Kolmogorov operators of Hamiltonian systems perturbed by noise

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Abstract. We consider a second order elliptic operator $\mathcal{K}$ arising from Hamiltonian systems with friction in $\mathbb{R}^{2n}$ perturbed by noise. An invariant measure for this operator is $\mu(dx,dy) = \exp(-2H(x,y))dx\,dy$, where H is the Hamiltonian. We study the realization $K : H^2(\mathbb{R}^{2n},\mu) \mapsto L^2(\mathbb{R}^{2n},\mu)$ of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n},\mu)$, proving that it is m-dissipative and that it generates an analytic semigroup.

1. Introduction

We consider a Hamiltonian system perturbed by noise,

$$\begin{cases} dX(t) = D_y H(X(t),Y(t))dt + \sqrt{\alpha}\, dW_1(t), \\ \\ dY(t) = -D_x H(X(t),Y(t))dt + \sqrt{\beta}\, dW_2(t), \end{cases} \tag{1.1}$$

where $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is the Hamiltonian which we assume to be regular, nonnegative but not necessarily Lipschitz continuous, W_1, W_2 are independent n–dimensional Brownian motions and α, β are positive constants. The Kolmogorov operator corresponding to (1.1) is

$$\mathcal{K}\varphi = \frac{\alpha}{2}\,\Delta_x\varphi + \frac{\beta}{2}\,\Delta_y\varphi + \langle D_y H, D_x\varphi\rangle - \langle D_x H, D_y\varphi\rangle, \tag{1.2}$$

where $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$, and Δ_x, Δ_y, D_x, D_y are the Laplacian and the gradient with respect to the variables x, y only. It is easy to see that

$$\int_{\mathbb{R}^{2n}} \mathcal{K}\varphi\, dx\, dy = 0$$

for each test function φ, which means that the Lebesgue measure in $\mathbb{R}^{2n}$ is invariant for $\mathcal{K}$. It is well known that, in order that an invariant probability measure exists, some *friction* term must be added, see e.g. [5]. In this paper we want to consider the case that the probability measure has density (with respect to the Lebesgue measure) proportional to $e^{-2H(x,y)}$. So, we assume that

$$Z := \int_{\mathbb{R}^{2n}} \exp(-2H)dx\,dy < +\infty \tag{1.3}$$

and we set

$$\mu(dx,dy) = Z^{-1}e^{-2H(x,y)}dxdy.$$

The simplest way to let μ be invariant is to consider the modified system

$$\begin{cases} dX(t) = D_yH(X(t),Y(t))dt - \alpha D_xH(X(t),Y(t))dt + \sqrt{\alpha}\,dW_1(t), \\ \\ dY(t) = -D_xH(X(t),Y(t))dt - \beta D_yH(X(t),Y(t))dt + \sqrt{\beta}\,dW_2(t). \end{cases} \tag{1.4}$$

In this case the Kolmogorov operator is

$$\mathcal{K}\varphi = \frac{\alpha}{2}\,\Delta_x\varphi + \frac{\beta}{2}\,\Delta_y\varphi + \langle D_yH - \alpha D_xH, D_x\varphi\rangle - \langle D_xH + \beta D_yH, D_y\varphi\rangle, \tag{1.5}$$

and

$$\int_{\mathbb{R}^{2n}} \mathcal{K}\varphi\, e^{-2H(x,y)}dxdy = 0,$$

for each test function φ. Therefore, the measure μ is invariant for $\mathcal{K}$.

Our aim is to study the realization K of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n},\mu)$. The main result is that if the second order derivatives of H are bounded, then

$$K : D(K) = H^2(\mathbb{R}^{2n},\mu) \mapsto L^2(\mathbb{R}^{2n},\mu), \quad K\varphi = \mathcal{K}\varphi$$

is an m-dissipative operator. Therefore, it generates a strongly continuous contraction semigroup in $L^2(\mathbb{R}^{2n},\mu)$.

