How to use interpolation in PDE's.

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Lecture 1

Preliminaries: real interpolation and semigroups

For the general theory of interpolation the reader is referred to the classical books [1, 4, 16, 31], and to the lecture notes [24] for a more elementary exposition.

Let X and Y be real or complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively, such that $Y \subset X$ is continuously embedded in X. The so called K-method lets us construct scales of Banach spaces that are interpolation spaces between X and Y.

An *intermediate space* between X and Y is any Banach space E such that

$$X \subset E \subset Y,$$

with continuous embeddings.

An *interpolation space* between X and Y is any intermediate space such that for every $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ (that is, for every $T \in \mathcal{L}(X)$ whose restriction to Y belongs to $\mathcal{L}(Y)$), the restriction of T to E belongs to $\mathcal{L}(E)$.

Definition 1.0.1 *For every $x \in X$ and $t > 0$, set*

$$K(t, x, X, Y) = \inf_{x=a+b,\ a\in X,\ b\in Y} \|a\|_X + t\|b\|_Y. \tag{1.1}$$

If there is no danger of confusion, we shall write $K(t, x)$ instead of $K(t, x, X, Y)$.

If I is any interval contained in $(0, +\infty)$, we denote by $L^p_*(I)$ the Lebesgue space $L^p(I, dt/t)$. In particular, $L^\infty_*(I) = L^\infty(I)$.

Definition 1.0.2 *Let $0 < \theta < 1$, $1 \le p \le \infty$, and set*

$$\begin{cases} (X, Y)_{\theta,p} = \{x \in X : \ t \mapsto t^{-\theta} K(t, x, X, Y) \in L^p_*(0, +\infty)\}, \\ \\ \|x\|_{(X,Y)_{\theta,p}} = \|t \mapsto t^{-\theta} K(t, x, X, Y)\|_{L^p_*(0,\infty)}; \end{cases} \tag{1.2}$$

Such spaces are called real interpolation spaces.

The mapping $x \mapsto \|x\|_{(X,Y)_{\theta,p}}$ is easily seen to be a norm in $(X, Y)_{\theta,p}$. If no confusion may arise, we shall write $\|x\|_{\theta,p}$ instead of $\|x\|_{(X,Y)_{\theta,p}}$. It is not hard to see that $(X, Y)_{\theta,p}$ is a Banach space.

Note that $K(t, x) \le \|x\|_X$ for every $x \in X$, so that $t \mapsto t^{-\theta} K(t, x) \in L^p_*(a, \infty)$ for all $a > 0$, and $\lim_{t\to\infty} t^{-\theta} K(t, x) = 0$. Therefore, only the behavior near $t = 0$ of $t^{-\theta} K(t, x)$ plays a role in the definition of $(X, Y)_{\theta,p}$. Indeed, one could replace the halfline $(0, +\infty)$ by any interval $(0, a)$ in definition 1.0.2, obtaining equivalent norms.

In the trivial case $X = Y$ we have $K(t, x) \le \min\{t, 1\}\|x\|$. Therefore, $(X, X)_{\theta,p} = X$ for $0 < \theta < 1$ and $1 \le p \le \infty$, with equivalence of the respective norms.

The inclusion properties of the real interpolation spaces are stated below.

Proposition 1.0.3 *For $0 < \theta < 1$, $1 \le p_1 \le p_2 \le \infty$ we have*

$$Y \subset (X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset (X,Y)_{\theta} \subset (X,Y)_{\theta,\infty} \subset \overline{Y} \subset X, \tag{1.3}$$

with continuous embeddings. Here $\overline{Y}$ is the closure of Y in X. Moreover for $0 < \theta_1 < \theta_2 < 1$ we have

$$(X,Y)_{\theta_2,\infty} \subset (X,Y)_{\theta_1,1}. \tag{1.4}$$

Therefore, $(X,Y)_{\theta_2,p} \subset (X,Y)_{\theta_1,q}$ for every p, $q \in [1,\infty]$.

