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ON THE $\kappa - \theta$ MODEL OF CELLULAR FLAMES: EXISTENCE IN THE LARGE AND ASYMPTOTICS

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Abstract. We consider the $\kappa - \theta$ model of flame front dynamics introduced in [6]. We show that a space-periodic problem for the latter system of two equations is globally well-posed. We prove that near the instability threshold the front is arbitrarily close to the solution of the Kuramoto–Sivashinsky equation on a fixed time interval if the evolution starts from close configurations. The dynamics generated by the model is illustrated by direct numerical simulation.

1. **Introduction.** Premixed gas combustion represents a rather intricate physical system involving fluid dynamics, multistep chemical kinetics, as well as molecular and radiative heat transfer. Its basic propagation mode exhibits two main mechanisms of destabilization: one due to the thermal expansion of the gas known as the

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hydrodynamic instability, and the thermal-diffusive instability which is a result of the competition between the exothermic reaction and the heat diffusion.

The thermal-diffusive instability manifests itself by generating a cellular structure, which in turn exhibits chaotic dynamics. It transpires, however, that the cellular instability may be described by a free interface problem [9][4][3]. Moreover, near the instability threshold it is possible to (asymptotically) separate the spatial and the temporal coordinates, and further reduce the system to a single geometrically invariant surface dynamics equation. In suitably chosen units it reads [7]:

$$V_n = 1 + (\alpha - 1)\kappa + \kappa_{ss}. \tag{1}$$

Here V_n is the normal velocity of the flame sheet, s is the arc-length along the interface, and κ is its curvature. The parameter α reflects the physico-chemical characteristics of the combustible; the cellular instability occurs when α exceeds unity.

The coordinate-free model (1) and especially its weakly-nonlinear approximation, the Kuramoto-Sivashinsky (KS) equation [10], appear in a variety of physical systems typically including free interface (see e.g [5]). However, (1) is not a unique low-dimensional model generating cellular instability. As has been demonstrated recently ([6]) the effect may also be exhibited by a system of coordinate-free equations

$$\begin{aligned} V_n &= 1 + \kappa + \Theta, \\ D_t\,\Theta &= \Theta_{ss} - \alpha\kappa - \Theta, \end{aligned} \tag{2}$$

where $D_t\Theta$ is the Lagrangian time derivative of the (reduced) interface temperature Θ along the "flow" generated by the normal velocity field $V_n\mathbf{n}$.

One should expect that including the interface temperature is likely to provide a deeper contact with the original reaction-diffusion system. Moreover, the $\kappa - \theta$ model may serve as a basis for the description of the flame interaction with the background flow-field. The $\kappa - \theta$ model may be derived either by using the appropriate gradient expansion in the intrinsic coordinates, or, as in [6], may be constructed as a *geometrically-invariant extrapolation* consistent with the relation provided by the linear stability analysis of the planar flame.

In a recent paper [2] we studied a truncated version of the $\kappa - \theta$ system - the *quasi-steady* (QS) model[1]:

$$\begin{aligned} V_n &= 1 + \kappa + \Theta, \\ \Theta_{ss} - \alpha\kappa - \Theta &= 0. \end{aligned} \tag{3}$$

More precisely we have demonstrated that Cauchy problem for its weakly nonlinear version

$$\Phi_t + \frac{1}{2}\Phi_x^2 = \left[I - \alpha(I - \partial_x^2)^{-1}\right]\Phi_{xx}, \tag{4}$$

has a unique solution in the Sobolev spaces of periodic functions, it is uniformly close to the solution of the KS equation

$$\Phi_t + \frac{1}{2}\Phi_x^2 = (1-\alpha)\Phi_{xx} - \Phi_{xxxx}, \tag{5}$$

for a fixed time interval and small instability parameter : $0 < \alpha - 1 \ll 1$.

[1] Although the QS model was introduced *ad hoc* to serve as it were as a "bridge" between the full $\kappa - \theta$ model and the GI equation (1), it represents in our opinion an interesting dynamical system in its own right (see the discussion in [1]).

In this paper we present similar results for the weakly non-linear version of the the $\kappa - \theta$ system:

$$\Phi_t + \frac{1}{2}\Phi_x^2 = \Phi_{xx} - \Theta, \tag{6}$$

$$\Theta_t + \Phi_x\Theta_x = \Theta_{xx} + \alpha\Phi_{xx} - \Theta. \tag{7}$$

It is obtained by assuming a slightly distorted planar flame propagating along the y-axis; then the flame surface can be described by an explicit function $y = -t + \Phi(x,t)$ of the transversal coordinate x, and, consequently, $ds \simeq dx$, $V_n \simeq 1 - \Phi_t - \frac{1}{2}\Phi_x^2$, $\kappa \simeq \Phi_{xx}$, $D_t\Theta \simeq \Theta_t + \Phi_x\Theta_x$.which yields (6-7).

All three of these systems (5), (4) and (6-7) share the same basic quality revealed by linear stability analysis, namely long-wave destabilization, which is suppressed by the dominant dissipative principal term for small wave lengths. The respective dispersion relations read:

$$\omega(k) = (\alpha - 1)k^2 - k^4, \tag{KS}$$

$$\omega(k) = -k^2 + \frac{\alpha k^2}{1+k^2} = (\alpha - 1)k^2 - \alpha^2 k^4 + \cdots, \tag{QS}$$

$$\omega(k) = \sqrt{1/4 + \alpha k^2} - 1/2 - k^2 = (\alpha - 1)k^2 - \alpha^2 k^4 + \cdots \tag{8}$$

As one can easily see, in the long-wave part of the spectrum $k \ll 1$, the dispersion relation for QS ([2]) and $\kappa - \theta$ (8) is similar to that of KS, while for the short waves $k \gg 1$ their decay (in contrast to KS) is $\omega \sim -k^2$.

