Data $f(x)=e^x$, si dimostra che $1+x \le e^x \le 1+xe^x \quad \forall x\in\mathbb{R}$

ed essendo $e^x \nearrow$, si ha che $1+x \le e^x \le 1+x\cdot e \quad \forall x\le 1$

Esercizio

Provare che i) $\lim_{x\to 0} e^x = 1$ ii) $\lim_{x\to x_0} e^x = e^{x_0} \quad \forall x_0\in\mathbb{R}$

iii) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ iv) $\lim_{x\to x_0} x^n = x_0^n$

dim

$$\frac{1-\cos x}{x^2} = \frac{1-\cos 2\cdot\frac{x}{2}}{x^2} = \frac{1-\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}{x^2} =$$

$$= \frac{\sin^2\frac{x}{2}+\sin^2\frac{x}{2}}{x^2} = 2\,\frac{\sin^2 x/2}{x^2} = 2\cdot\frac{1}{4}\cdot\frac{\sin^2 x/2}{\left(\frac{x}{2}\right)^2} =$$

$$= \frac{1}{2}\cdot\left(\frac{\sin\frac{x}{2}}{x/2}\right)^2 \xrightarrow[x/2\to 0]{} \frac{1}{2}\cdot 1^2 = \frac{1}{2}$$

Come abbiamo osservato $1+x \le e^x \le 1+e\cdot x \quad \forall x\le 1$

da cui segue $\lim_{x\to 0}(1+x) \le \lim_{x\to 0} e^x \le \lim_{x\to 0}(1+e\cdot x)$

" " " $\lim_{x\to 0} e^x = 1$

Inoltre $e^x - e^{x_0} = e^{x_0}\left(e^{x-x_0}-1\right)$ e dunque

$\lim_{x\to x_0} e^x = e^{x_0} \iff \lim_{x\to x_0} e^{x_0}\left(e^{x-x_0}-1\right) = 0 \iff \lim_{x\to x_0} e^{x-x_0} = 1$

$\iff \lim_{x-x_0\to 0} e^{x-x_0} = 1 \iff \lim_{y\to 0} e^y = 1$

Proviamo che $\lim_{x\to x_0} x^2 = x_0^2 \iff \lim_{x\to x_0}\left(x^2-x_0^2\right) = \lim_{x\to x_0}(x-x_0)(x+x_0) = 0$

Ma $\lim_{x\to x_0}(x+x_0) = 2x_0 \in\mathbb{R} \Rightarrow \exists\, U\in\mathcal{U}_{x_0}: \ |x+x_0|\le k \ \forall x\in U$

Ma $\lim_{x\to x_0}(x-x_0) = 0$

$\Rightarrow \lim_{x\to x_0}(x-x_0)(x+x_0) = 0$ (prodotto tra una f.ne limitata ed una infinitesima)

Suppongo di aver provato che $\lim_{x\to x_0} x^n = x_0^n$

Vogliamo provare che $\lim_{x\to x_0} x^{n+1} = x_0^{n+1}$ cioè

" " " $\lim_{x\to x_0}\left(x^{n+1}-x_0^{n+1}\right) = \lim_{x\to x_0}(x-x_0)\left(x^n + x^{n-1}x_0 + \dots + x_0^n\right) \overset{?}{=} 0$

Ma $\lim_{x\to x_0}\left(x^n + x^{n-1}x_0 + \dots + x\cdot x_0^{n-1} + x_0^n\right) = n\, x_0^n \in \mathbb{R}$

e dunque $\exists\, U\in\mathcal{U}_{x_0}: |f(x)| = |x^n + x^{n-1}x_0 + \dots + x_0^n| \le k \quad \forall x\in U$

