Primes in almost all short intervals and the distribution of the zeros of the Riemann zeta-function

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Dedicated to Prof. K. Ramachandra
on the occasion of his seventieth birthday

Abstract

We study the relations between the distribution of the zeros of the Riemann zeta-function and the distribution of primes in "almost all" short intervals. It is well known that a relation like $\psi(x)-\psi(x-y) \sim y$ holds for almost all $x \in [N, 2N]$ in a range for y that depends on the width of the available zero-free regions for the Riemann zeta-function, and also on the strength of density bounds for the zeros themselves. We also study implications in the opposite direction: assuming that an asymptotic formula like the above is valid for almost all x in a given range of values for y, we find zero-free regions or density bounds.

1 Introduction

We investigate the relations between the distribution of the zeros of the Riemann zeta-function and the distribution of primes in "almost all" short intervals. Here and in the sequel "almost all" means that the number of integers $n \le x$ which do not have the required property is $o(x)$ as $x \to \infty$. It is known since Hoheisel's work in the 1930's [8] that a relation of the type

$$\psi(x) - \psi(x - y) \sim y \tag{1}$$

holds in a certain range for y whose width depends on the strength of both zero-free regions and density estimates for the zeros of the zeta function. The same relation is true, obviously in a much wider range for y, if we deal with the same problem but only almost everywhere, that is, allowing a small set of exceptions. Here we show that a partial converse is also true: if (1) holds almost everywhere in a wide region of values for y, then the zeros of the Riemann zeta-function can not be too dense near $\sigma = 1$, nor too close to the same line, in a fairly strong quantitative sense. The corresponding relation between the error term in the Prime Number Theorem and the width of the zero-free region for the zeta-function is a classical result of Turán [19].

2 Prime numbers in all short intervals

For $x \geq 2$ let π and ψ denote the usual Chebyshev functions and set

$$R_1(x) \stackrel{\text{def}}{=} \sup_{u \in [2,x]} \left| \pi(u) - \int_2^u \frac{\mathrm{d}t}{\log t} \right|$$

$$R_2(x) \stackrel{\text{def}}{=} \sup_{u \in [2,x]} \bigl| \psi(u) - u \bigr|.$$

The Prime Number Theorem (PNT), in the sharpest known form due to Vinogradov and Korobov (see Titchmarsh [18, §6.19]), asserts that

$$R_j(x) \ll x \exp\Bigl\{ -c_j (\log x)^{3/5} (\log\log x)^{-1/5} \Bigr\}$$

for suitable $c_j > 0$, $j = 1$, 2. The additivity of the main terms for both π and ψ suggests the truth of the statements

$$\pi(x) - \pi(x-y) \sim \int_{x-y}^x \frac{\mathrm{d}t}{\log t} \sim \frac{y}{\log x}, \tag{2}$$

$$\psi(x) - \psi(x-y) \sim y, \tag{3}$$

for $y \leq x$ (provided, of course, that y is not too small with respect to x), and these relations are trivial for large y, that is $y / \bigl(R_j(x) \log x\bigr) \to \infty$. The unproved assumption that all the zeros $\beta + \mathrm{i}\gamma$ of the Riemann zeta-function with real part $\beta \in (0,1)$ actually have $\beta = \frac{1}{2}$ is known as the Riemann Hypothesis (RH). It implies that $R_j(x) \ll x^{1/2} (\log x)^j$ for $j = 1$, 2, and this is essentially best possible since Littlewood [11] proved in 1914 that

$$\limsup_{x \to \infty} \frac{R_1(x) \log x}{x^{1/2} \log\log\log x} > 0.$$

Assuming the RH, Selberg [17] proved in 1943 that (2) holds, provided that $y/(x^{1/2}\log x)\to\infty$. The best result to-date is due to Heath-Brown [6], and is described in the following section.