In order to understand which dispersion relation better reflects the physical reality, we should turn to the original free-interface (FI) combustion problem [9]. The dispersion relation for the free-interface problem reads (see e.g. [4]):

$$\left(1 + \omega + k^2 - \alpha\right)\left(1 - \sqrt{(1 + \omega + k^2)}\right) - \frac{1}{2}\alpha\omega = 0. \tag{FI}$$

To compare the latter dispersion relation with (KS) and (QS) we turn to the scaling employed in the derivation of the KS equation [10]: $\varepsilon := \alpha - 1 = o(1)$, $k^2 = \mathcal{O}(\varepsilon)$, and $\omega = \mathcal{O}(\varepsilon^2)$. Then, up to ε^2, (FI) yields:

$$\omega = (\alpha - 1)k^2 - k^4.$$

It is easy to see, however, that for $k \gg 1$, (FI) yields $\omega = -k^2$. Thus, while in the long-wave approximation the behavior of the exact dispersion relation of the free-interface problem is qualitatively correctly mimicked by such of KS, QS and $\kappa - \theta$ its decay at infinity is identical to QS and $\kappa - \theta$.

Therefore, the (linear parts of) QS and $\kappa - \theta$ equation present *uniform approximations of the spectrum* of the original combustion problem. Fig. 1 illustrates our remarks concerning the comparative linear stability analysis of the three equations on the one hand, and the free-interface problem on the other hand. One can see that (8) is the best approximation of (FI). This may be of some importance for certain more subtle features of long term dynamics (on the attractor).

There is another reason to favor the weak $\kappa - \theta$ system (6-7) over the other two models: the ease of numerical simulation. Indeed, from the computational point of view it represents a rather ordinary well-behaved reaction-diffusion system that can be solved using a simple explicit finite-difference method. One particularly appreciates this simplicity when dealing with more than one spacial dimension.

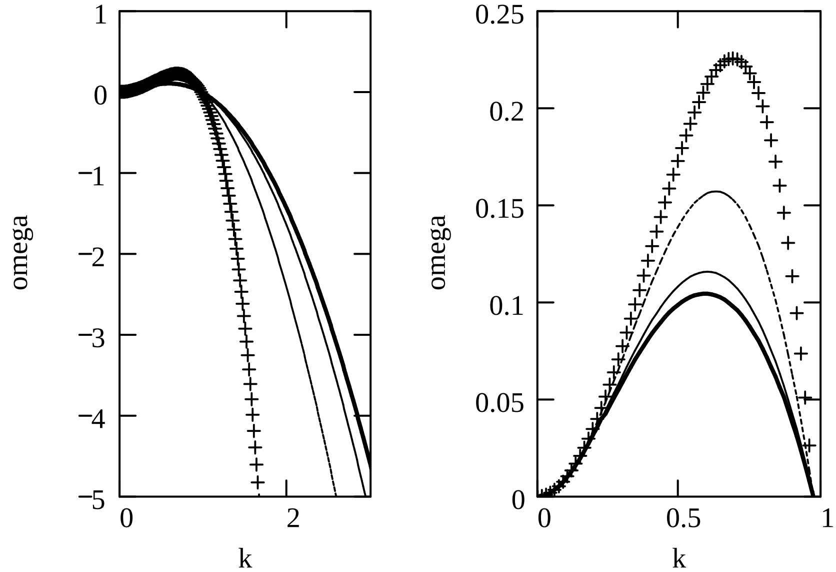

FIGURE 1. Fig. 1. Dispersion relations for the KS (pluses), the $\kappa-\theta$ (thin line), the QS (dots) and the original free-interface problem (thick line) for $\alpha = 1.5$. A magnification of the long wave region is on the right.

The objective of the paper is twofold. First we consider the Cauchy problem for (6-7) over the interval $[-L_/2, L_/2]$ with periodic initial conditions

$$\Phi(x,0) = \Phi_0(x), \quad \Theta(x,0) = \Theta_0(x), \tag{9}$$

showing existence in the large for positive time. This is not *a priori* obvious for parabolic systems with quadratic nonlinearities in the gradient. Here we get sup-norm estimates for Θ and Φ_x that yield existence in the large. Such estimates are obtained via a maximum principle argument, that works because of the particular structure of (6-7).

Second, extending the results of [2], we want to establish a rigorous link between the weakly nonlinear $\kappa - \theta$ system (6-7) and the KS equation. Indeed, setting $0 < \alpha - 1 := \varepsilon \ll 1$, we observe that in the long-wave range $k = O(\sqrt{\varepsilon})$, $\omega = O(\varepsilon^2)$ the dispersion relation (8) is asymptotically identical to that for the KS equation. Therefore, it seems plausible that the asymptotic (by ε) dynamics of the front for $t = O(\varepsilon^{-2})$ and $x = O(\varepsilon^{-1/2})$ is the same as that of the KS equation. More precisely, in the coordinates $\tau = t\varepsilon^2$, $\xi = x\sqrt{\varepsilon}$, we anticipate $\Phi \sim \varepsilon U$, where U solves the K–S equation

$$U_\tau + \frac{1}{2}(U_\xi)^2 + U_{\xi\xi\xi\xi} + U_{\xi\xi} = 0. \tag{10}$$

Using more rigorous terms, let U be the periodic solution of (10) on a fixed time interval $[0, T_0]$ with initial condition U_0 of period $L_0 > 0$. We prove that, for $0 < \varepsilon \le \varepsilon_0$, there exist a unique $L_0/\sqrt{\varepsilon}$-periodic solution Φ, Θ of the initial value problem (6-7-9) with special initial conditions compatible with U_0 and (10) at $\tau = 0$

(hence we do need to consider boundary layer terms), such that

$$\max |\Phi(x,t) - \varepsilon U(x\sqrt{\varepsilon}, t\varepsilon^2)| \le C\,\varepsilon^2, \tag{11}$$

for $0 \le t \le T_0/\varepsilon^2$.