Note that K is not symmetric, but we can show that for all $\varphi \in H^2(\mathbb{R}^{2n},\mu)$, $\psi \in H^1(\mathbb{R}^{2n},\mu)$ we have

$$\begin{aligned} \int_{\mathbb{R}^{2n}} K\varphi\,\psi\,d\mu = &-\frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha\langle D_x\varphi, D_x\psi\rangle + \beta\langle D_y\varphi, D_y\psi\rangle)d\mu \\ &+\frac{1}{2}\int_{\mathbb{R}^{2n}} (\langle 2H_y - \alpha H_x, D_x\varphi\rangle + \langle -2H_x + \beta H_y, D_y\varphi\rangle)\psi\,d\mu. \end{aligned} \tag{1.6}$$

However, taking $\psi = \varphi$ in (1.6) and manipulating the last integral (or else, integrating the equality $\mathcal{K}(\varphi^2) = 2\varphi\mathcal{K}\,\varphi + \alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2$ with respect to μ and recalling that $\int_{\mathbb{R}^{2n}} \mathcal{K}(\varphi^2)d\mu = 0$), we get

$$\int_{\mathbb{R}^{2n}} K\varphi\,\varphi\,d\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \tag{1.7}$$

which is a crucial formula for our analysis. It implies immediately that K is dissipative.

A typical procedure for showing m–dissipativity is to define a (dissipative) realization K_0 of $\mathcal{K}$ in $L^2(\mathbb{R}^{2n},\mu)$ with a small domain, say for instance $C^2_b(\mathbb{R}^{2n})$ (the space of the continuous bounded functions with continuous and bounded first and second order derivatives), and to prove that the range of $\lambda\mathbb{I} - K_0$ is dense in

$L^2(\mathbb{R}^{2n},\mu)$ for some $\lambda > 0$. Then by the Lumer–Phillips theorem the closure $\overline{K}_0$ of K_0 is an m-dissipative operator. But the problem of the characterization of the domain of $\overline{K}_0$ still remains open. So, we take the stick from the other side. We define the above realization K of $\mathcal{K}$ on $H^2(\mathbb{R}^{2n},\mu)$, and we show that K is m-dissipative. Since the H^2 norm is equivalent to the graph norm of K, from general density results in weighted Sobolev spaces it follows that $C_b^2(\mathbb{R}^{2n})$ and $C_0^\infty(\mathbb{R}^{2n})$ are cores for K.

From the point of view of the theory of elliptic PDE's, we show that for each $\lambda > 0$ and $f \in L^2(\mathbb{R}^{2n},\mu)$, the equation

$$\lambda u - \mathcal{K}u = f$$

has a unique solution $u \in H^2(\mathbb{R}^{2n},\mu)$. So, we have a maximal regularity result in weighted L^2 spaces. In fact, we prove something more: if a function $u \in H^2_{loc}(\mathbb{R}^{2n})$ satisfies $\lambda u - \mathcal{K}u = f \in L^2(\mathbb{R}^{2n},\mu)$, then $u \in H^2(\mathbb{R}^{2n},\mu)$ and

$$\|u\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^{2n},\mu)},$$

which is the dissipativity estimate,

$$\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \le \frac{2}{\lambda}\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

which comes from (1.7),

$$\int_{\mathbb{R}^{2n}} |D^2\varphi|^2 d\mu \le C\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

which is not obvious. Here C depends on λ and on the sup norm of the second derivatives of H.

After the elliptic regularity result we prove also some properties of the semigroup $T(t)$ generated by K. First, we show that it is an analytic semigroup. In general, elliptic operators with Lipschitz continuous unbounded coefficients do not generate analytic semigroups in L^2 spaces with respect to invariant measures. See a counterexample in [9]. In our case we can prove that the L^2 norm of the second order space derivatives of $T(t)\varphi$ blows up as $Ct^{-1}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$, as $t \to 0$, and since the graph norm of K is equivalent to the H^2 norm, then $\|tKT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$ is bounded by *const.* $\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}$ near $t = 0$.

Second, we address to asymptotic behavior of $T(t)$, proving that $T(t)\varphi$ weakly converges to the mean value $\overline{\varphi}$ as $t \to +\infty$ for each $\varphi \in L^2(\mathbb{R}^{2n},\mu)$ (strongly mixing property).