The proof of the inclusion $(X,Y)_{\theta,\infty} \subset \overline{Y}$ comes immediately from the definition of the function K. Indeed, let $x \in (X,Y)_{\theta,\infty}$. Then we have

$$K(t,x) = \inf_{x=a+b} \|a\|_X + t\|b\|_Y \le t^\theta \|x\|_{\theta,\infty}, \;\; t > 0,$$

so that for every $n \in \mathbb{N}$ (taking $t = 1/n$) there are $a_n \in X$, $b_n \in Y$ such that $x = a_n + b_n$, and

$$\|a_n\|_X + \frac{1}{n}\|b_n\|_Y \le \frac{2}{n^\theta}\|x\|_{\theta,\infty}.$$

In particular, $\|x - b_n\|_X = \|a_n\|_X \le 2\|x\|_{\theta,\infty} n^{-\theta}$, so that the sequence $\{b_n\}$ goes to x in X as $n \to \infty$.

This shows the connection between interpolation theory and approximation theory. The rate of convergence of b_n to x and the rate of blowing up of $\|b_n\|_Y$ are described precisely by the fact that $x \in (X,Y)_{\theta,\infty}$ (or $x \in (X,Y)_{\theta,p} \subset (X,Y)_{\theta,\infty}$).

The spaces $(X,Y)_{\theta,p}$ enjoy the following important interpolation property.

Theorem 1.0.4 *Let Y_1, Y_2 be continuously embedded in X_1, X_2, respectively. If $T \in \mathcal{L}(X_1,X_2) \cap \mathcal{L}(Y_1,Y_2)$, then $T \in \mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})$ for every $\theta \in (0,1)$ and $p \in [1,\infty]$. Moreover,*

$$\|T\|_{\mathcal{L}((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \le (\|T\|_{\mathcal{L}(X_1,X_2)})^{1-\theta}(\|T\|_{\mathcal{L}(Y_1,Y_2)})^{\theta}. \tag{1.5}$$

Proof — We may assume that $\|T\|_{\mathcal{L}(X_1,X_2)} \ne 0$. Let $x \in (X_1,Y_1)_{\theta,p}$: then for every $a \in X_1$, $b \in Y_1$ such that $x = a + b$ and for every $t > 0$ we have

$$\|Ta\|_{X_2} + t\|Tb\|_{Y_2} \le \|T\|_{\mathcal{L}(X_1,X_2)}\left(\|a\|_{X_1} + t\frac{\|T\|_{\mathcal{L}(Y_1,Y_2)}}{\|T\|_{\mathcal{L}(X_1,X_2)}}\|b\|_{Y_1}\right),$$

so that

$$K(t,Tx,X_2,Y_2) \le \|T\|_{\mathcal{L}(X_1,X_2)} K\left(t\frac{\|T\|_{\mathcal{L}(Y_1,Y_2)}}{\|T\|_{\mathcal{L}(X_1,X_2)}}, x, X_1, Y_1\right). \tag{1.6}$$

Setting $s = t\frac{\|T\|_{\mathcal{L}(Y_1,Y_2)}}{\|T\|_{\mathcal{L}(X_1,X_2)}}$ we get $Tx \in (X_2,Y_2)_{\theta,p}$, and

$$\|Tx\|_{(X_2,Y_2)_{\theta,p}} \le \|T\|_{\mathcal{L}(X_1,X_2)}\left(\frac{\|T\|_{\mathcal{L}(Y_1,Y_2)}}{\|T\|_{\mathcal{L}(X_1,X_2)}}\right)^\theta \|x\|_{(X_1,Y_1)_{\theta,p}},$$

and (1.5) follows. $\square$

Taking $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$ in 1.0.4, it follows that $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ are interpolation spaces between X and Y. Another important consequence is the next corollary.