2.1 Density Estimates

The connection with density estimates arises from the following well-known fact: if there exist constants $C\ge 2$ and $B\ge 0$ such that

$$\begin{aligned} N(\sigma,T) &\stackrel{\text{def}}{=} |\{\varrho=\beta+\mathrm{i}\gamma\colon \zeta(\varrho)=0,\ \beta\ge\sigma,\ |\gamma|\le T\}| \\ &\ll T^{C(1-\sigma)}(\log T)^B \end{aligned} \tag{4}$$

for $\sigma\in\left[\frac{1}{2},1\right]$, then it is comparatively easy to prove that, for any fixed $\varepsilon>0$, both (2) and (3) hold in the range

$$y\ge x^{1-C^{-1}+\varepsilon}. \tag{5}$$

Huxley [9] showed that (4) holds with $C=12/5$, and therefore $7/12$ is an admissible exponent in (5). Heath-Brown [6] has improved on (5), showing that if $\varepsilon(x)>0$ for all $x\ge 1$, then a quantitative form of (2) holds for

$$x^{7/12-\varepsilon(x)}\le y\le \frac{x}{(\log x)^4}$$

provided that $\varepsilon(x)\to 0$ as $x\to\infty$. In fact, he proved that

$$\pi(x)-\pi(x-y)=\frac{y}{\log x}\Bigl(1+\mathcal{O}\bigl(\varepsilon^4(x)\bigr)+\mathcal{O}\Bigl(\frac{\log\log x}{\log x}\Bigr)^4\Bigr).$$

The Density Hypothesis (DH) is the conjecture that (4) holds with $C=2$ and some $B\ge 0$; in view of the Riemann–von Mangoldt formula (see Titchmarsh [18, Theorem 9.4]), the Density Hypothesis is optimal apart from the value of B, and it yields the exponent $1/2$ in (5). Thus DH and RH have almost the same consequence as far as this problem is concerned.

2.2 Negative results

Maier [12] startled the world in 1985 by proving that (2) is false for $y=(\log x)^\lambda$ for any fixed $\lambda>1$. Hildebrand & Maier [7] improved on this in 1989 but their result is too complicated to be stated here. Later Friedlander, Granville, Hildebrand & Maier [3] extended these results to arithmetic progressions.

3 Prime numbers in almost all short intervals

For technical convenience we define the Selberg integral by means of

$$J(x,\theta) \stackrel{\text{def}}{=} \int_x^{2x} |\Delta(t,\theta)|^2 \,\mathrm{d}t,$$

where $\Delta(x,\theta) := \psi(x) - \psi(x-\theta x) - \theta x$. The natural expectation is that the relation

$$J(x,\theta) = o\bigl(x^3\theta^2\bigr) \tag{6}$$

holds in a much wider range for $y = \theta x$ than (5), since we now allow some exceptions to (3). We remark that, when $x^{\varepsilon-1} \le \theta \le 1$, the Brun–Titchmarsh inequality (see for example Montgomery & Vaughan [13]) yields $J(x,\theta) \ll_\varepsilon x^3\theta^2$. Ingham [10] proved in 1937 that (6) holds for $x^{-2C^{-1}+\varepsilon} \le \theta \le 1$. Hence (6) is known to hold in the ranges

$$\begin{cases} x^{-5/6+\varepsilon} \le \theta \le 1 & \text{unconditionally, and} \\ x^{-1+\varepsilon} \le \theta \le 1 & \text{assuming the DH.} \end{cases} \tag{7}$$

Assuming the RH, Selberg [17] gave the stronger bound

$$J(x,\theta) \ll x^2\theta \log^2\bigl(2\theta^{-1}\bigr),$$

uniformly for $x^{-1} \le \theta \le 1$. Goldston & Montgomery [5], using both the RH and the Pair Correlation Conjecture, showed that if $y = x^\alpha$ then

$$J\bigl(x, yx^{-1}\bigr) \sim (1-\alpha)xy\log x$$

uniformly for $0 \le \alpha \le 1-\varepsilon$, and Goldston [4], assuming the Generalized Riemann Hypothesis, gave the lower bound

$$J(x, yx^{-1}) \ge \left(\frac{1}{2} - 2\alpha - \varepsilon\right) xy\log x$$

uniformly for $0 \le \alpha < \frac{1}{4}$.