2. m-dissipativity of K

Throughout this section we assume that $H\colon \mathbb{R}^{2n} \mapsto \mathbb{R}$ is a nonnegative C^2 function with bounded second order derivatives, such that (1.3) holds.

The Hilbert spaces $H^1(\mathbb{R}^{2n},\mu)$, $H^2(\mathbb{R}^{2n},\mu)$ are defined as the sets of all $u \in H^1_{loc}(\mathbb{R}^{2n})$ (respectively, $u \in H^2_{loc}(\mathbb{R}^{2n})$), such that u and its first order derivatives (resp. u and its first and second order derivatives) belong to $L^2(\mathbb{R}^{2n},\mu)$. It is easy to see that $C_0^\infty(\mathbb{R}^{2n})$, the space of the smooth functions with compact support, is dense in $L^2(\mathbb{R}^{2n},\mu)$, in $H^1(\mathbb{R}^{2n},\mu)$, and in $H^2(\mathbb{R}^{2n},\mu)$. A proof is in [4, Lemma 2.1].

The main result of this paper is that

$$K\colon D(K) := H^2(\mathbb{R}^{2n},\mu) \mapsto L^2(\mathbb{R}^{2n},\mu), \quad K\varphi = \mathcal{K}\varphi$$

is m-dissipative. To this aim we need an embedding lemma.

Lemma 2.1. *For every $\varphi \in H^1(\mathbb{R}^{2n},\mu)$ and $i = 1,\dots,n$ the functions $\varphi D_{x_i}H$ and $\varphi D_{y_i}H$ belong to $L^2(\mathbb{R}^{2n},\mu)$. Moreover*

$$\int_{\mathbb{R}^{2n}} (\varphi D_{x_i}H)^2 d\mu \le \int_{\mathbb{R}^{2n}} (\|D_{x_ix_i}H\|_\infty \varphi^2 + (D_{x_i}\varphi)^2) d\mu,$$

$$\int_{\mathbb{R}^{2n}} (\varphi D_{y_i}H)^2 d\mu \le \int_{\mathbb{R}^{2n}} (\|D_{y_iy_i}H\|_\infty \varphi^2 + |D_{y_i}\varphi|^2) d\mu,$$

for all $\varphi \in H^1(\mathbb{R}^{2n},\mu)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^{2n})$. For every $i = 1,\dots,n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} (\varphi D_{x_i}H)^2 d\mu &= \frac{1}{2Z}\int_{\mathbb{R}^{2n}} \varphi^2 D_{x_i}H(-D_{x_i}e^{-2H})dx\,dy \\
&= \frac{1}{2Z}\int_{\mathbb{R}^{2n}} (\varphi^2 D_{x_ix_i}H + 2\varphi\, D_{x_i}\varphi\, D_{x_i}H)e^{-2H}dx\,dy \\
&\le \frac{\|D_{x_ix_i}H\|_\infty}{2}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|\varphi D_{x_i}H\|_{L^2(\mathbb{R}^{2n},\mu)}\|D_{x_i}\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \\
&\le \frac{\|D_{x_ix_i}H\|_\infty}{2}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \frac{1}{2}\|\varphi D_{x_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \frac{1}{2}\|D_{x_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}
\end{aligned}$$

so that

$$\|\varphi D_{x_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \|D_{x_ix_i}H\|_\infty \|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|D_{x_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}.$$

Similarly,

$$\|\varphi D_{y_i}H\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \|D_{y_iy_i}H\|_\infty \|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)} + \|D_{y_i}\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)},$$

and since $C_0^\infty(\mathbb{R}^{2n})$ is dense in $H^1(\mathbb{R}^{2n},\mu)$ the statement follows. □

The lemma has several important consequences. It implies that for every $\varphi \in H^2(\mathbb{R}^{2n},\mu)$ the drift $\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle$ belongs to $L^2(\mathbb{R}^{2n},\mu)$. It implies also (taking $\varphi \equiv 1$) that $|DH| \in L^2(\mathbb{R}^{2n},\mu)$. Moreover, in the case that $|DH| \to +\infty$ as $|(x,y)| \to +\infty$, the estimate

$$\int_{\mathbb{R}^{2n}} \varphi^2 |DH|^2 d\mu \le C\|\varphi\|_{H^1(\mathbb{R}^{2n},\mu)}, \quad \varphi \in H^1(\mathbb{R}^{2n},\mu),$$

implies easily that $H^1(\mathbb{R}^{2n},\mu)$ is compactly embedded in $L^2(\mathbb{R}^{2n},\mu)$. See e.g. [7, Prop. 3.4].

