Abstract of paper [1].

Let $k \geq 2$ be a fixed integer and $p$ denote a prime. For any $n \in \mathbb{N}$ such that the polynomial $x^k - n$ is irreducible over $\mathbb{Q}$ let

$$R_k(n) := \sum_{h + m^k = n} \Lambda(h), \quad \rho_k(n, p) := |\{m \mod p : m^k \equiv n \mod p\}|$$

and

$$\mathfrak{S}_k(n) := \prod_p \left(1 - \frac{\rho_k(n, p) - 1}{p - 1}\right) = \prod_p \left(1 - \frac{\rho_k(n, p)}{p}\right) / \left(1 - \frac{1}{p}\right).$$

One expects that for the integers $n$ that we are considering one has $R_k(n) \sim n^{1/k} \mathfrak{S}_k(n)$ as $n \to \infty$. Let $L := \log N,$

$$R_k^*(n) := \sum_{h + m^k = n \atop N - Y \leq h \leq N \atop Y / 2 \leq m^k \leq 3 / 2 Y} \Lambda(h), \quad \text{and} \quad P_k^*(n) := \frac{1}{k} \sum_{h + m^k = n \atop N - Y \leq h \leq N \atop Y / 2 \leq m^k \leq 3 / 2 Y} m^{1/k - 1}.$$

Building on previous work by Perelli & Pintz in the case $k = 2$, we prove

**Theorem 1** Let $k \geq 3$, $\varepsilon > 0$, $N^{7/12+\varepsilon} \leq Y \leq N$ and $\max(Y^{1-1/k+\varepsilon}, N^{1/2+\varepsilon}) \leq H \leq Y$. Then

$$\sum^*_{N \leq n \leq N + H} |R_k^*(n) - P_k^*(n) \mathfrak{S}_k(n)|^2 \ll_{\varepsilon, A, k} HY^{2/k} L^{-A},$$

where the $*$ means that the sum is over $n \in \mathbb{N}$ such that $x^k - n$ is irreducible over $\mathbb{Q}$.

**Theorem 2** Assume the Generalized Riemann Hypothesis and let $k \geq 2$, $\varepsilon > 0$. Then

$$E_k(N) := |\{N \leq n \leq 2N : n \neq p + m^k\}| \ll_{k, \varepsilon} N^{1+\varepsilon-2/(kK)}$$

where $K := 2^{k-1}$.

Theorem 2 is obtained by means of Hardy & Littlewood’s circle method, and Weyl’s inequality, by a suitable treatment of the singular series.

References