For technical reasons, our result is stated in terms of a modified Selberg integral with the function π in place of ψ, and represents the improvement in the known range of validity for y corresponding to Heath-Brown's result quoted above. The detailed proof can be found in [21].

Theorem 3.1 *If $x^{-5/6-\varepsilon(x)} \le \theta \le 1$, where $0 \le \varepsilon(x) \le \frac{1}{6}$ and $\varepsilon(x) \to 0$ when $x \to \infty$, then*

$$\begin{aligned} I(x,\theta) &\stackrel{\text{def}}{=} \int_x^{2x} \left|\pi(t) - \pi(t-\theta t) - \frac{\theta t}{\log t}\right|^2 \mathrm{d}t \\ &\ll \frac{x^3\theta^2}{(\log x)^2}\left(\varepsilon(x) + \frac{\log\log x}{\log x}\right)^2 \end{aligned}$$

In particular, the assumptions imply that $I(x,\theta) = o(x^3\theta^2(\log x)^{-2})$. A simple consequence is that for θ in the above range, the interval $[t-\theta t, t]$ contains the expected number of primes $\sim \theta t(\log t)^{-1}$ for almost all integers $t \in [x, 2x]$.

3.1 Outline of the proof

Since the details of the proof are rather complicated, we start with a weaker, but somewhat easier, result and deal with J instead of I. First we briefly sketch the classical proof of

$$J(x,\theta) = \int_x^{2x} \bigl|\psi(t) - \psi(t-\theta t) - \theta t\bigr|^2 \,\mathrm{d}t = o\bigl(x^3\theta^2\bigr) \tag{8}$$

uniformly for $x^{-5/6+\varepsilon} \le \theta \le 1$, by means of the Density bound (4) and then give some details of a different proof in the same spirit as Heath-Brown's paper [6]. What follows should be taken with a grain of salt: it is not supposed to be the literal truth, but rather an approximation to it. The interested reader is referred to Saffari & Vaughan [16] for all the details. For brevity, we set $\mathcal{L}_1 := \log x$. By the explicit formula for ψ (see for example Davenport [2, §17]) we have

$$\psi(x) = x - \sum_{|\gamma|<T} \frac{x^\varrho}{\varrho} + \mathcal{O}\Bigl(\frac{x}{T}(\log xT)^2\Bigr),$$

with the usual convention for the sums over the zeros of the zeta function. Hence

$$J(x,\theta) \ll \int_x^{2x} \Bigl|\sum_{|\gamma|<T} t^\varrho \frac{1-(1-\theta)^\varrho}{\varrho}\Bigr|^2 \,\mathrm{d}t + \frac{x^3}{T^2}(\log xT)^4.$$

The last term is harmless provided that $T < x$ and $T^{-1} = o\bigl(\theta\mathcal{L}_1^{-2}\bigr)$ which we now assume. We now skip all details until the very last step. The integral is easily rearranged into

$$\sum_{|\gamma_1|<T,}\sum_{|\gamma_2|<T} \frac{1-(1-\theta)^{\varrho_1}}{\varrho_1}\,\frac{1-(1-\theta)^{\overline{\varrho}_2}}{\overline{\varrho}_2} \int_x^{2x} t^{\varrho_1+\overline{\varrho}_2}\,\mathrm{d}t.$$

Since $\bigl(1-(1-\theta)^\varrho\bigr)\varrho^{-1} \ll \theta$, we have that the last expression is

$$\begin{aligned}\ll \theta^2 \sum_{|\gamma|<T} x^{1+2\beta} &= x\theta^2 \sum_{|\gamma|<T} \Bigl\{2\mathcal{L}_1 \int_{1/2}^{\beta} x^{2\sigma}\,\mathrm{d}\sigma + x\Bigr\} \\ &= x\theta^2 \Bigl\{2\mathcal{L}_1 \int_{1/2}^{1} N(\sigma,T)x^{2\sigma}\,\mathrm{d}\sigma + xN\Bigl(\frac{1}{2},T\Bigr)\Bigr\}.\end{aligned}$$

