangle\geq\int_0^T\langle D_0U(v_\varepsilon(s)),X(s)-v_\varepsilon(s)\rangle ds.$$

Since $X-v_\varepsilon=\varepsilon(X-v)$, we obtain

$$\int_0^T\langle d\eta(s),X(s)-v(s)\rangle\geq\int_0^T\langle D_0U(v_\varepsilon(s)),X(s)-v(s)\rangle ds.$$

Since $\langle D_0U(v_\varepsilon(s)),v_\varepsilon(s)-y\rangle\geq U(v_\varepsilon(s))-U(y)$ for each $y\in\mathbb{R}^N$, then

$$\begin{aligned}&\langle D_0U(v_\varepsilon(s)),X(s)-v(s))\rangle=\langle D_0U(v_\varepsilon(s)),v_\varepsilon(s)-(v_\varepsilon(s)+v(s)-X(s))\rangle\\&\geq U(v_\varepsilon(s))-U(v_\varepsilon(s)+v(s)-X(s))\\&=U(v_\varepsilon(s))-U[v(s)+\varepsilon(v(s)-X(s))].\end{aligned}$$

Consequently,

$$\int_0^T\langle d\eta(s),X(s)-v(s)\rangle\geq\int_0^T\{U(v_\varepsilon(s))-U[v(s)+\varepsilon(v(s)-X(s))]\}ds,$$

so that, letting $\varepsilon\to 0$,

$$\begin{aligned}&\int_0^T\langle d\eta(s),X(s)-v(s)\rangle\geq\liminf_{\varepsilon\to 0}\int_0^T\{U(v_\varepsilon(s))-U[v(s)+\varepsilon(v(s)-X(s))]\}ds\\&\geq\int_0^T(U(X(s))-U(v(s)))ds.\end{aligned}$$

□

We shall solve problem (3.1) by approximation, considering

$$\left\{\begin{array}{l} dX_\alpha(t) + DU_\alpha(X_\alpha(t))dt = dW(t), \\ \\ X_\alpha(0) = x \in \mathbb{R}^N, \end{array}\right. \tag{3.4}$$

whose solution we denote by $X_\alpha(t)$. Here U_α are the Moreau-Yosida approximations of U defined in (2.10). We recall that U_α is differentiable in $\mathbb{R}^N$ with Lipschitz continuous gradient.

We shall find some *a priori* estimates on the solution X_α to (3.4) which will allow us to find a solution of (3.1) letting $\alpha \to 0$. First we need a lemma.

Lemma 3.3. *There exist $\rho > 0$, $k > 0$ and $x_0 \in \mathcal{O}$ such that*

$$\langle DU_\alpha(x), x\rangle \geq \rho|DU_\alpha(x)| + \langle DU_\alpha(x), x_0\rangle - k(1 + |x|), \quad x \in \mathbb{R}^N. \tag{3.5}$$

Proof. Let $x_0 \in \mathcal{O}$ and $\rho > 0$ be such that the closed ball centered at x_0 with radius ρ is contained in $\mathcal{O}$. By the monotonicity of DU_α it follows that for each $z \in \mathbb{R}^N$ with $|z| = 1$ we have

$$\langle DU_\alpha(x) - DU_\alpha(x_0 + \rho z), x - (x_0 + \rho z)\rangle \geq 0, \quad x \in \mathbb{R}^N.$$

Consequently

$$\begin{aligned} \langle DU_\alpha(x), x\rangle \quad &\geq \quad \rho\langle DU_\alpha(x), z\rangle + \langle DU_\alpha(x), x_0\rangle + \langle DU_\alpha(x_0 + \rho z), x - (x_0 + \rho z)\rangle \\ &\geq \quad \rho\langle DU_\alpha(x), z\rangle + \langle DU_\alpha(x), x_0\rangle - k(1 + |x|), \end{aligned}$$

for a suitable k, since DU is bounded in the ball $B(x_0, \rho)$ and $|DU_\alpha(x)| \leq |D_0U(x)|$.

