Esercizio Calcolare

$$\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n+n}}+\frac{1}{\sqrt{n+(n-1)}}+\dots+\frac{1}{\sqrt{n+2}}+\frac{1}{\sqrt{n+1}}\right)$$

divo

$$\lim_{n\to+\infty}\frac{1}{\sqrt{n+k}}=0 \qquad k=1,2,3,\dots,n$$

Non si può concludere che la somma tende a 0 infatti

$$\underbrace{n\frac{1}{\sqrt{n+n}}}_{a_n}\le\underbrace{\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\dots+\frac{1}{\sqrt{n+n}}}_{n \text{ termini}}\le\underbrace{n\cdot\frac{1}{\sqrt{n+1}}}_{c_n}$$

$$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}\frac{n}{\sqrt{n}}\cdot\frac{1}{\sqrt{2}}=\lim_{n\to+\infty}\frac{\sqrt{n}}{\sqrt{2}}=+\infty$$

$$\lim_{n\to+\infty}c_n=\lim_{n\to+\infty}\frac{n}{\sqrt{n}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n}}=+\infty\cdot 1=+\infty$$

$$\Rightarrow\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\dots+\frac{1}{\sqrt{n+n}}\right)=+\infty$$

Esercizio Calcolare $\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^3+1}}+\frac{1}{\sqrt{n^3+2}}+\dots+\frac{1}{\sqrt{n^3+n}}\right)$

divo

$$\underbrace{\frac{n^2}{\sqrt{n^3+n}}}_{a_n}\le\underbrace{\frac{1}{\sqrt{n^3+1}}+\frac{1}{\sqrt{n^3+2}}+\dots+\frac{1}{\sqrt{n^3+n}}}_{n \text{ termini}}\le\underbrace{\frac{n^2}{\sqrt{n^3+1}}}_{c_n}$$

$$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}\frac{n}{n^{3/2}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n}}=0$$

$$\lim_{n\to+\infty}c_n=\lim_{n\to+\infty}\frac{n}{n^{3/2}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n^3}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n^3}}=0$$

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$$\Rightarrow\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^3+1}}+\dots+\frac{1}{\sqrt{n^3+n}}\right)=0$$

FONDAMENTALE $\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e \in]2,3[$

OSS: questa è una forma indeterminata del tipo

$1 \leq 1^\infty \leq +\infty$

Teorema (criterio della radice n-esima)

$(a_n)_n$ successione reale, $a_n \geq 0 \ \forall n$, $\exists \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$

1) $0 \leq \ell < 1 \Rightarrow \lim_{n\to+\infty} a_n = 0$

ii) $1 < \ell \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

dim

i) $0 \leq \ell < 1$ $\quad \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$

$\forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N} : \forall n > \bar{n} \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon$

$\frac{\ell+1}{2} = \ell + \varepsilon$

$\varepsilon = \frac{\ell+1}{2} - \ell = \frac{1-\ell}{2}$

$\Rightarrow \varepsilon = \frac{1-\ell}{2} \ \exists \bar{n} = \bar{n}\left(\frac{1-\ell}{2}\right) \in \mathbb{N} \ \forall n > \bar{n} \quad 0 < \sqrt[n]{a_n} < \frac{\ell+1}{2} = \ell + \frac{1-\ell}{2}$

$\Rightarrow \forall n > \bar{n} \quad 0 < a_n < \left(\frac{\ell+1}{2}\right)^n$

ma $\lim_{n\to+\infty} \left(\frac{\ell+1}{2}\right)^n = 0$ $\quad \Rightarrow \lim_{n\to+\infty} a_n = 0$

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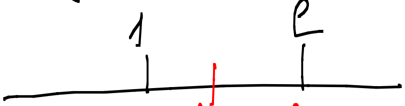
ii) considero $\ell > 1$ $\quad \ell \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N} \ \forall n > \bar{n} \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon$



$\ell - \varepsilon = \frac{1+\ell}{2}$

$\varepsilon = \ell - \frac{1+\ell}{2} = \frac{\ell-1}{2}$

$\Rightarrow \varepsilon = \frac{\ell-1}{2} \ \exists \bar{n} \in \mathbb{N} : \forall n > \bar{n} \quad \underset{\ell - \frac{\ell-1}{2}}{\overset{\shortparallel}{\frac{1+\ell}{2}}} < \sqrt[n]{a_n}$