The last summand is negligible if we slightly strengthen our demand to $T = o\big(x\mathcal{L}_1^{-1}\big)$, since $N\big(\frac{1}{2}, T\big) \ll T \log T$ by the Riemann–von Mangoldt formula. It is clear that we achieve our goal if we can prove that

$$\int_{1/2}^{1} N(\sigma, T) x^{2\sigma} \, \mathrm{d}\sigma = o\big(x^2 \mathcal{L}_1^{-1}\big). \tag{9}$$

It is important to stress the fact that simply plugging (4) into this integral does not suffice, since it would only yield the bound $\ll x^2 \mathcal{L}_1^B$ for the integral in question. Actually, what is needed for Theorem 3.1 is a zero-free region for the Riemann zeta-function of the shape

$$\sigma > \sigma_0(x) \stackrel{\text{def}}{=} 1 - \frac{c}{\mathcal{L}_1^A} \tag{10}$$

for some fixed $A \in (0, 1)$, and Korobov & Vinogradov proved that one can take any $A > \frac{2}{3}$ (see Titchmarsh [18, §6.19]). In fact, (10) clearly implies that the upper limit for the integral in (9) can be replaced by $\sigma_0(x)$, and this gives the desired bound provided that T is chosen (optimally) satisfying $T^C = o\big(x^2 \mathcal{L}_1^{-4}\big)$, and then it easily follows that (6) holds in the range (7).

3.2 The full result

In order to get the full result given by Theorem 3.1 we need a more involved argument. We use the Linnik–Heath-Brown identity: For $z > 1$

$$\log\big(\zeta(s)\Pi(s)\big) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \big(\zeta(s)\Pi(s) - 1\big)^k = \sum_{k \geq 1} \sum_{p \geq z} \frac{1}{kp^{ks}} = \sum_{n \geq 1} \frac{c_n}{n^s},$$

say, where $\Pi(s) = \prod_{p<z} (1 - p^{-s})$. Hence, if $z \leq \frac{1}{2}x$, $\theta \leq \frac{1}{2}$ and $t \in [x, 2x]$ then

$$\pi(t) - \pi(t - \theta t) = \sum_{t - \theta t < n \leq t} c_n + \mathcal{O}\big(\theta x^{1/2}\big).$$

For $k \geq 1$ put

$$\big(\zeta(s)\Pi(s) - 1\big)^k = \sum_{n \geq 1} \frac{a_k(n)}{n^s},$$

so that $a_k(n) = a_1^{*k}(n)$, a_1^{*k} being the k-fold Dirichlet convolution of a_1 with itself, where $a_1(1) = 0$ and $a_1(n) = 0$ unless all prime factors of n are $\geq z$, in which case $a_1(n) = 1$. If $z > x^{1/k_0}$ then $\big(\zeta(s)\Pi(s) - 1\big)^k$ has no non-zero coefficient for $n \le x$ and $k \ge k_0$. We will eventually choose $k_0 = 4$. It is far from easy to approximate $\zeta(s)\Pi(s) - 1$ with Dirichlet polynomials. We have

$$\pi(t) - \pi(t - \theta t) = \sum_{1 \le k < k_0} \frac{(-1)^{k-1}}{k} \sum_{t - \theta t < p \le t} a_k(n) + \mathcal{O}\big(\theta x^{1/2}\big).$$

The goal is to find a function $\mathfrak{M}_k(t)$ which is independent of θ, such that

$$\Sigma_k(t, \theta) \stackrel{\text{def}}{=} \sum_{t - \theta t < p \le t} a_k(n) = \theta \mathfrak{M}_k(t) + \mathfrak{R}_k(t, \theta),$$

where $\mathfrak{R}_k(t, \theta)$ is "small" in L^2 norm over $[x, 2x]$.