Inoltre $\lim_{x\to x_0}(x-x_0)=0$

$\Rightarrow \lim_{x\to x_0}\left(x^{n+1}-x_0^{n+1}\right)=0$ ovvero la Tesi ///

Oss: proviamo, attraverso la definizione, che
$\lim_{x\to x_0}(x^2-x_0^2)=\lim_{x\to x_0}(x-x_0)(x+x_0)=0$. Devo provare

$$\forall\varepsilon>0\ \exists\,\delta=\delta(\varepsilon,x_0)>0 : \forall x\in\mathbb{R}\quad 0<|x-x_0|<\delta \Rightarrow |x^2-x_0^2|<\varepsilon$$

ora, se $0<x-x_0<\delta$ allora $x_0-\delta+x_0 < x+x_0 < x_0+\delta+x_0 = 2x_0+\delta$

allora $2x_0-\delta<x+x_0<2x_0+\delta$ allora $|x+x_0|\le \max\{|2x_0-\delta|,|2x_0+\delta|\}$

allora preso $0<\delta<1$ (δ può essere scelto piccolo a piacere)

$$|x+x_0|\le\max\{|2x_0-1|,|2x_0+1|\}=k$$

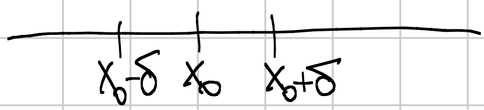
e dunque

$$\forall\varepsilon>0\ \exists\,\delta>0,\delta<1 \mid 0<|x-x_0|<\delta \Rightarrow |x^2-x_0^2| = |x-x_0|\,|x+x_0| \le k\cdot|x-x_0| < k\cdot\delta=\varepsilon$$

ovvero

$$\forall\varepsilon>0\ \exists\,\delta\in]0,1[: \quad 0<|x-x_0|<\delta\Rightarrow |x^2-x_0^2|<k\delta=\varepsilon$$

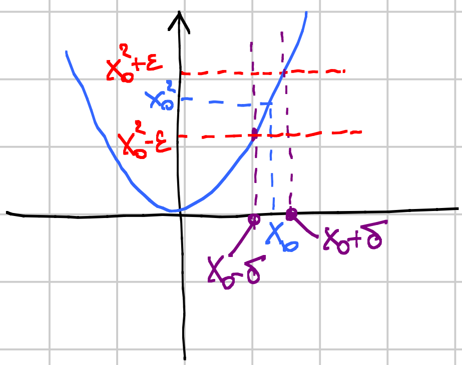
$$\delta=\varepsilon/k$$



$x\in]x_0-\delta,x_0+\delta[$

$\iff x_0-\delta<x<x_0+\delta$

$\iff x_0+x_0-\delta<x+x_0<x_0+\delta+x_0$



ovvero faccio in modo che

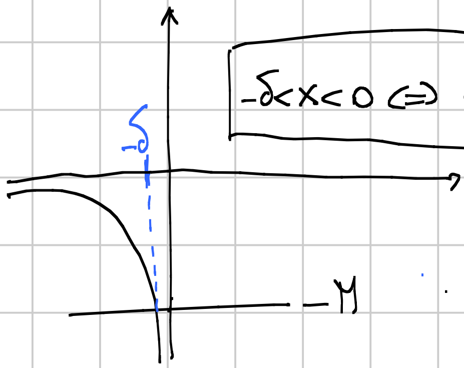
$\{(x,f(x)): x\in]x_0-\delta,x_0+\delta[\}\subseteq\,]x_0-\delta,x_0+\delta[\,\times\,]x_0^2-\varepsilon,x_0^2+\varepsilon[$

Esercizio Provare che $\nexists \lim_{x\to 0} 1/x$

dim

Cominciamo osservare che $\underbrace{\lim_{x\to 0^-} \frac{1}{x} = -\infty}_{(i)} \neq \underbrace{+\infty = \lim_{x\to 0^+} \frac{1}{x}}_{(ii)}$

$(i) \Leftrightarrow \forall M>0\ \exists \delta>0 : -\delta<x<0 \Rightarrow \frac{1}{x} < -M$