The conclusion follows choosing

$$z = \frac{DU_\alpha(x)}{|DU_\alpha(x)|}.$$

□

Theorem 3.4. *For any $\alpha > 0$ problem (3.4) has a unique continuous solution X_α. For each $x \in \mathcal{O}$ there exist the limits*

$$\lim_{\alpha\to 0} X_\alpha(t) =: X(t) \quad \text{in} \quad C([0, T]; \mathbb{R}^N), \quad \mathbb{P}\text{-a.s.}, \tag{3.6}$$

and

$$\lim_{\alpha\to 0} \int_0^t DU_\alpha(X_\alpha(s))ds =: \eta(t) \in \partial U(X(t)), \quad \forall\, t \in [0, T]. \tag{3.7}$$

Moreover, X is the unique solution to problem (3.1).

Proof. We set $Y_\alpha(t) = X_\alpha(t) - W(t)$. Then equation (3.4) is equivalent to the ordinary differential equation with random coefficients

$$\left\{\begin{array}{l} \dfrac{d}{dt}\, Y_\alpha(t) + DU_\alpha(Y_\alpha(t) + W(t)) = 0, \\ \\ Y_\alpha(0) = x, \end{array}\right. \tag{3.8}$$

which has a unique C^1 solution Y_α for $\mathbb{P}$-a.e. ω, because DU_α is Lipschitz continuous. We need three estimates.

First estimate. There exists $C = C(\omega, x)$ such that

$$|Y_\alpha(t)|^2 + \int_0^t |DU_\alpha(X_\alpha(s))|ds \leq C. \tag{3.9}$$

Multiplying scalarly both sides of equation in (3.8) by $Y_\alpha(t)$ yields

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)|^2 + \langle DU_\alpha(X_\alpha(t)), X_\alpha(t)\rangle = \langle DU_\alpha(X_\alpha(t)), W(t)\rangle. \tag{3.10}$$

By (3.10) and (3.5) it follows that

$$\frac{1}{2}\frac{d}{dt}|Y_\alpha(t)|^2+\rho|DU_\alpha(X_\alpha(t))|\le\langle DU_\alpha(X_\alpha(t)),W(t)-x_0\rangle+k(1+|X_\alpha(t)|).$$

Integrating with respect to t we get

(3.11)

$$\begin{aligned}&|Y_\alpha(t)|^2+2\rho\int_0^t|DU_\alpha(X_\alpha(s))|ds\\&\le|x|^2+2\int_0^t\langle DU_\alpha(X_\alpha(s)),W(s)-x_0\rangle ds+2k\int_0^t(1+|X_\alpha(s)|)ds\\&\le|x|^2+2\int_0^t\langle DU_\alpha(X_\alpha(s)),W(s)-x_0\rangle ds+2k\int_0^t|Y_\alpha(s)|ds+2kT(1+\|W\|_\infty).\end{aligned}$$

Now we estimate the first integral in the right hand side of (3.11).

We fix $\omega\in\Omega$ and consider $\delta=\delta(\omega)>0$ such that

$$s,t\in[0,T],\ |t-s|\le\delta\Rightarrow|W(t)-W(s)|\le\frac{1}{4}\rho.$$

Then we take a decomposition $\{0=t_0<t_1<t_n=t\}$ of $[0,t]$ with $t\le T$ such that $\max_{1\le i\le n}|t_i-t_{i-1}|\le\delta$. We have

$$\begin{aligned}&\left|\int_0^t(DU_\alpha(X_\alpha(s)),W(s)-x_0)ds\right|\le\sum_{k=1}^n\left|\int_{t_{k-1}}^{t_k}(DU_\alpha(X_\alpha(s)),W(s)-W(t_{k-1}))ds\right|\\&+\sum_{k=1}^n\left|\int_{t_{k-1}}^{t_k}(DU_\alpha(X_\alpha(s)),W(t_{k-1})-x_0)ds\right|\\&\le\frac{1}{4}\rho\int_0^t|DU_\alpha(X_\alpha(s))|ds+\sum_{k=1}^n\left|\int_{t_{k-1}}^{t_k}(Y_\alpha'(s),W(t_{k-1})-x_0)ds\right|\\&\le\frac{1}{4}\rho\int_0^t|DU_\alpha(X_\alpha(s))|ds+2nC\sup_{s\in[0,t]}|Y_\alpha(s)|,\end{aligned}$$

with $C=\sup_{s\in[0,T]}|W(s)-x_0|$. Using this estimate in (3.11) gives

$$|Y_\alpha(t)|^2+\frac{3}{2}\rho\int_0^t|DU_\alpha(X_\alpha(s))|ds\le|x|^2+(4nC+2kT)\sup_{s\in[0,t]}|Y_\alpha(s)|+2kT(1+\|W\|_\infty).$$

