Abstract of paper [1].

Let $k \geq 2$ be a fixed integer and p denote a prime. For any $n \in \mathbb{N}$ such that the polynomial $x^k - n$ is irreducible over $\mathbb{Q}$ let

$$R_k(n) := \sum_{h+m^k=n} \Lambda(h), \qquad \rho_k(n,p) := |\{m \mod p \colon m^k \equiv n \mod p\}|$$

and

$$\mathfrak{S}_k(n) := \prod_p \left(1 - \frac{\rho_k(n,p)-1}{p-1}\right) = \prod_p \left(1 - \frac{\rho_k(n,p)}{p}\right) \Big/ \left(1 - \frac{1}{p}\right).$$

One expects that for the integers n that we are considering one has $R_k(n) \sim n^{1/k}\mathfrak{S}_k(n)$ as $n \to \infty$. Let $L := \log N$,

$$R_k^*(n) := \sum_{\substack{h+m^k=n \\ N-Y \leq h \leq N \\ Y/2 \leq m^k \leq 3/2\, Y}} \Lambda(h), \qquad \text{and} \qquad P_k^*(n) := \frac{1}{k} \sum_{\substack{h+m=n \\ N-Y \leq h \leq N \\ Y/2 \leq m \leq 3/2\, Y}} m^{\frac{1}{k}-1}.$$

Building on previous work by Perelli & Pintz in the case $k = 2$, we prove

Theorem 1 *Let $k \geq 3$, ε, $A > 0$, $N^{7/12+\varepsilon} \leq Y \leq N$ and $\max\left(Y^{1-1/k+\varepsilon}, N^{1/2+\varepsilon}\right) \leq H \leq Y$. Then*

$$\sum_{N \leq n \leq N+H}^{\star} |R_k^*(n) - P_k^*(n)\mathfrak{S}_k(n)|^2 \ll_{\varepsilon,A,k} HY^{2/k}L^{-A},$$

where the $\star$ means that the sum is over $n \in \mathbb{N}$ such that $x^k - n$ is irreducible over $\mathbb{Q}$.

Theorem 2 *Assume the Generalized Riemann Hypothesis and let $k \geq 2$, $\varepsilon > 0$. Then*

$$E_k(N) := |\{N \leq n \leq 2N \colon n \neq p + m^k\}| \ll_{k,\varepsilon} N^{1+\varepsilon-2/(kK)}$$

where $K := 2^{k-1}$.

Theorem 2 is obtained by means of Hardy & Littlewood's circle method, and Weyl's inequality, by a suitable treatment of the singular series.

References

[1] A. Perelli and A. Zaccagnini. On the sum of a prime and a k-th power. *Izv. Ross. Akad. Nauk, Ser. Math.*, 59:185–200, 1995.