3.3 Reduction to mean-value bounds

We skip the tedious, detailed description of the decomposition into Dirichlet polynomials. Essentially, we can truncate the Dirichlet series for both ζ and Π at height $2x$ without changing the sum we are interested in. Simplifying details, we have

$$\Sigma_k(t, \theta) = \sum_{t - \theta t < p \le t} b_k(n) \qquad \text{where} \qquad \sum_{n \ge 1} \frac{b_k(n)}{n^s} = \prod_{h=1}^{k} f_h(s)$$

for suitable Dirichlet polynomials f_h. Thus we can write $\Sigma_k(t, \theta)$ as a contour integral by means of the truncated Perron formula: neglecting the error term we have

$$\Sigma_k(t, \theta) \sim \frac{1}{2\pi \mathrm{i}} \int_{\frac{1}{2} - \mathrm{i}T_0}^{\frac{1}{2} + \mathrm{i}T_0} \big(\zeta^*(s)\Pi^*(s) - 1\big)^k \, \frac{t^s - (t - \theta t)^s}{s} \, \mathrm{d}s,$$

the stars meaning that we have truncated the Dirichlet series at $2x$. Here $T_0 = x^{5/6+\beta}$, where $\beta > 0$ is very small but fixed. A short range near 0 of the form $|\Im(s)| \le T_1$ gives the main term of Σ, in the very convenient form $\theta \mathfrak{M}_k(t)$, with $\mathfrak{M}_k(t)$ independent of θ, plus a manageable error term, provided that θ is not too close to 1. For brevity, we do not give the precise definition of T_1 (which depends on the details of the decomposition into Dirichlet polynomials referred to above), and only remark that our final choice satisfies $T_1 = x^{o(1)}$.

We finally have the estimate for the L^2 norm of $\mathfrak{R}_k(t, \theta)$

$$\int_x^{2x} |\mathfrak{R}_k(t, \theta)|^2 \, \mathrm{d}t \ll x^2 \theta^2 (\log x) \max_{T \in [T_1, T_0]} \int_T^{2T} |\zeta^*(s)\Pi^*(s) - 1|^{2k} \, \mathrm{d}t$$

so that we need to prove that the rightmost integral above is $o(x(\log x)^{-1})$ uniformly for $T \in [T_1, T_0]$.

The tools needed for completing the proof include the Korobov–Vinogradov zero-free region, the Halász method and Ingham's fourth power moment estimate for the Riemann zeta-function.

Here some of the difficulties arise from the fact that not all Dirichlet polynomial involved have a fixed, bounded number of factors. We had to make a different choice for the Dirichlet polynomials from Heath-Brown, because that choice leads to too large error terms since we have a larger z than Heath-Brown and a much smaller h. This is due to the fact that we need z to be almost $x^{1/3}$, as opposed to $x^{1/6}$. The slight additional difficulty is more than compensated by the fact that we only have to save a little over the estimate given by Montgomery's theorem, since our problem leads naturally to estimating the mean-square of a Dirichlet polynomial. For the details see the author's paper [21].

4 The inverse problem

What can one say about the zeros of the Riemann zeta-function, given bounds for $J(x, \theta)$ (or for $I(x, \theta)$) uniformly in a suitable range for θ? In other words, does a bound for J imply something like a density theorem or a zero-free region for the zeta function? The answer is positive and the general philosophy is that good estimates for J in a sufficiently wide range of uniformity for θ yield good zero-free regions for the zeta function, just as one would expect. It should be remarked, however, that we do not establish a real equivalence between density bounds and bounds for J. This is very clearly illustrated by Corollary 4.2 below.

Our main result is the following

Theorem 4.1 *Assume that*

$$J(x, \theta) \ll \frac{x^3 \theta^2}{F(\theta x)} \qquad \textit{uniformly for} \qquad G(x)^{-1} \le \theta \le 1 \tag{11}$$

where F and G are positive, strictly increasing functions, unbounded as x tends to infinity. There exist absolute constants $B_0 \ge 1$ and $C_0 \ge 1$ such that if F and G are as above with $G(x) = x^\beta$ for some fixed $\beta \in (0, 1]$, then for any $B \ge \max\bigl(B_0, \beta^{-1}\bigr)$ and any $C > C_0$ we have

$$N(\sigma, T) \ll_{B,C} \frac{T^{BC(1-\sigma)}}{\min\bigl(F\bigl(T^{B-1}\bigr), T\bigr)}$$

There is a more general form of this Theorem which gives interesting results also in the case $G(x) = o_\varepsilon(x^\varepsilon)$ for every $\varepsilon > 0$: see Corollary 4.4. We note that the proof gives the numerical bounds $B_0 \le 40000$ and $C_0 \le 2000\log(16e)$, though these values can without doubt be improved upon, and that the Riemann Hypothesis implies that one can take $F(x) = x(\log x)^{-2}$ and $G(x) = x^{1-\varepsilon}$.