The integration formula (1.7) implies immediately that K is dissipative. What is not trivial is m-dissipativity. To prove m-dissipativity we have to solve the resolvent equation

$$\lambda\varphi - \mathcal{K}\varphi = f \tag{2.1}$$

for each $f \in L^2(\mathbb{R}^{2n}, \mu)$ and $\lambda > 0$, and show that the solution φ belongs to $H^2(\mathbb{R}^{2n}, \mu)$. That is, we have to prove an existence and maximal regularity result for an elliptic equation with unbounded coefficients. Of course it is enough to prove that for each $f \in C_0^\infty(\mathbb{R}^{2n})$ the resolvent equation has a unique solution φ in $H^2(\mathbb{R}^{2n}, \mu)$, and $\|\varphi\|_{H^2(\mathbb{R}^{2n},\mu)} \leq C\|f\|_{L^2(\mathbb{R}^{2n},\mu)}$.

Since the coefficients of $\mathcal{K}$ are regular enough, existence of a solution may be proved in several ways. For instance, the problem in the whole $\mathbb{R}^{2n}$ may be approached by a sequence of Dirichlet problems in the balls $B(0, k)$, and using classical interior estimates for solutions of second order elliptic problems we arrive at a solution $\varphi \in H^2_{loc}(\mathbb{R}^{2n})$. See e.g. [8, Theorem 3.4]. Or else, we may use the stochastic characteristics method, that gives a solution to (2.1) as

$$\varphi = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[f(X(t), Y(t))]dt$$

where $(X(t), Y(t))$ is the solution to (1.4) with initial data $X(0) = x$, $Y(0) = y$. See e.g. [3, Thms. 1.2.5, 1.6.6]. The assumptions of [3] are satisfied because the second order derivatives of H are bounded, and consequently the first order derivatives of H have at most linear growth.

Uniqueness of the solution in $H^2(\mathbb{R}^{2n}, \mu)$ follows immediately from dissipativity. Estimates for the second order derivatives in $L^2(\mathbb{R}^{2n}, \mu)$ are less obvious. They are proved in the next theorem.

Theorem 2.2. *Let $\varphi \in H^2_{loc}(\mathbb{R}^{2n})$ satisfy (2.1), with $f \in L^2(\mathbb{R}^{2n}, \mu)$. Then*

$$\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \leq \frac{1}{\lambda}\|f\|_{L^2(\mathbb{R}^{2n},\mu)} \tag{2.2}$$

$$\int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2)d\mu \leq \frac{2}{\lambda}\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)} \tag{2.3}$$

$$\int_{\mathbb{R}^{2n}} |D^2\varphi|^2 d\mu \leq C\|f\|^2_{L^2(\mathbb{R}^{2n},\mu)} \tag{2.4}$$

where C does not depend on f and φ.

Proof. Without loss of generality we may assume that $f \in C_0^\infty(\mathbb{R}^{2n})$. Then $\varphi \in H^3_{loc}(\mathbb{R}^{2n})$, by local elliptic regularity. Moreover by the Schauder estimates of [6], φ and its first and second order derivatives are bounded and Hölder continuous. In particular, $\varphi \in H^2(\mathbb{R}^{2n}, \mu)$.

