2015-04-30- Correzione della prova in itinere di Analisi 1 del 30 aprile 2015

1) Determinare, al variare di $\alpha \in \mathbb{R}$, l'ordine di infinitesimo e la parte principale della funzione

$$f(x) = \frac{1}{1+2x} - e^{-2x} + \cos(2x) - 1 + \alpha x^3$$

$$\frac{1}{1+y} = \frac{1}{1-(-y)} = 1 + (-y) + (-y)^2 + (-y)^3 + (-y)^4 + o(y^4) \qquad y \to 0$$

$$= 1 - y + y^2 - y^3 + y^4 + o(y^4)$$

$$\Rightarrow \frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 + 16x^4 + o(y^4)$$

Analogamente $\quad e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + o(y^4) \quad y \to 0$

$$\Rightarrow e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + o(x^4) \quad x \to 0$$

Infine $\quad \cos(y) = 1 - \frac{y^2}{2} + \frac{y^4}{24} + o(y^5) \quad y \to 0$

$$\Rightarrow \cos(2x) = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5) \quad x \to 0$$

Dunque

$$f(x) = \cancel{1} \cancel{-2x} \cancel{+4x^2} - 8x^3 + 16x^4 - \left(\cancel{1} \cancel{-2x} \cancel{+2x^2} - \frac{4}{3}x^3 + \frac{2}{3}x^4\right)$$
$$+ \cancel{1} \cancel{-2x^2} + \frac{2}{3}x^4 \cancel{-1} + \alpha x^3 + o(x^4)$$
$$= x^3\left(-8 + \frac{4}{3} + \alpha\right) + x^4\left(16 - \frac{2}{3} + \frac{2}{3}\right) + o(x^4)$$
$$= x^3\left(-\frac{20}{3} + \alpha\right) + 16x^4 + o(x^4)$$

$$\text{ordine}(f) = \begin{cases} 3 & \alpha \neq \frac{20}{3} \\ 4 & \alpha = \frac{20}{3} \end{cases}$$

$$\text{PP}(f) = \begin{cases} \left(\alpha - \frac{20}{3}\right)x^3 & \alpha \neq \frac{20}{3} \\ 16x^4 & \alpha = \frac{20}{3} \end{cases}$$

2) i) Determinare tutte le soluzioni del seguente sistema

$$\begin{cases} w+\bar{z}^2=z^2 \\ z\cdot\bar{z}+4i-5=w \end{cases}$$

ii) Determinare tutte le soluzioni dell'equazione

$$z^5+iz^3-z^2-i=0$$

i) $\begin{cases} w=(z-\bar{z})(z+\bar{z}) \\ \text{"} \end{cases} \Rightarrow \begin{cases} w=4i(\operatorname{Re} z)(\operatorname{Im} z) \\ |z|^2-5=w-4i \end{cases} \Rightarrow$

posto $z=x+iy$ troviamo (ovviamente $z\neq 0$!)

$$\begin{cases} w=4i \\ xy=1 \\ x^2+y^2=5 \end{cases} \qquad \begin{cases} w=4i \\ xy=1 \\ x^2+1/x^2=5 \end{cases} \qquad \begin{cases} \text{"} \\ x^4-5x^2+1=0 \end{cases}$$

$$x^2=\frac{5\pm\sqrt{25-4}}{2}=\frac{5\pm\sqrt{21}}{2}$$

$$z_1=\sqrt{\frac{5+\sqrt{21}}{2}}+i\sqrt{\frac{2}{5+\sqrt{21}}} \qquad z_2=-z_1$$

$$z_3=\sqrt{\frac{5-\sqrt{21}}{2}}+i\sqrt{\frac{2}{5-\sqrt{21}}} \qquad z_4=-z_3$$

e dunque trovo le 4 soluzioni $(4i, z_k)$ $k=1,2,3,4$

ii) $z^5+iz^3-z^2-i=0$ equivale a

$$z^3(z^2+i)-(z^2+i)=(z^2+i)(z^3-1)=0$$

$$\Leftrightarrow z^2=-i \quad \text{o} \quad z^3=1$$

$$z^2=i \Leftrightarrow z^2=\cos\frac{\pi}{2}+i\operatorname{sen}\frac{\pi}{2} \Rightarrow z_1=\cos\frac{\pi}{4}+i\operatorname{sen}\frac{\pi}{4}=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$$

$$z_2=\cos\frac{5}{4}\pi+i\operatorname{sen}\frac{5}{4}\pi=-z_1$$

$$z^3=1 \Leftrightarrow z^3=\cos(0)+i\operatorname{sen}(0) \Rightarrow z_3=\cos(0)+i\operatorname{sen}(0)=1$$

$$z_4=\cos\left(\tfrac{2}{3}\pi\right)+i\operatorname{sen}\left(\tfrac{2}{3}\pi\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$z_5=\cos\left(\tfrac{4}{3}\pi\right)+i\operatorname{sen}\left(\tfrac{4}{3}\pi\right)=\overline{z_4}$$