Therefore,

$$\sup_{s\in[0,T]}|Y_\alpha(s)|^2\le|x|^2+(4nC+2kT)\sup_{s\in[0,T]}|Y_\alpha(s)|+2kT(1+\|W\|_\infty)$$

and (3.9) follows.

Second estimate. There is $C=C(\omega,x)$ such that for any $h\in[0,T]$ we have

$$|Y_\alpha(h)-x|^2\le C\Big(h+\sup_{s\in[0,h]}|W(s)|\Big).\tag{3.12}$$

Multiplying scalarly by $(Y_\alpha(t)-x)$ both sides of the equality

$$\frac{d}{dt}(Y_\alpha(t)-x)+DU_\alpha(X_\alpha(t))=0$$

yields

$$\frac{1}{2}\frac{d}{dt}|Y_\alpha(t)-x|^2+\langle DU_\alpha(X_\alpha(t)),Y_\alpha(t)-x\rangle=0,$$

which is equivalent to

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)-x|^2+\langle DU_\alpha(X_\alpha(t))-DU_\alpha(x),X_\alpha(t)-x\rangle$$

$$=\langle DU_\alpha(X_\alpha(t)),W(t)\rangle-\langle DU_\alpha(x),X_\alpha(t)-x\rangle.$$

Taking into account the monotonicity of DU_α we get

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)-x|^2\le\langle DU_\alpha(X_\alpha(t)),W(t)\rangle+|DU(x)|\,|X_\alpha(t)-x|.$$

By (3.9) the functions X_α are bounded in $[0,T]$ by a constant independent of α. Hence, there is $C>0$ such that

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t)-x|^2\le C+\sup_{s\in[0,t]}|W(s)|\,|DU_\alpha(X_\alpha(t))|.$$

Integrating with respect to t and using again (3.9) we arrive at (3.12).

Third estimate. There is $C=C(\omega,x)$ such that for any $t\in[0,T]$ and $h\in[0,T-t]$ we have

$$|Y_\alpha(t+h)-Y_\alpha(t)|^2\le C\Big(h+\sup_{s\in[0,h]}|W(s)|\Big)+C\sup_{s\in[0,t]}|W(s+h)-W(s)|. \tag{3.13}$$

Write for $h>0$

$$\frac{d}{dt}\,(Y_\alpha(t+h)-Y_\alpha(t))+DU_\alpha(X_\alpha(t+h))-DU_\alpha(X_\alpha(t))=0.$$

Multiplying scalarly both sides by $Y_\alpha(t+h)-Y_\alpha(t)$ yields

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t+h)-Y_\alpha(t)|^2+\langle DU_\alpha(X_\alpha(t+h))-DU_\alpha(X_\alpha(t)),Y_\alpha(t+h)-Y_\alpha(t)\rangle=0,$$

which can be rewritten as

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t+h)-Y_\alpha(t)|^2+\langle DU_\alpha(X_\alpha(t+h))-DU_\alpha(X_\alpha(t)),X_\alpha(t+h)-X_\alpha(t)\rangle$$

$$=\langle DU_\alpha(X_\alpha(t+h))-DU_\alpha(X_\alpha(t)),W(t+h)-W(t)\rangle.$$

By the monotonicity of DU_α we get

$$\frac{1}{2}\,\frac{d}{dt}\,|Y_\alpha(t+h)-Y_\alpha(t)|^2\le\langle DU_\alpha(X_\alpha(t+h))-DU_\alpha(X_\alpha(t)),W(t+h)-W(t)\rangle$$

$$\le(|DU_\alpha(X_\alpha(t+h))|+|DU_\alpha(X_\alpha(t))|)|W(t+h)-W(t)|.$$

Integrating with respect to t yields

$$\frac{1}{2}\,|Y_\alpha(t+h)-Y_\alpha(t)|^2\le\frac{1}{2}\,|Y_\alpha(h)-x|^2$$

$$+\int_0^t(|DU_\alpha(X_\alpha(s+h))|+|DU_\alpha(X_\alpha(s))|)ds\,\sup_{s\in[0,t]}|W(s+h)-W(s)|.$$

(3.13) follows now from (3.9) and (3.12).