$\boxed{-\delta<x<0 \Leftrightarrow -\frac{1}{\delta} > \frac{1}{x}}$

$\forall M>0\ \exists \delta = \frac{1}{M}\quad -\delta<x<0$

$\Rightarrow \frac{1}{x} < -\frac{1}{\delta} = -M$

ASINTOTO

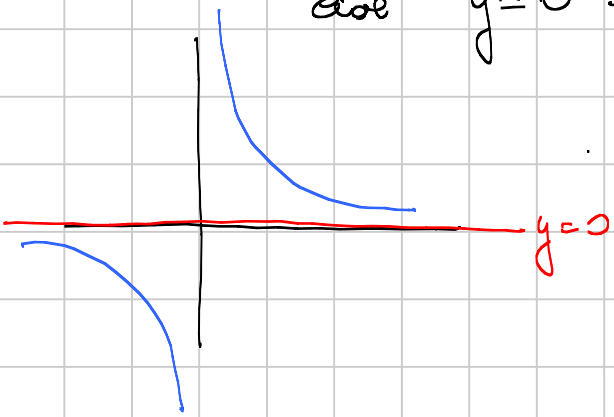
f ha un asintoto $r(x)$ (retta) per $x \to x_0$ se

$$\lim_{x \to x_0} \left(f(x) - r(x) \right) = 0$$

Asintoto orizzontale

$f(x) = \frac{1}{x}$ $\quad \lim_{x \to \pm\infty} \frac{1}{x} = 0$, cioè $\lim_{x \to \pm\infty} \left(\frac{1}{x} - 0 \right) = 0$

cioè $y = 0$ è asintoto orizzontale



Asintoto obliquo

$f(x) = \frac{2x^3}{1+x^2+|x|}$ proveremo che $\lim_{x \to +\infty} \left(f(x) - 2x + 2 \right) = 0$

$$\lim_{x \to -\infty} \left(f(x) - 2x - 2 \right) = 0$$

ovvero, posto $r(x) = 2x - 2$ $\quad \lim_{x \to +\infty} \left(f(x) - r(x) \right) = 0$

$t(x) = 2x + 2$ $\quad \lim_{x \to -\infty} \left(f(x) - t(x) \right) = 0$

Asintoto verticale

$f(x) = \frac{1}{x}$ $\quad x = 0$ è asintoto verticale

poiché $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ $\quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$

Ricerca dell'asintoto obliquo

Calcoliamo l'eq. dell'asintoto obliquo di

$$f(x)=\frac{2x^3}{1+|x|+x^2}$$

per $x \to +\infty$ $(-\infty)$

dim

$\boxed{+\infty}$ cerco, se esiste, una retta $r(x)=ax+b$ t.c.

$$\lim_{x\to+\infty}\left(f(x)-ax-b\right)=0$$

$$\Leftrightarrow \lim_{x\to+\infty} x\left(\frac{f(x)}{x}-a-\frac{b}{x}\right)=0 \qquad \text{ma } \lim_{x\to+\infty} x=+\infty$$

$$\Rightarrow \lim_{x\to+\infty}\left(\frac{f(x)}{x}-a-\frac{b}{x}\right)=0 \Rightarrow \lim_{x\to+\infty}\frac{f(x)}{x}=a$$

(ho scoperto che se $\exists \lim_{x\to+\infty}\left(f(x)-ax-b\right)=0$ allora $\lim_{x\to+\infty}\frac{f(x)}{x}=a$)

$$\lim_{x\to+\infty}\frac{2x^{\cancel{3}\,2}}{1+|x|+x^2}\cdot\frac{1}{\cancel{x}}=\lim_{x\to+\infty}\frac{2\cancel{x^2}}{\cancel{x^2}\left(1+\frac{1}{|x|}+\frac{1}{x^2}\right)}=2$$

dunque $\boxed{a=2}$

Se $\lim_{x\to+\infty}\left(f(x)-2x-b\right)=0$

allora $\lim_{x\to+\infty}\left(f(x)-2x\right)=b$

Calcolo, se esiste, $\lim_{x\to+\infty}\left(\frac{2x^3}{1+|x|+x^2}-2x\right)=\lim_{x\to+\infty}\frac{\cancel{2x^3}-2x-2x|x|-\cancel{2x^3}}{1+|x|+x^2}$

$$=\lim_{x\to+\infty} 2\cancel{x^2}\cdot\frac{-1-1/x}{\cancel{x^2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)}=2\cdot\frac{-1}{1}=-2$$