2. **Existence in the large.** Problem (6-7) is a parabolic system with smooth nonlinearities. Any theory of semilinear parabolic systems yields existence and uniqueness of a regular solution of the initial-boundary value problem

$$\Phi(0,\ \cdot) = \Phi_0,\ \ \Theta(0,\cdot) = \Theta_0, \tag{12}$$

$$\Phi(t,-L/2) = \Phi(t,L/2),\ \ \Theta(t,-L/2) = \Theta(t,L/2), \tag{13}$$

for (6-7), for periodic initial data Φ_0, Θ_0 with some minimal regularity properties (say, for instance, Φ_0, $\Theta_0 \in C^1([-L/2,L/2])$). The solution is defined on $[0,T) \times [-L/2,L/2]$, where T depends on the initial data, and it is smooth with respect to (t,x) for $t > 0$. Since we are not interested in the behavior near $t = 0$ we may assume from the very beginning that Φ_0, $\Theta_0 \in C^\infty([-L/2,L/2])$.

In the next section we need that $T = +\infty$, for any initial data. It is well known that the solutions of parabolic equations with quadratic nonlinearities in the gradient exist for all $t > 0$, but this is not true in general for parabolic systems. In our case we can show existence in the large because of the particular structure of our system. That is, after differentiating the Φ-equation with respect to x, the coefficients of $\Psi_x := \Phi_{xx}$ and of Θ_x are the same (equal to Φ_x), in the equation for Ψ and in the equation for Θ.

Theorem 1. *For each $\Phi_0,\ \Theta_0 \in C^1([-L/2,L/2])$ such that $\Phi_0(-L/2) = \Phi_0(L/2)$, $\Theta_0(-L/2) = \Theta_0(L/2)$, the solution to $(6,7,12,13)$ exists in the large.*

Proof of Theorem 1. The function

$$v(t,x) := \Theta^2(t,x) + \Phi_x^2(t,x),\ \ 0 \le t < T,\ -L/2 \le x \le L/2,$$

is continuous in $[0,T) \times [-L/2,L/2]$ and smooth in $(0,T) \times [-L/2,L/2]$. For $0 < t < T$ and $-L/2 \le x \le L/2$ we have

$$v_x = 2(\Theta\,\Theta_x +\ \Phi_x\Phi_{xx}),\ \ v_{xx} = 2(\Theta_x^2 +\ \Theta\,\Theta_{xx} +\ \Phi_{xx}^2 + \Phi_x\Phi_{xxx}),$$
$$v_t =\ 2(\Theta\,\Theta_t +\ \Phi_x\Phi_{tx})\ ,$$

so that

$$\begin{aligned} v_t &= 2\Theta(\Theta_{xx} + \alpha\Phi_{xx} - \Theta_x\ \Phi_x - \Theta) + 2\Phi_x(\Phi_{xxx} - \Phi_x\Phi_{xx} - \Theta_x) \\ &= v_{xx} - \Phi_x v_x\ + g(t,x), \end{aligned} \tag{14}$$

where

$$g(t,x) := -2\Theta_x^2 - 2\Phi_{xx}^2 - 2\Theta^2 + 2\alpha\Theta\Phi_{xx} - 2\Phi_x\Theta_x.$$

Since

$$|2\alpha\Theta\Phi_{xx}| \le \frac{\alpha^2}{2}\Theta^2 + 2\Phi_{xx}^2,\ \ |2\Phi_x\Theta_x| \le \frac{1}{2}\Phi_x^2 + 2\Theta_x^2,$$

we get

$$g(t,x) \le \left(\frac{\alpha^2}{2} - 2\right)\Theta^2 + \frac{1}{2}\Phi_x^2 \le \max\left\{\frac{\alpha^2}{2} - 2, \frac{1}{2}\right\}v.$$

By the maximum principle,

$$0 \le v(t,x) \le \|v(0,\cdot)\|_\infty\, e^{\lambda t},\ \ 0 \le t < T,\ -L/2 \le x \le L/2,$$

with $\lambda = \max\{\ \alpha^2/2 - 2, 1/2\}$. Then existence in the large follows from a standard bootstrap argument, using regularity results for linear equations. Indeed, since $|\Theta|$

and $|\Phi_x|$ are bounded by $e^{\lambda T_1/2}$ in the rectangle $[0,T_1]\times[-L/2,L/2]$ for every $T_1<T$, the general theory of linear parabolic equations [8] for equation (14) yields that for every $\theta\in(0,1)$ the norm $\|\Phi\|_{C^{1/2+\theta/2,1+\theta}([0,T_1]\times[-L/2,L/2])}$ is bounded by a constant $C(T_1,\theta)$ that does not blow up for finite time. In particular, the norm $\|\Phi_x\|_{C^{\theta/2,\ \theta}([0,T_1]\times[-L/2,L/2])}$ is bounded by a constant that does not blow up for finite time. Using this fact, applying now the general theory of linear parabolic equations [8] to equation (7) with periodic boundary condition, it follows that the norm $\|\Theta\|_{C^{1\ +\theta/2,2+\theta}([0,T_1]\times[-L/2,L/2])}$ is bounded by a constant that does not blow up for finite time, and replacing in (6) the same holds for Φ. This is enough to continue the solution beyond T, if T were finite. □