Limit as $\alpha\to0$. Since W is continuous in $[0,T]$, by (3.13) it follows that $\{Y_\alpha\}$ is equicontinuous. By the Arzelà-Ascoli Theorem there is a sequence (α_n) such that

$$Y_{\alpha_n}(t)\to Y(t),\quad \text{uniformly on } [0,T]$$

and so

$$X_{\alpha_n}(t)\to X(t),\quad \text{uniformly on } [0,T].$$

On the other hand by (3.9) the sequence (η_{α_n}) defined by

$$\eta_{\alpha_n}(t)=\int_0^t DU_{\alpha_n}(X_{\alpha_n}(s))ds,\quad t\in[0,T],$$

satisfies the estimate

$$\int_0^t |\eta'_{\alpha_n}(s)|ds \le C, \quad t\in[0,T]. \tag{3.14}$$

It follows that (η_{α_n}) has uniformly bounded variation in $[0,T]$, i.e.,

$$|\eta_{\alpha_n}|_{BV([0,T];\mathbb{R}^N)} \le C, \quad n\in\mathbb{N}.$$

Then by Helly's Theorem (see e.g. [10, Thm.4 p.370, Thm.5 p.372]) there is a subsequence, again denoted by (η_{α_n}), such that

$$\eta_{\alpha_n}(t) \to \eta(t), \quad \forall\, t\in[0,T], \tag{3.15}$$

and

$$\int_0^T \langle d\eta_{\alpha_n}(t), z(t)\rangle \to \int_0^T \langle d\eta(t), z(t)\rangle, \quad \forall\, z\in C([0,T];\mathbb{R}^N), \tag{3.16}$$

where $\int_0^T \langle d\eta_{\alpha_n}(t), z(t)\rangle$ and $\int_0^T \langle d\eta(t), z(t)\rangle$ are the corresponding Stieltjes integrals.

Letting $n\to\infty$ in the identity

$$X_{\alpha_n}(t)+\eta_{\alpha_n}(t) = x + W(t),$$

we obtain that $(X(t),\eta(t))$ satisfies

$$X(t)+\eta(t) = x+W(t), \quad \forall\, t\in[0,T]. \tag{3.17}$$

Moreover, taking into account that

$$\int_0^T \langle DU_{\alpha_n}(X_{\alpha_n}(t)) - DU_{\alpha_n}(z(t)), X_\alpha(t)-z(t)\rangle dt \ge 0, \quad \forall\, z\in C([0,T];\mathcal{O}),$$

we obtain by (3.16) that $(X(t),\eta(t))$ satisfies (3.2).

Proof that $X(t)\in\overline{\mathcal{O}}$, $\mathbb{P}$-a.e. Set

$$J_\alpha(y) = (1+\alpha\partial U)^{-1}(y), \quad \alpha>0,\ y\in\mathbb{R}^N.$$

Then $J_\alpha(y)\in \operatorname{dom}\partial U = \mathcal{O}$, and

$$|J_\alpha(y)-y| \le \alpha |DU_\alpha(y)|, \quad \forall\, y\in\mathbb{R}^N. \tag{3.18}$$

Therefore, $\int_0^T |X_\alpha(s) - J_\alpha(X_\alpha(s))|ds \le \alpha\int_0^T |DU_\alpha(X_\alpha(s))|ds$, and by estimate (3.9) it follows that

$$\int_0^T |X_\alpha(s)-J_\alpha(X_\alpha(s))|ds \le C\alpha, \tag{3.19}$$

which implies that

$$\lim_{\alpha\to 0} X_\alpha(s) - J_\alpha(X_\alpha(s)) = 0, \quad \text{a.e.}$$

Since $X_\alpha(s)\to X(s)$ for each s and $J_\alpha(X_\alpha(s))\in\mathcal{O}$, then $X(s)\in\overline{\mathcal{O}}$ for almost all $s\in[0,T]$. Since X is continuous, then $X(s)\in\overline{\mathcal{O}}$ for all $s\in[0,T]$.