Estimates (2.2) and (2.3) follow in a standard way, multiplying both sides of (2.1) by φ and using the integration formula (1.7). To get (2.4) we differentiate

(2.1) with respect to x_i and y_i, obtaining

$$\begin{aligned}
&\lambda D_{x_i}\varphi - KD_{x_i}\varphi - \sum_{k=1}^{n}(D_{x_iy_k}H - \alpha D_{x_ix_k}H)D_{x_k}\varphi \\
&\qquad\qquad + \sum_{k=1}^{n}(D_{x_ix_k}H + \beta D_{x_iy_k}H)D_{y_k}\varphi \quad = D_{x_i}f, \\
&\lambda D_{y_i}\varphi - KD_{y_i}\varphi - \sum_{k=1}^{n}(D_{y_iy_k}H - \alpha D_{y_ix_k}H)D_{x_k}\varphi \\
&\qquad\qquad + \sum_{k=1}^{n}(D_{y_ix_k}H + \beta D_{y_iy_k}H)D_{y_k}\varphi \quad = D_{y_i}f.
\end{aligned}$$

Replacing the expressions of the derivatives of f in the equality

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu = &-\frac{1}{2}\int_{\mathbb{R}^{2n}} (\alpha\langle D_x\varphi, D_xf\rangle + \beta\langle D_y\varphi, D_yf\rangle)d\mu \\
&+ \int_{\mathbb{R}^{2n}} (\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu
\end{aligned}$$

that follows from (1.6), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu &= \int_{\mathbb{R}^{2n}} (\lambda\varphi - f)f d\mu \\
&= \int_{\mathbb{R}^{2n}} -\frac{\lambda}{2}(\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}(\alpha D_{x_i}\varphi\, KD_{x_i}\varphi + \beta D_{y_i}\varphi\, KD_{y_i}\varphi)d\mu \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^{2n}} -\langle SD\varphi, D\varphi\rangle + (\langle D_yH, D_x\varphi\rangle - \langle D_xH, D_y\varphi\rangle)f\, d\mu,
\end{aligned}$$

where the $2n \times 2n$ matrix S has entries

$$\begin{aligned}
S_{i,k} \qquad\qquad &= -\frac{\alpha}{2}(D_{x_iy_k}H - \alpha D_{x_ix_k}H), \\
S_{i,n+k} = S_{n+i,k} \quad &= \frac{\alpha}{2}D_{x_ix_k}H - \frac{\beta}{2}D_{y_iy_k}H + \alpha\beta D_{x_iy_k}H, \\
S_{n+i,n+k} \qquad &= \frac{\beta}{2}(D_{y_ix_k}H + \beta D_{y_iy_k}H),
\end{aligned}$$

for i, $k = 1, \ldots, n$.

Using formula (1.6) in the integrals $\int_{\mathbb{R}^{2n}} K\varphi\, f\, d\mu$, $\int_{\mathbb{R}^{2n}} D_{x_i}\varphi K D_{x_i}\varphi\, d\mu$, and $\int_{\mathbb{R}^{2n}} D_{y_i}\varphi K D_{y_i}\varphi\, d\mu$ we get

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^{2n}} (\alpha|D_x\varphi|^2 + \beta|D_y\varphi|^2) d\mu \\
&\quad + \int_{\mathbb{R}^{2n}} (\alpha^2 \sum_{i,k} (D_{x_i x_k}\varphi)^2 + 2\alpha\beta \sum_{i,k} (D_{x_i y_k}\varphi)^2 + \beta^2 \sum_{i,k} (D_{y_i y_k}\varphi)^2) d\mu \\
&\quad + \int_{\mathbb{R}^{2n}} (\langle SD\varphi, D\varphi\rangle + 2(\langle D_y H, D_x\varphi\rangle - \langle D_x H, D_y\varphi\rangle) f\, d\mu \\
&\quad = 2\int_{\mathbb{R}^{2n}} (f - \lambda\varphi) f d\mu \le 4\|f\|^2.
\end{aligned}$$

(We have used (2.2) in the last inequality). Since the second order derivatives of H are bounded, there is $c > 0$ such that

$$\left| \int_{\mathbb{R}^{2n}} \langle SD\varphi, D\varphi\rangle d\mu \right| \le c \int_{\mathbb{R}^{2n}} |D\varphi|^2 d\mu \le \frac{2c}{\lambda} \|f\|^2.$$