3. **Asymptotics.** In order to show uniform closeness of the $\kappa-\theta$ system to KS for small ε it is convenient to reduce it to a scalar equation which is obviously done by expressing

$$\Theta=\Phi_{xx}-\Phi_t-\frac{1}{2}\Phi_x^2$$

from (6), and substituting it into the second equation (7) to obtain the following ultra-parabolic equation:

$$\Phi_t+\Phi_{tt}+\Phi_{xxxx}+(\alpha-1)\Phi_{xx}-2\Phi_{xxt}+\frac{1}{2}[I-\partial_x^2+\partial_t]\Phi_x^2=\Phi_x[\Phi_{xx}-\Phi_t-\frac{1}{2}\Phi_x^2]_x. \tag{15}$$

The Cauchy problem for (15) requires two initial conditions:

$$\Phi(x,0)=\Phi_0(x),\qquad \Phi_t(x,0)=\Psi_0(x) \tag{16}$$

which are obviously related to (9) via (6) at $t=0$:

$$\Psi_0(x)=-\frac{1}{2}\left[\Phi_0'(x)\right]^2+\Phi_0''(x)-\Theta_0(x)$$

In this section we are interested in $\alpha=1+\varepsilon$, where ε is a (small) fixed positive number. We take the period and time interval to depend on ε, introducing a reference period $L_0>0$ and a reference time interval $[0,T_0]$ on which the solution U of KS lives. The reference space and time variables will be denoted by ξ and τ. We consider (15) in the class of periodic functions with period $L_0/\sqrt{\varepsilon}$ in x, on the time interval $[0,T_0/\varepsilon^2]$ in t.

We rescale

$$\Phi=\varepsilon\varphi,\ t=\tau/\varepsilon^2,\ x=\xi/\sqrt{\varepsilon},\ \Phi_0=\varepsilon\varphi_0,\ \Psi_0=\varepsilon^3\psi_0 \tag{17}$$

whence the Cauchy problem (15),(16) becomes after division by ε^3:

$$\varphi_\tau+\varepsilon^2\varphi_{\tau\tau}+\varphi_{\xi\xi\xi\xi}+\varphi_{\xi\xi}-2\varepsilon\varphi_{\xi\xi\tau}+\frac{1}{2}[1-\varepsilon\partial_\xi^2+\varepsilon^2\partial_\tau]\varphi_\xi^2=\varepsilon\varphi_\xi[\phi_{\xi\xi}-\varepsilon\varphi_\tau-\frac{1}{2}\varepsilon\varphi_\xi^2]_\xi. \tag{18}$$

$$\varphi(\xi,0)=\varphi_0(\xi),\qquad \varphi_\tau(\xi,0)=\psi_0(\xi). \tag{19}$$

We then look for φ as

$$\varphi=U+\varepsilon u, \tag{20}$$

so that with $\varepsilon=0$ in (18), we recover the 4th order KS equation

$$U_\tau+\frac{1}{2}U_\xi^2+U_{\xi\xi\xi\xi}+U_{\xi\xi}=0. \tag{21}$$

Our aim now is to establish a uniform bound on u which is valid for small positive ε, uniformly in ε. For u the equation reads (for convenience we keep the notation $\varphi = U + \varepsilon u$ to save space):

$$u_\tau + \varepsilon^2 u_{\tau\tau} + u_{\xi\xi\xi\xi} + u_{\xi\xi} - 2\varepsilon u_{\xi\xi\tau} + \frac{1}{2}[2u_\xi U_\xi + \varepsilon u_\xi^2] + \frac{1}{2}[-\partial_\xi^2 + \varepsilon\partial_\tau]\varphi_\xi^2 =$$
$$\varphi_\xi[\varphi_{\xi\xi} - \varepsilon\varphi_\tau - \frac{1}{2}\varepsilon\varphi_\xi^2]_\xi + 2U_{\xi\xi\tau} - \varepsilon U_{\tau\tau}. \tag{22}$$

We now consider so-called "well-prepared" L_0-periodic initial data of the form

$$\varphi(0,\xi) = \varphi_0(\xi) = U_0(\xi) + \varepsilon u_0(\xi), \tag{23}$$

$$\psi_0(\xi) = U_\tau(0,\xi) + \varepsilon u_\tau(0,\xi) = -\frac{1}{2}\Big[(U_0)_\xi\Big]^2 - (U_0)_{\xi\xi} - (U_0)_{\xi\xi\xi\xi} + \varepsilon v_0(0,\xi). \tag{24}$$

Thus U_0, u_0 are the initial data for U, u. We assume as much smoothness as needed, for simplicity take U_0, u_0 in $C^\infty([-L_0/2, L_0/2])$. Hence all the derivatives of U involved in the sequel are bounded in space and time. Note we can set $u_0(\xi) = 0$ thus assuming that both (21) and (18) start from the same initial configuration.

The basic estimate in this section comes from testing (22) with u_τ, which yields (we assume the integration to be over the main period from $-L_0/2$ to $L_0/2$ unless otherwise stated and omit the limits):

$$|u_\tau|^2 + \frac{1}{2}\frac{d}{d\tau}(\varepsilon^2|u_\tau|^2 + |u_{\xi\xi}|^2) + 2\varepsilon|u_{\xi\tau}|^2 \tag{25}$$
$$+ \int u_{\xi\xi}u_\tau d\xi + \int \frac{1}{2}(2u_\xi U_\xi + \varepsilon u_\xi^2)u_\tau\, d\xi + \int\{\frac{1}{2}[-\partial_\xi^2 + \varepsilon\partial_\tau]\varphi_\xi^2\}u_\tau d\xi$$
$$= \int\{\varphi_\xi[\varphi_{\xi\xi} - \varepsilon\varphi_\tau - \frac{1}{2}\varepsilon\varphi_\xi^2]_\xi + 2U_{\xi\xi\tau} - \varepsilon U_{\tau\tau}\}u_\tau d\xi. \tag{26}$$

We use the following notations:

$$|v|^2 = \int v(\xi)^2 d\xi, \quad |v|_\infty = \sup_\xi |v(\xi)|,$$

and we use several times the inequality

$$\|v_\xi\|_\infty \le \sqrt{L_0}|v_{\xi\xi}|,$$

which holds for each periodic $v \in H^2(-L_0, L_0)$, since for each x we have $|v_\xi(x)| = |\int_{x_0}^x v_{\xi\xi} d\xi| \le \sqrt{L_0}|v_{\xi\xi}|$, where x_0 is any point such that $v_\xi(x_0) = 0$.