Corollary 4.2 *If* (11) *holds with* $F(x) = x^\alpha$ *and* $G(x) = x^\beta$ *for some fixed* α, $\beta \in (0,1]$ *then*

$$\Theta \stackrel{\text{def}}{=} \sup\{\Re(\varrho)\colon \zeta(\varrho) = 0\} \le 1 - \frac{1}{C_0}\eta$$

where C_0 *is the same constant as in Theorem* 4.1, *and*

$$\eta = \begin{cases} B_0^{-1} & \text{if } \beta \ge B_0^{-1} \text{ and } \alpha \ge (B_0-1)^{-1} \\ \alpha(1+\alpha)^{-1} & \text{if } \beta \ge B_0^{-1} \text{ and } \alpha \le (B_0-1)^{-1} \\ \beta & \text{if } \beta < B_0^{-1} \text{ and } \alpha \ge \beta(1-\beta)^{-1} \\ \alpha(1+\alpha)^{-1} & \text{if } \beta < B_0^{-1} \text{ and } \alpha \le \beta(1-\beta)^{-1}. \end{cases}$$

Thus, if one could prove that Theorem 4.1 holds with $B_0 = 2$ and $C_0 = 1$, then from (11) with $\alpha = \beta = 1-\varepsilon$ one would recover the Riemann Hypothesis, though a simpler, direct argument suffices (see the beginning of §2 in [22]). If instead $F(x) \ll_\varepsilon x^\varepsilon$ for every $\varepsilon > 0$, then our result is the following:

Corollary 4.3 *If the hypotheses of Theorem* 4.1 *hold with* $F(x) \ll x^\varepsilon$ *for every* $\varepsilon > 0$ *and* $G(x) \ge F(x)$, *then for every* $B > B_0$ *and* $t > 2$ *the Riemann zeta-function has no zeros in the region*

$$\sigma > 1 - \frac{(B-1)\log F(t)}{BC_0 \log t}.$$

Finally, the general version referred to above yields the following result for a special but interesting choice of slowly growing functions F and G.

Corollary 4.4 *Let* B_0 *and* C_0 *denote the constants in Theorem* 4.1. *Then if* (11) *holds with* $F(x) = \exp\bigl(\log x\bigr)^\alpha$ *and* $G(x) = \exp\bigl(\log x\bigr)^\beta$ *for some fixed* α, $\beta \in (0,1]$, *then the Riemann zeta-function has no zeros in the region*

$$\sigma > 1 - \frac{1 + o(1)}{B_0 C_0 \bigl(\log(2+|t|)\bigr)^{r(\alpha,\beta)}}$$

where $r(\alpha,\beta) := \bigl(1 - \min(\alpha,\beta)\bigr)\beta^{-1}$.

We observe that if $F(x) = (\log x)^A$ then from Corollary 4.3 we recover Littlewood's zero-free region (which is needed in the proof), while arguing as in the proof of Corollary 4.4 we can show that one recovers the Korobov–Vinogradov zero-free region, provided that one can take $F(x) = G(x) = \exp\bigl\{(\log x)^{3/5}(\log\log x)^{-1/5}\bigr\}$.

Recall that $\Delta(x,\theta) := \psi(x) - \psi(x-\theta x) - \theta x$. In order to put our results into proper perspective, we recall that Pintz [14], [15] proved that if $\varrho_0 = \beta_0 + \mathrm{i}\gamma_0$ is any zero of the Riemann zeta-function, then

$$\limsup_{x\to\infty} \frac{|\Delta(x,1)|}{x^{\beta_0}} \geq \frac{1}{|\varrho_0|}$$

and also the more precise inequality $\Delta(x,1) = \Omega\bigl(x\exp\{-(1-\varepsilon)\omega(x)\}\bigr)$ where $\omega(x) := \min\bigl\{(1-\beta)\log x + \log|\gamma| \colon \varrho = \beta + \mathrm{i}\gamma\bigr\}$.