By lemma 2.1, the function $\langle D_y H, D_x\varphi\rangle - \langle D_x H, D_y\varphi\rangle$ belongs to $L^2(\mathbb{R}^{2n}, \mu)$, and its norm does not exceed $(C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2)^{1/2}$. Therefore,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^{2n}} (\langle D_y H, D_x\varphi\rangle - \langle D_x H, D_y\varphi\rangle) f\, d\mu \right| \le (C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2)^{1/2}\|f\| \\
&\quad \le \frac{\varepsilon}{2}(C\|\,|D\varphi|^2\,\|^2 + \|\,|D^2\varphi|\,\|^2) + \frac{1}{2\varepsilon}\|f\|^2,
\end{aligned}$$

for every $\varepsilon > 0$ (we have used (2.3) in the last inequality). Choosing ε small enough, in such a way that

$$\varepsilon\|\,|D^2\varphi|\,\|^2 \le \int_{\mathbb{R}^{2n}} (\alpha^2 \sum_{i,k} (D_{x_i x_k}\varphi)^2 + 2\alpha\beta \sum_{i,k} (D_{x_i y_k}\varphi)^2 + \beta^2 \sum_{i,k} (D_{y_i y_k}\varphi)^2) d\mu$$

estimate (2.4) follows. □

Remark 2.3. In the estimate of the second order derivatives of φ we have not used the full assumption that H has bounded second order derivatives, but only its consequence $\langle S\xi, \xi\rangle \ge \kappa|\xi|^2$, for some $\kappa \in \mathbb{R}$ and all $\xi \in \mathbb{R}^{2n}$. This shows that the assumption that all the second order derivatives of H could be weakened if we only want to prove that the domain of K is contained in $H^2(\mathbb{R}^{2n}, \mu)$.

3. Further properties of K and $T(t)$

The operator $\mathcal{K}$ belongs to a class of elliptic operators with unbounded coefficients recently studied in [1, 2],

$$\varphi \mapsto \mathrm{Tr}(QD^2\varphi) + \langle F, D\varphi\rangle.$$

In our case the coefficients of the matrix Q are constant, and the derivatives of every component of F are bounded. Therefore, the assumptions of [1, Prop. 4.3] are satisfied, and it yields pointwise estimates of the space derivatives of $T(t)\varphi$. Precisely,

$$|DT(t)\varphi(x,y)|^2 \le \frac{\sigma}{2\nu(1-e^{-\sigma t})}\,(T(t)\varphi^2)(x,y), \;\; \varphi \in L^2(\mathbb{R}^{2n},\mu),\; t>0,$$

where $\sigma = 2\sup_{\xi\in\mathbb{R}^{2n}\setminus\{0\}}\langle DF\cdot\xi,\xi\rangle/|\xi|^2$, DF being the Jacobian matrix of F, and ν is the ellipticity constant which is the minimum between $\alpha/2$ and $\beta/2$ in our case. Integrating over $\mathbb{R}^{2n}$ we get, for each $t>0$,

$$\|\,|DT(t)\varphi|\,\|^2_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{\sigma}{2\nu(1-e^{-\sigma t})}\int_{\mathbb{R}^{2n}} T(t)\varphi^2 d\mu = \frac{\sigma}{2\nu(1-e^{-\sigma t})}\|\varphi\|^2_{L^2(\mathbb{R}^{2n},\mu)}, \tag{3.1}$$

which implies

$$\begin{aligned} &\text{(i)}\; \|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{C}{\sqrt{t}}\,\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\;\; 0<t\le 1,\\ &\text{(ii)}\; \|\,|DT(t)\varphi|\,\|_{L^2(\mathbb{R}^{2n},\mu)} \le C\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\;\; t\ge 1. \end{aligned} \tag{3.2}$$

An important consequence of the gradient estimates (3.2) and of lemma 2.1 is analyticity of $T(t)$.

Proposition 3.1. *$T(t)$ maps $L^2(\mathbb{R}^{2n},\mu)$ into $H^2(\mathbb{R}^{2n},\mu)$ for every $t>0$, and there is $C>0$ such that*

$$\|KT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{C}{t}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},\;\; 0<t\le 1,\; \varphi\in L^2(\mathbb{R}^{2n},\mu).$$