Now we have to estimate the integrals in (25) and in (26). Bilinear terms in u and its derivatives are easy. Let us consider the first ones: for each $\nu > 0$ we have

$$\left|\int u_{\xi\xi}u_\tau d\xi\right| \le \frac{\nu}{2}|u_\tau|^2 + \frac{1}{2\nu}|u_{\xi\xi}|^2,$$

and

$$\left|\int u_\xi U_\xi u_\tau d\xi\right| \le |U_\xi|_\infty |u_\xi|\,|u_\tau| \le |U_\xi|_\infty\sqrt{L_0}|u_\xi|_\infty|u_\tau|$$
$$\le L_0|U_\xi|_\infty|u_{\xi\xi}|\,|u_\tau| \le L_0|U_\xi|_\infty\left(\frac{\nu}{2}|u_\tau|^2 + \frac{1}{2\nu}|u_{\xi\xi}|^2\right).$$

The other bilinear terms are similar. We therefore focus our attention on trilinear/quadrilinear terms.

(a) By the Young inequality,

$$\left|\int \varepsilon u_\xi^2 u_\tau d\xi\right| \le \varepsilon|u_\xi^2||u_\tau| \le \frac{\nu}{2}|u_\tau|^2 + \varepsilon^2\frac{1}{2\nu}|u_\xi^2|^2$$

$$\le \frac{\nu}{2}|u_\tau|^2 + \varepsilon^2\frac{1}{2\nu}L_0|u_\xi|^4_\infty \le \frac{\nu}{2}|u_\tau|^2 + \varepsilon^2\frac{1}{2\nu}(L_0)^{3/2}|u_{\xi\xi}|^4$$

(alternative proof by taking the sup norm of u_τ out of the integral, the latter controlled by the L^2 norm of $u_{\xi\tau}$).

(b)

$$\int\left(\frac{1}{2}[-\partial_\xi^2 + \varepsilon\partial_\tau]\varepsilon^2 u_\xi^2\right)u_\tau d\xi$$

$$= \int \varepsilon^2 u_\xi u_{\xi\xi} u_{\xi\tau} d\xi + \int \varepsilon^3 u_\xi u_{\xi\tau} u_\tau d\xi;$$

$$(i)\quad \left|\int \varepsilon^{3/2} u_\xi u_{\xi\xi}\varepsilon^{1/2} u_{\xi\tau} d\xi\right| \le \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2 + \frac{1}{2\nu}\varepsilon^3|u_\xi u_{\xi\xi}|^2$$

$$\le \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2 + \frac{1}{2\nu}\varepsilon^3|u_\xi|^2_\infty|u_{\xi\xi}|^2 \le \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2 + \frac{1}{2\nu}\varepsilon^3 L_0|u_{\xi\xi}|^4;$$

$$(ii)\quad \left|\int \varepsilon^3 u_\xi u_{\xi\tau} u_\tau d\xi\right| = \left|\int \varepsilon^{5/2} u_\xi u_\tau \varepsilon^{1/2} u_{\xi\tau} d\xi\right|$$

$$\le \frac{1}{2\nu}\varepsilon^5|u_\xi u_\tau|^2 + \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2 \le \frac{1}{2\nu}\varepsilon^5|u_\xi|^2_\infty|u_\tau|^2 + \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2$$

$$\le \frac{1}{2\nu}\varepsilon^5 L_0|u_{\xi\xi}|^2|u_\tau|^2 + \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2 \le \frac{1}{\nu}\varepsilon^5 L_0|u_{\xi\xi}|^4 + \frac{1}{\nu}\varepsilon^5 L_0|u_\tau|^4 + \frac{\nu}{2}\varepsilon|u_{\xi\tau}|^2.$$

(c)

$$\int\ [\varepsilon u_\xi(\varepsilon u_{\xi\xi} - \varepsilon^2 u_\tau - \frac{1}{2}\varepsilon^3 u_\xi^2)_\xi]u_\tau d\xi$$

$$= -\int \varepsilon u_{\xi\xi}[\varepsilon u_{\xi\xi} - \varepsilon^2 u_\tau - \frac{1}{2}\varepsilon^3 u_\xi^2]\ u_\tau d\xi$$

$$-\int \varepsilon u_\xi[\varepsilon u_{\xi\xi} - \varepsilon^2 u_\tau - \frac{1}{2}\varepsilon^3 u_\xi^2]\}u_{\xi\tau} d\xi;$$

$$(i)\quad \left|\int\ \varepsilon^2 u_{\xi\xi}^2 u_\tau d\xi\right|$$

$$\le \varepsilon^2\ |u_\tau|_\infty|u_{\xi\xi}|^2 \le \varepsilon^{1/2}\sqrt{L_0}|u_{\xi\tau}|\varepsilon^{3/2}|u_{\xi\xi}|^2 \le \frac{\nu}{2}\varepsilon L_0|u_{\xi\tau}|^2 + \frac{1}{2\nu}\varepsilon^3|u_{\xi\xi}|^4;$$

$$(ii)\quad -\int \varepsilon^3 u_{\xi\xi} u_\tau^2 d\xi = 2\int \varepsilon^3 u_\xi u_\tau u_{\xi\tau} d\xi$$

as above;

$$(iii)\quad \left|\int \frac{1}{2}\varepsilon^4 u_{\xi\xi} u_\xi^2 u_\tau d\xi\right|$$