$\Rightarrow \forall n > \bar{n} \quad \left(\frac{1+\ell}{2}\right)^n < a_n$ $\xRightarrow{\text{Teorema Confronto}}$ $\lim_{n\to+\infty} a_n = +\infty$

ma $\lim_{n\to+\infty}\left(\frac{1+\ell}{2}\right)^n = +\infty$ poiché $\frac{1+\ell}{2} > 1$ $\square$

Esercizio Calcolare $\lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^n$

$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n^2}$

dim

Sono due limiti della forma 1^∞

$a_n = \left(1+\frac{1}{n^2}\right)^n$ osservo che $\left(1+\frac{1}{n^2}\right)^{n^2} = e_{n^2}$

dove $e_n = \left(1+\frac{1}{n}\right)^n$

$e_1 = (1+1)^1 \quad e_2 = \left(1+\frac{1}{2}\right)^2 \; \cdots\cdots \; e_n$

$e_1 = (1+1)^1 \quad e_4 = \left(1+\frac{1}{4}\right)^4$

Dunque $\left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)_n$ è una sottosuccessione di $\left(1+\frac{1}{n}\right)^n$

$\Rightarrow \exists \lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

$\Rightarrow \exists \lim_{n\to+\infty}\sqrt[n]{\left(1+\frac{1}{n^2}\right)^{n^2}} = \lim_{n\to+\infty}\sqrt[n]{e} = 1$

$\qquad \| $

$\lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^n$

oppure, ricordando che $\left(1+\frac{1}{n}\right)^n$ è crescente, si ha

$1 \leq \left(1+\frac{1}{n^2}\right)^{n^2} \leq e \qquad \forall n$

$\Downarrow$

$1 = \sqrt[n]{1} \leq \left(1+\frac{1}{n^2}\right)^n \leq \sqrt[n]{e} \qquad \forall n$

ed ora $\lim_{n\to+\infty} 1 = \lim_{n\to+\infty}\sqrt[n]{e} = 1$

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$$\Longrightarrow \lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^n = 1$$

Considero ora la successione $\left(\left(1+\frac{1}{n}\right)^{n^2}\right)_{n\in\mathbb{N}}$

$$\left(1+\frac{1}{n}\right)^{n^2} \geq 1+n^2\cdot\frac{1}{n} = 1+n \quad \forall n$$

Disug. di Bernoulli

ma $\lim_{n\to+\infty}(1+n) = +\infty$

$$\Longrightarrow \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n^2} = +\infty$$

Esercizio Calcolare $\lim_{n\to+\infty}\left(1+q+q^2+q^3+\dots+q^n\right)$

al variare di $q\in\mathbb{R}$

dim

$S_n = 1+q+q^2+\dots+q^n$

$$\begin{aligned}(1-q)S_n &= (1-q)\left(1+q+q^2+\dots+q^n\right)\\ &= 1+\cancel{q}+\cancel{q^2}+\dots+\cancel{q^n}+\\ &\quad -\cancel{q}-\cancel{q^2}-\dots-\cancel{q^n}-q^{n+1}\\ &= 1-q^{n+1}\end{aligned}$$

$q\neq 1$

$$\Longrightarrow S_n = \frac{1-q^{n+1}}{1-q}$$

$q=1 \quad S_n = 1+1^1+1^2+\dots+1^n = n+1$

$$\Longrightarrow S_n = \sum_{k=0}^{n} q^k = \begin{cases} n+1 & q=1 \\ \dfrac{1-q^{n+1}}{1-q} & q\neq 1\end{cases}$$

$q=1$ $\quad \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}(n+1) = +\infty$

$q>1$ $\quad \lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\frac{1-q^{n+1}}{1-q} = \lim_{n\to+\infty}\frac{q^{n+1}-1}{q-1} =$

$$= \frac{\left(\lim_{n\to+\infty} q^{n+1}\right)-1}{q-1} = \frac{+\infty}{q-1} = +\infty$$

$-1<q<1$ $\quad \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1-q^{n+1}}{1-q} = \frac{1-\lim_{n\to+\infty} q^{n+1}}{1-q} = \frac{1}{1-q}$