Proof. Let $\varphi \in H^2(\mathbb{R}^{2n},\mu)$. Then $u(t,\cdot) := T(t)\varphi$ satisfies $u_t = \mathcal{K}u$ for $t\ge 0$. By general regularity results for parabolic equations, u_t and the second order space derivatives of u have first order space derivatives in $L^2_{loc}(\mathbb{R}^{2n})$, and for $i=1,\dots,n$,

$$\begin{aligned} D_{x_i}u_t = D_tD_{x_i}u &= \mathcal{K}D_{x_i}u + \langle D_yD_{x_i}H, D_xu\rangle - \langle D_xD_{x_i}H, D_yu\rangle,\\ D_{y_i}u_t = D_tD_{y_i}u &= \mathcal{K}D_{y_i}u + \langle D_yD_{y_i}H, D_xu\rangle - \langle D_xD_{y_i}H, D_yu\rangle. \end{aligned}$$

Therefore, denoting by z_i either x_i or y_i, we have

$$D_t(D_{z_i}u - T(t)D_{z_i}\varphi) = \mathcal{K}(D_{z_i}u - T(t)D_{z_i}\varphi) + \langle D_yD_{z_i}H, D_xu\rangle - \langle D_xD_{z_i}H, D_yu\rangle$$

so that

$$\begin{aligned} &D_{z_i}T(t)\varphi - T(t)D_{z_i}\varphi\\ &\quad = \int_0^t T(t-s)\langle D_yD_{z_i}H, D_xT(s)\varphi\rangle - \langle D_xD_{z_i}H, D_yT(s)\varphi\rangle ds := J_{z_i}\varphi(t). \end{aligned}$$

Let C be the constant in (3.2), and let C_1 be the maximum of the sup norm of all the second order derivatives of H. Then for $0<t\le 1$ we have

$$\|J_{z_i}\varphi(t)\|_{L^2(\mathbb{R}^{2n},\mu)} \le C_1\int_0^t \frac{C}{\sqrt{s}}\,ds\;\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} = 2C_1C\sqrt{t}\|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)},$$

and, more important,

$$\begin{aligned} \| \, |DJ_{z_i}\varphi(t)| \, \|_{L^2(\mathbb{R}^{2n},\mu)} &\le C_1 \int_0^t \frac{C}{\sqrt{t-s}}\,\frac{C}{\sqrt{s}}\,ds\, \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}, \\ &= C_1 C^2 \int_0^1 \frac{1}{(1-\sigma)^{1/2}\sigma^{1/2}} d\sigma\, \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} := C_2 \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}. \end{aligned} \tag{3.3}$$

From the equality

$$\begin{aligned} D_{z_j z_i} T(t)\varphi &= D_{z_j} D_{z_i} T(t/2) T(t/2)\varphi \\ &= D_{z_j} T(t/2) D_{z_i} T(t/2)\varphi + D_{z_j} J_{z_i}(T(t/2)\varphi)(t/2) \end{aligned}$$

we get, using (3.2)(i) and (3.3),

$$\|D_{z_j z_i} T(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \left(\frac{C}{\sqrt{t/2}}\right)^2 \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} + C_2 \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}.$$

This estimate and lemma 2.1 yield

$$\|KT(t)\varphi\|_{L^2(\mathbb{R}^{2n},\mu)} \le \frac{C}{t} \|\varphi\|_{L^2(\mathbb{R}^{2n},\mu)}, \quad 0 < t \le 1.$$

Since $H^2(\mathbb{R}^{2n},\mu)$ is dense in $L^2(\mathbb{R}^{2n},\mu)$, the statement follows. □

We can improve estimate (3.2)(ii), showing that $\| \, |DT(t)\varphi| \, \|_{L^2(\mathbb{R}^{2n},\mu)}$ goes to 0 as $t \to +\infty$.

Lemma 3.2. *For all* $\varphi \in L^2(\mathbb{R}^{2n},\mu)$, $\lim_{t\to+\infty} \| \, |DT(t)\varphi| \, \|_{L^2(\mathbb{R}^{2n},\mu)} = 0$.