$$\le \frac{1}{2}\varepsilon^4|u_\xi^2|_\infty|u_{\xi\xi}||u_\tau| \le \frac{1}{2}\varepsilon^4 L_0|u_{\xi\xi}|^3|u_\tau| \le \frac{1}{4}\{\nu|u_\tau|^2 + \frac{1}{\nu}\varepsilon^8 L_0^2|u_{\xi\xi}|^6\};$$

$$(iv)\quad \int \varepsilon^2 u_\xi u_{\xi\xi} u_{\xi\tau} d\xi$$

as above;

$$(v)\quad \int \varepsilon^3 u_\xi u_\tau u_{\xi\tau} d\xi$$

as above;

$$(vi)\quad \left|\int \frac{1}{2}\varepsilon^4 u_\xi u_\xi^2 u_{\xi\tau} d\xi\right|$$

$$\le \frac{1}{2}\varepsilon^{7/2}|u_\xi|^3_\infty\varepsilon^{1/2}\sqrt{L_0}|u_{\xi\tau}| \le \frac{1}{2}\left(\frac{\nu}{2}\ \varepsilon|u_{\xi\tau}|^2 + \frac{L_0^4}{2\nu}\varepsilon^7|u_{\xi\xi}|^6\right).$$

Eventually, from (25), (26) we get

$$\begin{aligned}&|u_\tau|^2 + \frac{1}{2}\frac{d}{d\tau}(\varepsilon^2|u_\tau|^2 + |u_{\xi\xi}|^2) + 2\varepsilon|u_{\xi\tau}|^2 \\ &\le C_1\nu\left(|u_\tau|^2 + \varepsilon|u_{\xi\tau}|^2\right) + \frac{C_2}{\nu}\left(|u_{\xi\xi}|^2 + \varepsilon^2|u_{\xi\xi}|^4 + \varepsilon^7|u_{\xi\xi}|^6 + \varepsilon^5|u_\tau|^4\right).\end{aligned} \tag{27}$$

We need the following technical lemma.

Lemma 1. *Assume that a family of nonnegative functions* $A_\varepsilon \in C^1([0,T_0])$, $\varepsilon \in (0,1]$*, satisfies*

$$A_\varepsilon' \le C_0 + C_1A_\varepsilon + C_2\varepsilon A_\varepsilon^2 + C_3\varepsilon^2A_\varepsilon^3, \quad A_\varepsilon(0) \le A_0$$

with positive constants A_0*,* C_i*, independent of* ε*. Then there exist* $\varepsilon_0 > 0$*,* $K_0 > 0$ *such that* $A_\varepsilon(\tau) \le K_0$ *for all* $\tau \in [0,T_0]$ *whenever* $0 < \varepsilon \le \varepsilon_0$*.*

Proof of Lemma 1. Set $B(\tau) = A_\varepsilon(\tau)e^{-C_1\tau}$. (We drop the subindex ε for notational convenience). Then

$$B'(\tau) \le C_0e^{-C_1\tau} + C_2\varepsilon B^2(\tau)e^{C_1\tau} + C_3\varepsilon^2B^3(\tau)\, e^{2C_1\tau},$$

and integrating between 0 and τ we find

$$B(\tau) \le A_0 + \frac{C_0}{C_1}(1-e^{-C_1\tau}) + \varepsilon\frac{C_2}{C_1}(e^{C_2\tau}-1)\sup_{0\le s\le\tau}B^2(s) + \varepsilon^2\frac{C_3}{2C_1}(e^{2C_2\tau}-1)\sup_{0\le s\le\tau}B^3(s)$$

which implies

$$\|B\|_{L^\infty(0,T_1)} \le \left(A_0 + \frac{C_0}{C_1}\right) + \varepsilon\frac{C_2}{C_1}(e^{C_2T_1}-1)\|B\|^2_{L^\infty(0,T_1)} + \varepsilon^2\frac{C_3}{2C_1}(e^{2C_2T_1}-1)\|B\|^3_{L^\infty(0,T)}$$

for every $T_1 \le T_0$.

Now fix $\varepsilon_0 > 0$ such that the curve in the plane (x,y)

$$y = \left(A_0 + \frac{C_0}{C_1}\right) + \varepsilon_0\frac{C_2}{C_1}(e^{C_2T_0}-1)x^2 + \varepsilon_0^2\frac{C_3}{2C_1}(e^{2C_2T_0}-1)x^3$$

intersects the diagonal $y = x$ in two points (k_0,k_0) and (k_1,k_1), with $k_0 < k_1$. Then for every $T_1 \le T_0$ and $\varepsilon \le \varepsilon_0$ we have either $\|B\|_{L^\infty(0,T_1)} \le k_0$ or $\|B\|_{L^\infty(0,T_1)} \ge k_1$. But $\limsup_{T\to 0}\|B\|_{L^\infty(0,T)} \le A_0 + C_0/C_1 < k_0$ and $T \mapsto \|B\|_{L^\infty(0,T)}$ cannot have jumps, therefore $\|B\|_{L^\infty(0,T_1)} \le k_0$ for every $T_1 \le T_0$. In particular, $\|B\|_{L^\infty(0,T_0)} \le k_0$ and consequently $\|A_\varepsilon\|_{L^\infty(0,T_0)} \le k_0e^{C_1T_0} := K_0$, provided $\varepsilon \le \varepsilon_0$. □

Here we can take

$$\begin{aligned}A_\varepsilon(\tau) &= \varepsilon^2|u_\tau(\tau,\cdot)|^2 + |u_{\xi\xi}(\tau,\cdot)|^2,\\ A_\varepsilon(0) &= \varepsilon^2|u_\tau(0,\cdot)|^2 + |u_{\xi\xi}(0,\cdot)|^2 = \varepsilon^2|v_0|^2 + |u_0''|^2,\end{aligned}$$

choosing $\nu > 0$ small enough in (27). The L^2 estimate for $u_{\xi\xi}$ then implies a sup-norm estimate, that yields the following theorem.