Uniqueness. Assume that there are two solutions (X_1,η_1), (X_2,η_2). Fix any $x_0\in\mathcal{O}$ and set, for $0<\varepsilon<1$,

$$X_1^\varepsilon(s) = (1-\varepsilon)X_1(s)+\varepsilon x_0,\ X_2^\varepsilon(s) = (1-\varepsilon)X_2(s)+\varepsilon x_0, \quad 0\le s\le T.$$

Then X_1^ε and X_2^ε have values in $\mathcal{O}$, and (3.3) implies

$$\int_0^T \langle d\eta_1(s), X_1(s)-X_2^\varepsilon(s)\rangle \ge \int_0^T [U(X_1(s)-U(X_2^\varepsilon(s))]ds,$$

$$\int_0^T \langle d\eta_2(s), X_2(s)-X_1^\varepsilon(s)\rangle \ge \int_0^T [U(X_2(s)-U(X_1^\varepsilon(s))]ds.$$

Summing up,

$$\begin{aligned}
&\int_0^T \langle d\eta_1(s) - d\eta_2(s), X_1(s) - X_2(s)\rangle \\
&+\varepsilon \int_0^T \langle d\eta_1(s), X_2(s) - x_0\rangle + \varepsilon \int_0^T \langle d\eta_2(s), X_1(s) - x_0\rangle \\
\geq\ & \int_0^T [U(X_1(s)) - U(X_1^\varepsilon(s))]ds + \int_0^T [U(X_2(s)) - U(X_2^\varepsilon(s))]ds.
\end{aligned}$$

Letting $\varepsilon \to 0$, both integrals in the right hand side go to 0: indeed, for almost all s $X_i(s) \in \mathcal{O}$ so that $U(X_i^\varepsilon(s)) \to U(X_i(s))$, moreover $U(X_i^\varepsilon(s)) \leq (1-\varepsilon)U(X_1(s)) + \varepsilon U(x_0)$, so that

$$\varepsilon(U(X_i(s)) - U(x_0)) \leq U(X_i(s)) - U(X_i^\varepsilon(s)) \leq U(X_i(s)) - \min U,$$

and $U(X_i(\cdot)) \in L^1(0,T)$ by Proposition 3.2, for $i = 1, 2$. Therefore, recalling that $X_1 - X_2 = -(\eta_1 - \eta_2)$,

$$\int_0^T \langle d\eta_1(s) - d\eta_2(s), X_1(s) - X_2(s)\rangle = -\int_0^T \langle d\eta_1(s) - d\eta_2(s), \eta_1(s) - \eta_2(s)\rangle \geq 0.$$

It follows that

$$\int_0^T d(|\eta_1(t) - \eta_2(t)|^2) \leq 0$$

and since $\eta_1 - \eta_2$ is continuous, this implies $\eta_1 = \eta_2$ and $X_1 = X_2$.

So, we have proved the statement, with convergence in (3.6) and in (3.7) for a sequence (α_n). But uniqueness implies that (X_α, η_α) converges. □

Proposition 3.5. *Let (X, η) be the solution of* (3.1). *If $U \in C^1(\mathcal{O})$, then $d\eta(t) = DU(X(t))dt$ for all $t \in [0,T]$ such that $X(t) \in \mathcal{O}$.*