Proof. Let $\varphi \in H^2(\mathbb{R}^{2n},\mu)$. From the equality

$$\frac{d}{dt}\|T(t)\varphi\|^2 = 2\int_{\mathbb{R}^{2n}} T(t)\varphi\, KT(t)\varphi\, d\mu$$

we obtain

$$\|T(t)\varphi\|^2 - \|\varphi\|^2 = 2\int_0^t \int_{\mathbb{R}^{2n}} T(s)\varphi\, KT(s)\varphi\, d\mu\, ds$$

and using (1.7) we get

$$\|T(t)\varphi\|^2 + \int_0^t \int_{\mathbb{R}^{2n}} (\alpha|D_x T(s)\varphi|^2 + \beta|D_y T(s)\varphi|^2) d\mu\, ds = \|\varphi\|^2, \quad t > 0. \tag{3.4}$$

Therefore, the function

$$\chi_\varphi(s) := \int_{\mathbb{R}^{2n}} (\alpha|D_x T(s)\varphi|^2 + \beta|D_y T(s)\varphi|^2) d\mu, \quad s \ge 0,$$

is in $L^1(0,+\infty)$, and its L^1 norm does not exceed $\|\varphi\|^2$. Its derivative is

$$\chi_\varphi'(s) = \int_{\mathbb{R}^{2n}} (2\alpha\langle D_x T(s)\varphi, D_x T(s)K\varphi\rangle + 2\beta\langle D_y T(s)\varphi, D_y T(s)K\varphi\rangle) d\mu$$

so that

$$\begin{aligned} |\chi_\varphi'(s)| \le\; & 2\Big(\int_{\mathbb{R}^{2n}} \alpha|D_xT(s)\varphi|^2 d\mu\Big)^{1/2}\Big(\int_{\mathbb{R}^{2n}} \alpha|D_xT(s)K\varphi|^2 d\mu\Big)^{1/2} \\ & + 2\Big(\int_{\mathbb{R}^{2n}} \beta|D_yT(s)\varphi|^2) d\mu\Big)^{1/2}\Big(\int_{\mathbb{R}^{2n}} \beta|D_yT(s)K\varphi|^2) d\mu\Big)^{1/2} \\ \le\; & \chi_\varphi(s) + \chi_{K\varphi}(s). \end{aligned}$$

Therefore, also χ_φ' is in $L^1(0,+\infty)$. This implies that $\lim_{s\to+\infty}\chi_\varphi(s) = 0$, and the statement holds for every $\varphi \in H^2(\mathbb{R}^{2n},\mu)$. For general $\varphi \in L^2(\mathbb{R}^{2n},\mu)$ the statement follows approaching φ by a sequence of functions in $H^2(\mathbb{R}^{2n},\mu)$ and using estimate (3.2)(ii). □

The lemma has some interesting consequences.

First, the kernel of K consists of constant functions. This is because if $\varphi \in \text{Ker } K$ then $T(t)\varphi = \varphi$ for each $t > 0$, so that $D\varphi = DT(t)\varphi$ goes to 0 as t goes to $+\infty$, hence $D\varphi = 0$ and φ is constant. So, we have a sort of Liouville theorem for K.

Second, for each $\varphi \in L^2(\mathbb{R}^{2n},\mu)$, $T(t)\varphi$ converges weakly to the mean value $\overline{\varphi}$ of φ as $t\to+\infty$. This is because $T(t)\varphi$ is bounded, hence there is a sequence $t_n \to +\infty$ such that $T(t_n)\varphi$ converges weakly to some limit $g \in L^2(\mathbb{R}^{2n},\mu)$, each derivative $D_{z_i}T(t_n)\varphi$ converges strongly to 0, and since the derivatives are closed with respect to the weak topology too, then $D_{z_i}g = 0$ and g is constant. The constant has to be equal to the mean value of φ because μ is invariant, and hence

$$g = \int_{\mathbb{R}^{2n}} g\, d\mu = \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} T(t_n)\varphi\, d\mu = \lim_{n\to\infty}\int_{\mathbb{R}^{2n}} \varphi\, d\mu = \overline{\varphi}.$$

Strong convergence is not obvious in general. Of course if 0 is isolated in the spectrum of K (for instance, if $H^2(\mathbb{R}^{2n},\mu)$ is compactly embedded in $L^2(\mathbb{R}^{2n},\mu)$) then $T(t)\varphi$ converges exponentially to the spectral projection of φ on the kernel of K, which is just $\overline{\varphi}$.

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