Theorem 2. *There exists* $\varepsilon_0 > 0$ *such that, whenever* $0 < \varepsilon < \varepsilon_0$*,* $|u(\xi,\tau)| \le C$ *for all* (ξ,τ) *in* $[-L_0/2, L_0/2] \times [0,T_0]$*. The number* ε_0 *and the bound* C *depend only on* U_0*,* u_0*,* L_0 *and* T_0*.*

Returning to the original problem, $\Phi = \varepsilon\varphi$, $\varphi = U + \varepsilon u$, we state the main result of the paper:

Corollary 1.

$$\max|\Phi(x,t) - \varepsilon U(x\sqrt{\varepsilon}, t\varepsilon^2)| \le C\,\varepsilon^2 \tag{28}$$

for

$$|x| \le \frac{L_0}{2\sqrt{\varepsilon}},\ 0 \le t \le \frac{T_0}{\varepsilon^2}. \tag{29}$$

4. **Numerical simulation.** The direct numerical simulation presented in this section was intended primarily to observe at least a qualitative closeness of the solutions of the weakly nonlinear κ-θ system (6-7) to the solutions of KS for moderately small values of ε, and, secondly, to show that (6-7) is capable of generating a variety of dynamics including cellular structures and turbulence.

Note that the asymptotic convergence to KS by Theorem 2 is in terms of the instability parameter ε for a fixed time interval, and not in terms of $t \to \infty$. Since the spatial interval $L = L_0/\sqrt{\varepsilon}$ and the instability parameter in (11) depend on $\varepsilon \ll 1$, the time that it takes for a given solution to sufficiently approach the *attractor* may be extremely large. Therefore, we cannot at this point make any rigorous claims concerning the final pattern that is generated, and, therefore, we need the numerical solution to investigate the similarity of fully developed dynamics between (6-7) and the KS equation. It is also interesting to observe the behavior exhibited by the system for $\varepsilon \sim 1$ where the result of Sec. 3 is not valid.

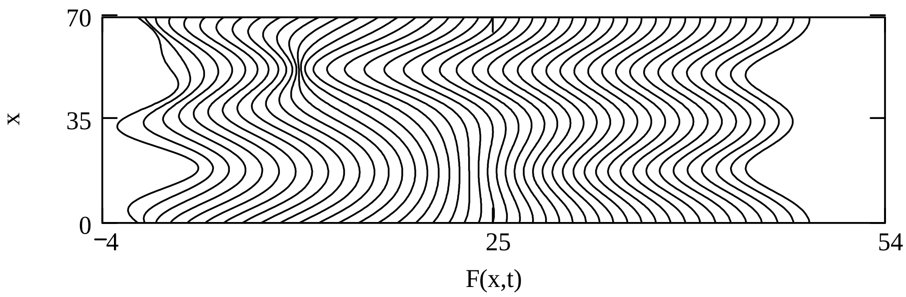

FIGURE 2. Front evolution generated by (6-7) for $\varepsilon = .04$

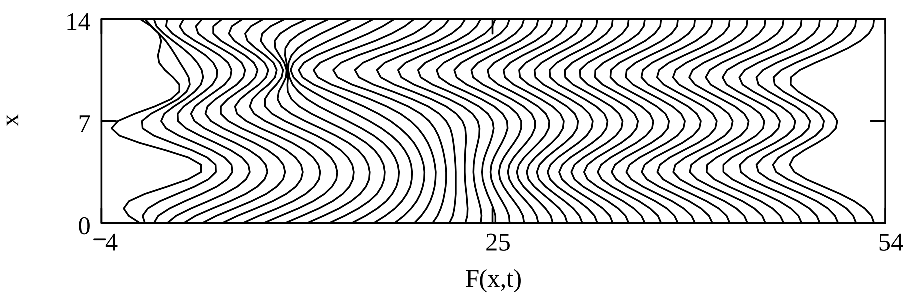

FIGURE 3. Front evolution generated by KS equation

Again, one of the advantages of system (6-7) in comparison with either QS or KS equations is its extreme simplicity from the computational point of view. We employed a straightforward explicit finite-difference scheme for the preliminary numerical simulations of (6-7) presented here. The numerical solution of KS was "borrowed" from [2].

The initial configuration $\Phi_0(x)$ of the front was set to be identical with that of KS, while the initial (reduced) temperature was chosen to be exactly compatible with KS, i.e.

$$\Theta_0(x) = \alpha\Phi_0''(x) + \Phi_0''''(x)$$

Numerical solutions were carried out on the interval $L_0 = 14$ for KS, and for the appropriately rescaled (as prescribed by the asymptotics in Sec 3 intervals $L = L_0/\sqrt{\varepsilon}$ in the physical coordinates for the $\kappa - \theta$ system. The fixed time interval for KS was chosen $T_0 = 20$, the time interval for the $\kappa-\theta$ system was also appropriately rescaled.