$q=-1$ $\quad \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1-(-1)^{n+1}}{1+1} = \frac{1-\lim_{n\to+\infty}(-1)^{n+1}}{2} \quad \nexists$

infatti, $\quad (-1)^{2n+1} = -1 \xrightarrow[n\to+\infty]{} -1$

$(-1)^{2n+2} = 1 \xrightarrow[n\to+\infty]{} 1 \quad \Rightarrow \nexists \lim_{n\to+\infty} (-1)^{n+1}$

$q<-1$ $\quad \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1-q^{n+1}}{1-q} = \frac{1-\lim_{n\to+\infty} q^{n+1}}{1-q} \quad \nexists$

infatti, $\quad (q)^{2n+2} \longrightarrow +\infty$

$q^{2n+1} \longrightarrow -\infty \quad \Rightarrow \nexists \lim_{n\to+\infty} q^{n+1}$

$\boxed{VII}$

Pb: $0{,}999\ldots9\ldots \neq 1$??

$0{,}333\ldots3\ldots \neq 1/3$??

$$0{,}\bar{9} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n} + \cdots$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots + \left(\frac{1}{10}\right)^n + \cdots\right)$$

$$= \frac{9}{10} \lim_{n\to+\infty} \left[\sum_{k=0}^{n} \left(\frac{1}{10}\right)^k\right]$$

$$= \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{9}{10} \cdot \frac{10}{9} = 1$$

risultato precedente

$$0{,}\bar{3} = \frac{1}{3} \cdot 0{,}\bar{9} = \frac{1}{3}$$

$$2,\overline{11} = 2 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots =$$

$$\overset{\shortparallel}{2,\overline{1}} = 2 + \frac{1}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \dots\right)$$

$$= 2 + \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = 2 + \frac{1}{10} \cdot \frac{10}{9} = 2 + \frac{1}{9}$$

$$2,\overline{12} = 2 + \frac{12}{10^2} + \frac{12}{10^4} + \frac{12}{10^6} + \dots$$

$$= 2 + \frac{12}{10^2}\left(1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^4 + \left(\frac{1}{10}\right)^6 + \dots\right)$$

$$= 2 + \frac{12}{10^2} \cdot \frac{1}{1 - \frac{1}{10^2}} = 2 + \frac{12}{100} \cdot \frac{100}{99} = 2 + \frac{12}{99}$$

Successione di Erone

$$\begin{cases} a_1 = 2 \\ a_{n+1} = \underbrace{\frac{1}{2}\left(a_n + \frac{2}{a_n}\right)}_{\text{è la media tra } a_n \text{ e } \frac{2}{a_n}} \end{cases} \quad \forall n > 1$$

$$a_1 = 2 > a_2 = \frac{1}{2}(2+1) = \frac{3}{2} > a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2}\,\frac{17}{6}$$

Suppongo $a_{n-1} > a_n$

Voglio provare $a_n > a_{n+1}$

devo provare $a_n > \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

$$= \frac{1}{2}\left(\frac{a_n^2 + 2}{a_n}\right)$$

$$\Downarrow \quad \boxed{a_n > 0}$$

$$2a_n^2 > a_n^2 + 2$$

$$\boxed{a_n^2 > 2} \quad ?$$

Devo provare $a_n > \sqrt{2} \quad \forall n$

$a_1 = 2 > \sqrt{2}$ ok

suppongo $a_n > \sqrt{2}$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \overset{?}{>} \sqrt{2}$$

$$\frac{a^2+2}{a_n} > 2\sqrt{2}$$

$$a_n^2 + 2 > 2\sqrt{2}\, a_n$$

$$(a_n - \sqrt{2})^2 > 0$$

ma questo è vero poiché $a_n > \sqrt{2}$ per ipotesi induttiva

quindi $a_n > \sqrt{2}$
$(a_n)_n \downarrow$ $\Rightarrow \exists \lim_{n\to+\infty} a_n = \ell \geq \sqrt{2}$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$\Downarrow$$

$$\ell = \frac{1}{2}\left(\ell + \frac{2}{\ell}\right)$$

$$\Downarrow$$

$$\ell = \pm\sqrt{2} \Rightarrow \ell = \sqrt{2}$$