allora $b=-2$

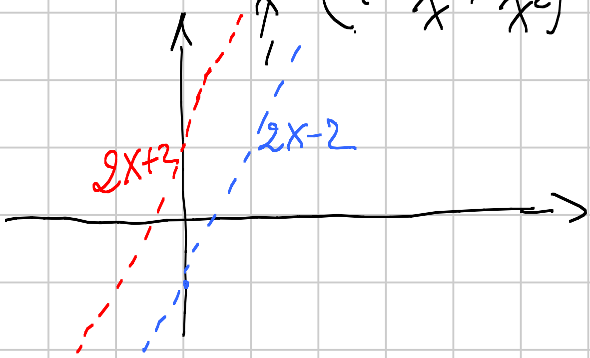
e dunque $r(x)=2x-2$ è l'asintoto obliquo di f per $x\to+\infty$

$\boxed{-\infty}$ $\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{2x^{3}}{1+|x|+x^2}\cdot\frac{1}{x}$

$= \lim_{x\to-\infty} \frac{2\cancel{x^2}}{\cancel{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}\right)} = 2 \Rightarrow \boxed{a=2}$

$\lim_{x\to-\infty}\left(f(x)-2x\right) = \lim_{x\to-\infty} \frac{\cancel{2x^3}-2x+2x^2-\cancel{2x^3}}{1-x+x^2}$

$= \lim_{x\to-\infty} \frac{2\cancel{x^2}\left(1-\frac{2}{x}\right)}{\cancel{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}\right)} = 2 \Rightarrow \boxed{b=2}$



Oss può accadere che $\exists \lim_{x\to+\infty} \frac{f(x)}{x} = a \in \mathbb{R}$

ma $\nexists \lim_{x\to+\infty} (f(x)-ax) = b \in \mathbb{R}$

per esempio $f(x) = x + \log(x^2+1)$

$\frac{f(x)}{x} = 1 + \frac{\log(1+x^2)}{x} \xrightarrow[x\to+\infty]{} 1+0 = 1 = a$

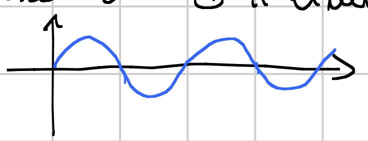
però

$\lim_{x\to+\infty} (f(x)-ax) = \lim_{x\to+\infty} \left(\cancel{x}+\log(1+x^2)-\cancel{x}\right) = +\infty \notin \mathbb{R}$

Esercizio provare che $\nexists \lim_{x\to+\infty} \operatorname{sen} x$

dim

$\operatorname{sen} x$ oscilla tra -1 e 1, e quindi non mi aspetto che $\exists$ il limite

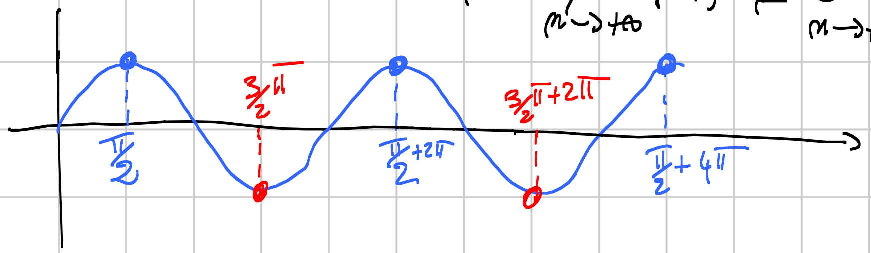


Il Teorema che lega il limite di funzioni al limite di 7
successioni si può utilizzare nella forma contronominale
ovvero

"$\nexists \lim_{x \to x_0} f(x) \iff \exists$ due successioni (x_n) e (y_n)"

t.c. $\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} y_n = x_0$

ma $\lim_{n \to +\infty} f(x_n) \neq \lim_{n \to +\infty} f(y_n)$



$(x_n) = \left(\frac{\pi}{2} + 2\pi \cdot n\right)$ $\qquad$ $(y_n) = \left(\frac{3}{2}\pi + 2\pi \cdot n\right)$