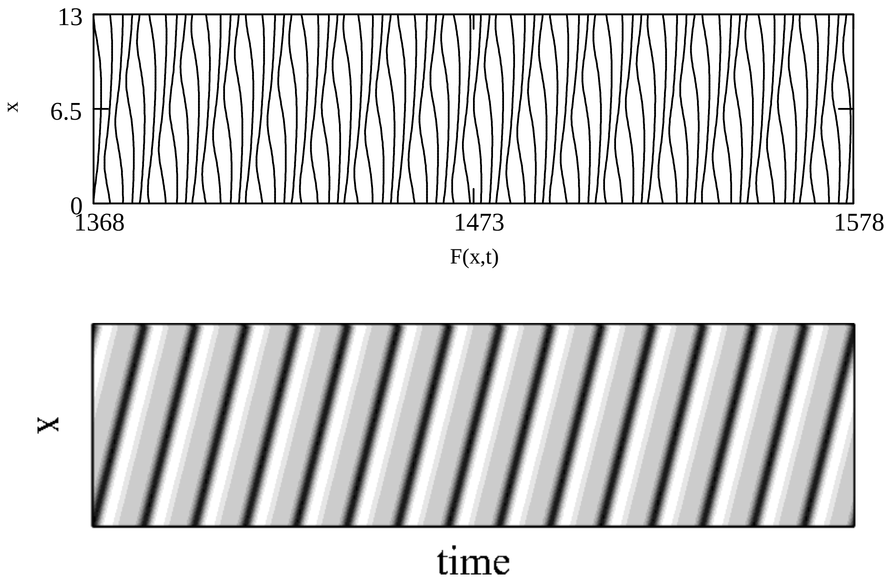

FIGURE 4. Front evolution (top), and interface temperature trace (bottom) generated by (6-7) for $\varepsilon = 1$, $L = 13$. Darker areas correspond to lower temperatures.

Fig. 2 shows consecutive front positions generated by (6-7) for $\varepsilon = .04$ while Fig. 3 describes evolution by KS (for comparison purposes the solution of KS is presented in the x, t coordinates)

Finally, Figs. 4-6 represent several dynamical regimes generated by the $\kappa - \theta$ system. First one obtains steady propagation of a single cell for the width (period) just above the instability threshold. Increasing the interval slightly we observe a spinning cell (Fig. 4); recall that the physical interpretation of the periodic problem is the dynamics on a cylindrical sample.

Next we observe a time periodic regime (Fig. 5) with the cells alternating their phases in a rather abrupt but entirely orderly fashion. Fig. 6 depicts a fully developed turbulent regime generated by the weakly nonlinear $\kappa - \theta$ model.

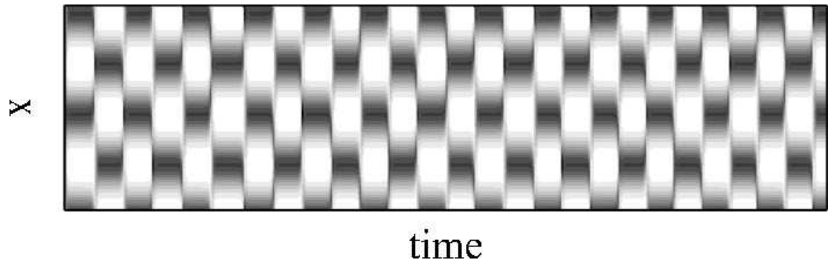

FIGURE 5. Interface temperature trace (bottom) generated by (6-7)for $\varepsilon = 1,\ L = 15$. Darker areas correspond to lower temperatures.

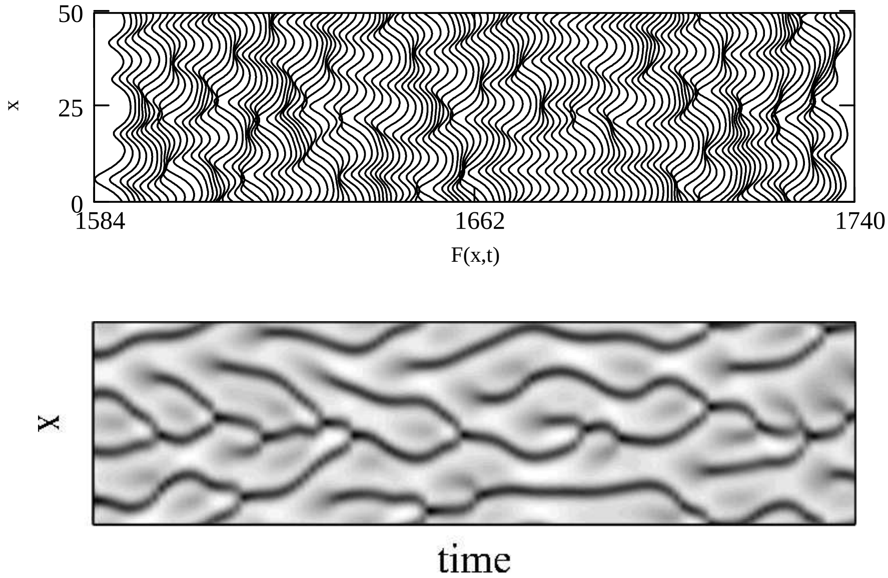

FIGURE 6. Turbulent dynamics generated by the weakly nonlinear $\kappa - \theta$ model: interface snapshots (top) and its temperature trace (bottom).

5. **Conclusion.** One of the advantages of the full $\kappa - \theta$ model or its quasi-steady approximation QS, as we argued above, is that including the interface temperature may better reflect the properties of the original free-interface problem modeling cellular flames, while, at the same time, as we hope, they may serve as a basis for the description of the flame interaction with the background flow in the full description of the premixed gas combustion.

The rigorous results in Sections 2 and 3 above show that for the weakly nonlinear case, the periodic problem is well-posed, and its solutions are asymptotically close to the solutions of the KS equation in the sense of the instability parameter ε for any fixed time interval. At the same time, we have demonstrated numerically a qualitative similarity as $t \to \infty$ of fully developed dynamical patterns between the $\kappa - \theta$ model and the KS equation. Indeed, the $\kappa - \theta$ model is capable of generating cellular structures and turbulence. A more detailed numerical study that we intend to present in the near future shows that the $\kappa - \theta$ model generates a remarkable variety of dynamical patterns.

It would be interesting to verify that dissipativity and existence of a compact attractor of finite Hausdorff dimension, that were established rigorously for KS [12], and QS [1], can be demonstrated for the weakly nonlinear $\kappa - \theta$ model as well.

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