4.1 Sketch of the proof

Since every zero $\varrho = \beta + \mathrm{i}\gamma$ of the Riemann zeta-function gives rise to a pole of the function

$$F(s) \stackrel{\text{def}}{=} -\frac{\zeta'}{\zeta}(s) - \zeta(s)$$

we can detect zeros by counting the "spikes" of the function F just to the right of the critical strip. It should be remarked that since ζ is of finite order (as a Dirichlet series), it does not cancel the contribution of the pole of $-\zeta'/\zeta$ at $s = \varrho$. Obviously F is related to the function $\Delta(x,\theta)$ defined above, which is the sum of the coefficients in its Dirichlet series in the interval $(x-\theta x, x]$. The function Δ, in its turn, appears in the Selberg integral. Since we are assuming that $|\Delta(t,\theta)|$ is usually small, that is, it is small in the L^2-norm over the interval $[x,2x]$, this piece of information can be used to show that the zeta function can not have too many zeros.

More precisely, if the zeta function has a zero in the circle $|s-1-\mathrm{i}t| \leq r$, say, then for a suitable integer k, we have that $F^{(k)}(1+r+\mathrm{i}t)$ is "large" in a strong quantitative sense. Taking all zeros into account, we find a lower bound for $N(\sigma,T)$. But the assumption on J can be used to give an upper bound for $|F^{(k)}(s)|$, and therefore for $N(\sigma,T)$.

From now on we write c_j for a suitable positive, absolute, effectively computable constant, $\mathcal{L}_2 = \log T$, $w = 1+\mathrm{i}t$ with $2 \leq |t| \leq T$, $r = c_7(1-\sigma)$, $s_0 = w + r$.

The crucial lower estimate is the following bound.

Lemma 4.5 *There exist absolute constants c_1 and $c_2 > 0$ with the following property: let $\mathcal{L}_2^{-1} \le r \le c_1/(16e)$ and $K \ge c_2 r \mathcal{L}_2$. If the Riemann zeta-function has a zero in the circle $|s - w| \le r$, then there exists an integer k such that $K \le k \le 2K$ and*

$$\frac{1}{k!} \left| F^{(k)}(w + r) \right| \ge \frac{2}{c_1} \left(\frac{c_1}{4r} \right)^{k+1}.$$

This can be proved as Lemma A in Bombieri [1, §6], applying a suitable form of Turán Second Main Theorem (see the Corollary of Theorem 8.1 in Turán [20]; the proof of the latter result yields $c_1 = 1/(8e)$) to the function $k!^{-1} \left| F^{(k)}(w + r) \right|$ using the development for the zeta-function

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\varrho - w| \le 1} \frac{1}{s - \varrho} + \mathcal{O}(\mathcal{L}_2),$$

given by Titchmarsh [18, Theorem 9.6], the Cauchy inequalities for the derivatives of holomorphic functions, the Phragmén–Lindelöf principle and the fact that ζ is of finite order in the half plane $\sigma > \frac{1}{2}$.

Lemma 4.6 *There exist absolute constants $A_0 \ge 1$, $B_0 \ge 1$ and $C_0 \ge 2$ with the following property. Let $\mathcal{L}_2^{-1} \le r \le c_1/(16e)$. If the zeta-function has a zero in the circle $|s - w| \le r$, then for all $x \ge T^{B_0}$ and $C > C_0$ we have*

$$\int_x^{x^{A_0}} \Bigl| \sum_{n \in [x, y]} \frac{\Lambda(n) - 1}{n^w} \Bigr|^2 \frac{\mathrm{d}y}{y} \gg_C (\log x)^3 x^{-Cr}.$$

The proof of this Lemma is close to Bombieri [1], §6, Lemma B, using our Lemma 4.5 in place of his Lemma A.