$x_n \xrightarrow[n \to +\infty]{} +\infty$ $\qquad$ $y_n \xrightarrow[n \to +\infty]{} +\infty$

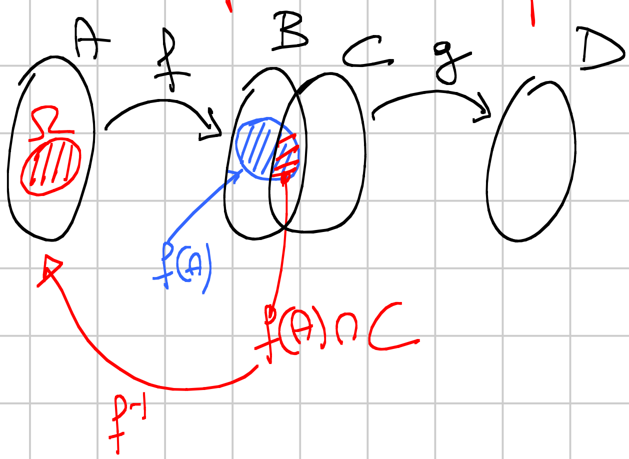
$$\lim_{n \to +\infty} \operatorname{sen}(x_n) = \lim_{n \to +\infty} \operatorname{sen}\left(\frac{\pi}{2} + n \cdot 2\pi\right) = \lim_{n \to +\infty} \operatorname{sen}\frac{\pi}{2} = 1$$

$$\neq$$

$$\lim_{n \to +\infty} \operatorname{sen}(y_n) = \lim_{n \to +\infty} \operatorname{sen}\left(\frac{3}{2}\pi + n\, 2\pi\right) = \lim_{n \to +\infty} \operatorname{sen}\frac{3}{2}\pi = -1$$

$$\Rightarrow \nexists \lim_{x \to +\infty} \operatorname{sen} x$$

Dominio funzione composta



$f: A \to B$

$g: C \to D$

$$\Omega = f^{-1}(f(A) \cap C) = \{x \in A : f(x) \in f(A) \cap C\} = \{x \in A : f(x) \in C\}$$

Esercizio Calcolare il dominio di $h(x)=\log\left(\sqrt{100-x^2}-1\right)$

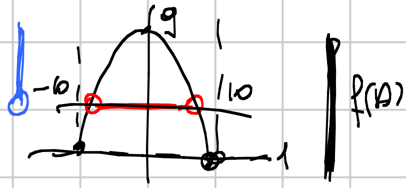
dim

$$h(x)=(g\circ f)(x) \qquad f(x)=\sqrt{100-x^2}-1 \qquad g(y)=\log(y)$$

$f: A \to \mathbb{R}$

$$\boxed{A=\{x : f(x)\in\mathbb{R}\}=\{x : \sqrt{100-x^2}-1\in\mathbb{R}\} = \{x : 100-x^2\geq 0\}=[-10,10]}$$

$$f(A)=[-1,9]$$



$g: C\to D \qquad C=\{x : \log x\in\mathbb{R}\} =]0,+\infty[$

$$f(A)\cap C =]0,9]$$

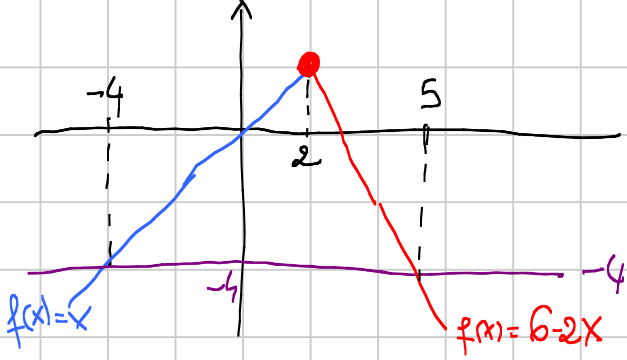
$$\Omega = f^{-1}(]0,9]) = \{x\in[-10,10] : \sqrt{100-x^2}-1>0\}$$
$$=\{x\in[-10,10] : \sqrt{100-x^2}>1\}$$
$$=\{x\in[-10,10] : 99>x^2\}$$
$$=]-\sqrt{99},\sqrt{99}[\qquad \square$$