Proof. By Proposition 3.2 we know that $X(t) \in \mathcal{O}$ for almost all $t \in [0,T]$. Let $t_0 \in [0,T]$ be such that $X(t_0) \in \mathcal{O}$. Then there exist $0 \leq a < b \leq T$ such that $a \leq t_0 \leq b$ and

$$X(s) \in \mathcal{O}, \quad \forall\, s \in [a,b].$$

Next we choose v of the form

$$v(s) = X(s) \pm \varepsilon\phi, \quad \phi \in C_c^\infty(a,b),$$

with ε small enough such that $v(s) \in \mathcal{O}$ for each $s \in [a,b]$. We substitute in (3.3) and we get

$$\mp \int_a^b \langle d\eta(s), \phi(s)\rangle \geq \mp \int_a^b \langle DU(X(s) \pm \varepsilon\phi(s)), \phi(s)\rangle ds.$$

Letting $\varepsilon \to 0$ we get

$$\mp \int_a^b \langle d\eta(s), \phi(s)\rangle \geq \mp \int_a^b \langle DU(X(s)), \phi(s)\rangle ds.$$

Hence,

$$d\eta(s) = DU(X(s))ds \quad \forall\, s \in [a,b].$$

□

3.1. **Identification of $T(t)$ with the transition semigroup.** Let $f \in C_b(\overline{\mathcal{O}})$ and let $\widetilde{f}$ be any extension of f belonging to $C_b(\mathbb{R}^N)$.

Define the transition semigroup of (3.1) by

$$P_t f(x) = \mathbb{E}(f(X(t))), \quad t > 0,\ f \in C_b(\overline{\mathcal{O}}),\ x \in \mathcal{O},$$

where X is the first component of the solution (X, η) of (3.1). By (3.6), $f(X(t)) = \widetilde{f}(X(t)) = \lim_{\alpha \to 0} \widetilde{f}(X_\alpha(t))$, $\mathbb{P}$-a.s. Hence,

$$P_t f(x) = \lim_{\alpha \to 0} \mathbb{E}(\widetilde{f}(X_\alpha(t))), \quad t > 0,\ x \in \mathcal{O}.$$

On the other hand, $\mathbb{E}(\widetilde{f}(X_\alpha(t))) = T_\alpha(t)\widetilde{f}(x)$. By Theorem 2.14, for each $t > 0$ $T_\alpha(t)\widetilde{f}(x)$ converges to $T(t)f(x)$ as $\alpha \to 0$. Therefore, $T(t)f(x) = P_t f(x)$ for each $x \in \mathcal{O}$.

4. Acknowledgements

We thank Viorel Barbu for useful discussions about stochastic variational inequalities.

References

[1] V. Barbu, G. Da Prato: *The generator of the transition semigroup corresponding to a stochastic variational inequality*, Preprint SNS, n.7 (2007).
[2] M. Bertoldi, S. Fornaro: *Gradient estimates in parabolic problems with unbounded coefficients*, Studia Math. **165** (2004), 221–254.
[3] M. Bertoldi, L. Lorenzi: *Analytical methods for Markov semigroups*, Chapman & Hall – CRC Press (2006).
[4] H. Brézis: *Opérateurs maximaux monotones*, North-Holland, Amsterdam (1973).
[5] E. Cépa: *Problème de Skorohod multivoque*, Ann. Probab. 26 (1998), 500–532.
[6] G. Da Prato, A. Lunardi: *On a class of elliptic operators with unbounded coefficients in convex domains*, Rend. Acc. Naz. Lincei Ser. IX **15** (2004), 315–326.
[7] G. Da Prato, A. Lunardi: *Elliptic operators with unbounded drift coefficients and Neumann boundary condition*, J. Diff. Eqns. **198** (2004), 35–52.
[8] G. Da Prato, A. Lunardi: *On a class of self-adjoint elliptic operators in L^2 spaces with respect to invariant measures*, J. Diff. Eqns. **234** (2007), 54–79.
[9] E.B. Davies: *Heat Kernels and Spectral Theory*, Cambridge Univ. Press (1989).
[10] A.N. Kolmogorov, S.V. Fomin: *Introductory real analysis.* Dover, New York (1970).
[11] T. Liggett: *Interacting particle systems.* Grundlehren der Mathematischen Wissenschaften, 276. Springer-Verlag, New York (1985).
[12] W. Rudin: *Real and Complex Analysis*, McGraw-Hill (1974).
[13] H.B. Stewart: *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions*, Trans. Amer. Math. Soc. **259** (1980), 299–310.

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