Lemma 4.7 *Uniformly for $x^{\varepsilon - 1} \le \theta \le \frac{1}{2}$ we have*

$$\int_x^{2x} \Bigl| \sum_{n \in (t - \theta t, t]} \frac{\Lambda(n) - 1}{n} \Bigr|^2 \frac{\mathrm{d}t}{t} \ll_\varepsilon x^{-3} J(x, \theta) + \theta^4.$$

It is a straightforward application of the Brun–Titchmarsh inequality.

Lemma 4.8 *For $\tau = \exp(\theta)$ we have*

$$\int_{-\theta^{-1}}^{\theta^{-1}} \Bigl| \sum_{n \in (x, y]} \frac{\Lambda(n) - 1}{n} n^{\mathrm{i}u} \Bigr|^2 \mathrm{d}u \ll \theta^{-2} \int_x^y \Bigl| \sum_{n \in (u, \tau u]} \frac{\Lambda(n) - 1}{n} \Bigr|^2 \frac{\mathrm{d}u}{u} + \theta.$$

This follows from Gallagher's Lemma (see Bombieri [1, Theorem 9]).

From Lemma 4.6 we have

$$\int_x^{x^{A_0}} \int_{\gamma-\frac{1}{2}r}^{\gamma+\frac{1}{2}r} \Big| \sum_{n\in(x,y]} \frac{\Lambda(n)-1}{n^{1+\mathrm{i}v}} \Big|^2 \,\mathrm{d}v \frac{\mathrm{d}y}{y} \gg r(\log x)^3 x^{-Cr},$$

for any $C > C_0$, and, summing over zeros,

$$N(\sigma,T) r(\log x)^3 x^{-Cr} \ll r\log T \int_x^{x^{A_0}} \int_{-T-r}^{T+r} \Big| \sum_{n\in(x,y]} \frac{\Lambda(n)-1}{n^{1+\mathrm{i}u}} \Big|^2 \,\mathrm{d}u \,\frac{\mathrm{d}y}{y},$$

since each point of the interval $(-T-r, T+r)$ belongs to at most $c_0 r\mathcal{L}_2$ intervals of type $\big(\gamma-\frac{1}{2}r, \gamma+\frac{1}{2}r\big)$, by the Density Lemma in Bombieri [1, §6].

Hence

$$\begin{aligned} N(\sigma,T) &\ll \frac{\log T}{(\log x)^3} x^{Cr} \int_x^{x^{A_0}} \int_{-T-r}^{T+r} \Big| \sum_{n\in(x,y]} \frac{\Lambda(n)-1}{n^{1+\mathrm{i}u}} \Big|^2 \,\mathrm{d}u \,\frac{\mathrm{d}y}{y} \\ &\ll \frac{\log T}{(\log x)^3} x^{Cr} \Big\{ \theta^{-2} \int_x^{x^{A_0}} \int_x^{y} \Big| \sum_{n\in(u,\tau u]} \frac{\Lambda(n)-1}{n} \Big|^2 \frac{\mathrm{d}u}{u} \,\frac{\mathrm{d}y}{y} + \theta\log x \Big\} \end{aligned}$$

by Lemma 4.8 with $T = \theta^{-1}$, $\tau = \exp\theta$. The inner integral is essentially

$$\ll \log x \max_{x\le t\le y} \int_t^{2t} \Big| \sum_{n\in(u,\tau u]} \frac{\Lambda(n)-1}{n} \Big|^2 \frac{\mathrm{d}u}{u} \ll \log x \max_{x\le t\le y} J\big(t, \tau-1\big) t^{-3}$$

by Lemma 4.7. By our hypothesis and our choice of T, we finally have

$$N(\sigma,T) \ll \frac{\log T}{\log x} x^{Cr} \left\{ \frac{1}{F(\theta x)} + \frac{\theta}{\log x} \right\} \ll \frac{\log T}{\log x} x^{Cr} \left\{ \frac{1}{F\big(xT^{-1}\big)} + T^{-1} \right\}$$

This is our main estimate, subject to the conditions $x \ge \max\big(T^{B_0}, G^{-1}(T)\big)$ and $\theta = T^{-1}$, and the proof of the various corollaries is fairly straightforward. For the details see the author's paper [22].

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