Esercizio Calcolare $(g\circ f)(x)$ dove

$$f(x)=\begin{cases} x & \text{se } x<2 \\ 6-2x & \text{se } x\geq 2\end{cases} \qquad g(x)=\begin{cases} \operatorname{sen} 2x & \text{se } x<-4 \\ \cos(x+1) & \text{se } x\geq -4\end{cases}$$

dim

$$(g\circ f)(x)=\begin{cases} \operatorname{sen}(2f(x)) & \text{se } f(x)<-4 \\ \cos(f(x)+1) & \text{se } f(x)\geq -4\end{cases}$$



$$f=\begin{cases} x & x<2 \\ 6-2x & x\geq 2\end{cases}$$

$$(g\circ f)(x)=\begin{cases} \operatorname{sen}(2x) & x<-4 \\ \cos(x+1) & -4\leq x<2 \\ \cos(6-2x+1) & 2\leq x<5 \\ \operatorname{sen}(2(6-2x)) & 5\leq x\end{cases}$$

9

può essere utile fare qualche esperimento

$$f=\begin{cases} x, & x<2 \\ 6-2x, & 2\leq x\end{cases} \qquad g(y)=\begin{cases} \operatorname{sen}(2y) & \text{se } y<-4 \\ \cos(y+1) & \text{se } -4\leq y\end{cases}$$

$$g(f(x))=\begin{cases} \operatorname{sen}(2f(x)) & f(x)<-4 \\ \cos(f(x)+1) & -4\leq f(x)\end{cases}$$

$x=-5<2 \Rightarrow f(-5)=-5<-4$ $\quad g(f(-5))=\operatorname{sen}(2f(-5))=\operatorname{sen}(2\cdot(-5))=\operatorname{sen}(-10)$

$x=1<2 \Rightarrow f(1)=1>-4$ $\quad g(f(1))=\cos(f(1)+1)=\cos(2)$

$x=4>2 \Rightarrow f(4)=6-2\cdot 4=-2>-4$ $\quad g(f(4))=\cos(f(4)+1)=\cos(-2+1)=\cos 1$

$x=6>2 \Rightarrow f(6)=6-2\cdot 6=-6<-4$ $\quad g(f(6))=\operatorname{sen}(2f(6))=\operatorname{sen}(-12)$

Esercizio (x casa)

Calcolare $\operatorname{arctg} x+\operatorname{arctg}\frac{1}{x}$ $\begin{cases} \text{se } x>0 \\ \text{se } x<0\end{cases}$

dim

si tratta di calcolare $\operatorname{arctg}\alpha+\operatorname{arctg}\beta$

$$\operatorname{Tg}(\alpha+\beta)=\frac{\operatorname{sen}(\alpha)\cos\beta+\operatorname{sen}\beta\cos\alpha}{\cos\alpha\cos\beta-\operatorname{sen}\alpha\operatorname{sen}\beta}=$$

$$=\frac{\operatorname{Tg}\alpha+\operatorname{Tg}\beta}{1-\operatorname{Tg}\alpha\operatorname{Tg}\beta} \Rightarrow \alpha+\beta=\operatorname{arctg}\left(\frac{\operatorname{Tg}\alpha+\operatorname{Tg}\beta}{1-\operatorname{Tg}\alpha\operatorname{Tg}\beta}\right)$$

e dunque, posto $\alpha=\operatorname{arctg} x$ $\beta=\operatorname{arctg}\frac{1}{x}$ si ha

$$\operatorname{arctg} x+\operatorname{arctg}\frac{1}{x}=\operatorname{arctg}\left(\frac{x+1/x}{1-x\cdot 1/x}\right)=\begin{cases} \frac{\pi}{2} & x>0 \\ -\frac{\pi}{2} & x<0\end{cases}$$

formalmente è poco corretto