3) Determinare i valori di $a,b\in\mathbb{R}$ per i quali il limite

$$\lim_{x\to0}\frac{e^{2x}-\log(1+x+x^2)-1-ax-bx^2}{\sin(x^3)+x^4}$$

è reale, e calcolarlo.

$$f(x)=\frac{e^{2x}-\log(1+x+x^2)-1-ax-bx^2}{\sin(x^3)+x^4}=$$

$$=\frac{\cancel{1}+2x+2x^2+\frac{4}{3}x^3+o(x^3)-\left(x+x^2-\frac{1}{2}(x+x^2)^2+\frac{1}{3}(x+x^2)^3\right)-\cancel{1}-ax-bx^2}{x^3+x^4+o(x^4)}$$

$$=\frac{x(2-1-a)+x^2\left(2-1+\frac{1}{2}-b\right)+x^3\left(\frac{4}{3}+1-\frac{1}{3}\right)+o(x^3)}{x^3+o(x^3)}$$

$$=\frac{x(1-a)+x^2\left(\frac{3}{2}-b\right)+2x^3+o(x^3)}{x^3+o(x^3)}\qquad x\to0$$

e dunque $\lim_{x\to0}f(x)=2\in\mathbb{R}$ se $a=1$ e $b=\frac{3}{2}$

4) i) Determinare tutte le primitive della funzione

$$f(x)=x\log(x^2+2x+2)$$

indicando esplicitamente l'insieme in cui sono definite

ii) Calcolare $\int_0^1 e^x\operatorname{arctg}(e^x)\,dx$

i) $$\int x\log(x^2+2x+2)\,dx \underset{\text{per parti}}{=} \frac{x^2}{2}\cdot\log(x^2+2x+2)-\int\frac{x^2}{\cancel{2}}\cdot\frac{\cancel{2}x+\cancel{2}}{x^2+2x+2}\,dx$$

$$=\frac{x^2}{2}\log(x^2+2x+2)-\int\left[(x-1)+\frac{2}{(x+1)^2+1}\right]dx$$

$$\begin{array}{l|l} x^3+x^2 & x^2+2x+2 \\ \hline x^3+2x^2+2x & x-1 \\ \hline // \quad -x^2-2x & \\ -x^2-2x-2 & \\ \hline \qquad 2 & \end{array}$$

$$=\frac{x^2}{2}\log(x^2+2x+2)-\frac{x^2}{2}+x-2\int\frac{dx}{(x+1)^2+1}$$

$$=\frac{x^2}{2}\log(x^2+2x+2)-\frac{x^2}{2}+x-2\operatorname{arctg}(x+1)+c$$

$c\in\mathbb{R}$

ed essendo $x^2+2x+2\geq1\ \forall x$, le primitive sono definite su $\mathbb{R}$

ii) $\int e^x \operatorname{arctg}(e^x)\, dx = \left(\int \cancel{y} \operatorname{arctg}(y) \frac{dy}{\cancel{y}} \right)_{y=e^x}$

$x = \log y$

$dx = \frac{dy}{y}$

ma $\int \operatorname{arctg}(y)\, dy \underset{\text{per parti}}{=} y \operatorname{arctg}(y) - \frac{1}{2}\int \frac{2y}{1+y^2}\, dy =$

$= y \operatorname{arctg}(y) - \frac{1}{2}\log(1+y^2) + c \qquad c \in \mathbb{R}$

$= \left(y \operatorname{arctg} y - \frac{1}{2}\log(1+y^2) + c \right)_{y=e^x} \qquad c \in \mathbb{R}$

$= e^x \operatorname{arctg}(e^x) - \frac{1}{2}\log(1+e^{2x}) + c \qquad c \in \mathbb{R}$

Ne segue che $\int_0^1 e^x \operatorname{arctg}(e^x)\, dx = \left[e^x \operatorname{arctg}(e^x) - \frac{1}{2}\log(1+e^{2x}) \right]_{x=0}^{x=1}$

$= e \operatorname{arctg}(e) - \frac{1}{2}\log(1+e^2) - \frac{\pi}{4} + \log(\sqrt{2})$