Corollary 1.0.5 *For $0 < \theta < 1$, $1 \le p \le \infty$ there is $c(\theta, p)$ such that*

$$\|y\|_{(X,Y)_{\theta,p}} \le c(\theta,p)\|y\|_X^{1-\theta}\|y\|_Y^{\theta} \ \ \forall y \in X \cap Y. \tag{1.7}$$

Proof — Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, according to the fact that X, Y are real or complex Banach spaces. Let $y \in X \cap Y$, and define T by $T(\lambda) = \lambda y$ for each $\lambda \in \mathbb{K}$. Then $\|T\|_{\mathcal{L}(\mathbb{K},X)} = \|y\|_X$, $\|T\|_{\mathcal{L}(\mathbb{K},Y)} = \|y\|_Y$, and $\|T\|_{\mathcal{L}(\mathbb{K},(X,Y)_{\theta,p})} = \|y\|_{(X,Y)_{\theta,p}}$. The statement follows now from theorem 1.0.4, through the equality $(\mathbb{K},\mathbb{K})_{\theta,p} = \mathbb{K}$. $\square$

Let us see some easy examples. $C_b(\mathbb{R}^n)$ is the space of the bounded continuous functions in $\mathbb{R}^n$, endowed with the sup norm $\|\cdot\|_\infty$; $C_b^1(\mathbb{R}^n)$ is the subset of the continuously differentiable functions with bounded derivatives, endowed with the norm $\|f\|_\infty + \sum_{i=1}^n \|D_i f\|_\infty$. For $\theta \in (0,1)$, $C^\theta(\mathbb{R}^n)$ is the set of the bounded and uniformly Hölder continuous functions, endowed with the norm

$$\|f\|_{C^\theta} = \|f\|_\infty + [f]_{C^\theta} = \|f\|_\infty + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\theta}.$$

For $\theta \in (0,1)$, $p \in [1,\infty)$, $W^{\theta,p}(\mathbb{R}^n)$ is the space of all $f \in L^p(\mathbb{R}^n)$ such that

$$[f]_{W^{\theta,p}} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{\theta p + n}} dx\, dy \right)^{1/p} < \infty.$$

It is endowed with the norm $\|\cdot\|_{L^p} + [\cdot]_{W^{\theta,p}}$.

Example 1.0.6 *For $0 < \theta < 1$, $1 \le p < \infty$ we have*

$$(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty} = C^\theta(\mathbb{R}^n), \tag{1.8}$$

$$(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p} = W^{\theta,p}(\mathbb{R}^n), \tag{1.9}$$

with equivalence of the respective norms.

Proof — Let us prove that the first statement holds. Let $f \in (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}$. We already know that

$$\|f\|_\infty \le K(1, f, C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n)) \le \|f\|_{\theta,\infty}.$$

For every decomposition $f = a + b$, with $a \in C_b(\mathbb{R}^n)$, $b \in C_b^1(\mathbb{R}^n)$, and for each x, $y \in \mathbb{R}^n$ we have

$$|f(x) - f(y)| \le |a(x) - a(y)| + |b(x) - b(y)| \le 2\|a\|_\infty + \|b\|_{C^1}|x-y|,$$

so that, taking the infimum over all a, b we get

$$|f(x) - f(y)| \le 2K(|x-y|, f, C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n)) \le 2|x-y|^\theta \|f\|_{\theta,\infty}.$$

Therefore f is θ-Hölder continuous and $\|f\|_{C^\theta} = \|f\|_\infty + [f]_{C^\theta} \le 3\|f\|_{\theta,\infty}$.

Conversely, let $f \in C^\theta(\mathbb{R}^n)$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a compactly supported function, with support contained in $B(0,1)$, such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For every $f \in C^\theta(\mathbb{R}^n)$ and $t > 0$ set

$$b_t(x) = \frac{1}{t^n} \int_{\mathbb{R}^n} f(y)\varphi\left(\frac{x-y}{t}\right)dy, \ \ a_t(x) = f(x) - b_t(x), \ \ x \in \mathbb{R}^n. \tag{1.10}$$

Then

$$a_t(x) = \frac{1}{t^n}\int_{\mathbb{R}^n}(f(x) - f(x-y))\varphi(y/t)dy$$

so that

$$\|a_t\|_\infty \leq [f]_{C^\theta}\frac{1}{t^n}\int_{\mathbb{R}^n}|y|^\theta\varphi(y/t)dy = t^\theta[f]_{C^\theta}\int_{\mathbb{R}^n}|w|^\theta\varphi(w)dw.$$

Moreover, $\|b_t\|_\infty \leq \|f\|_\infty$, and

$$D_ib_t(x) = \frac{1}{t^{n+1}}\int_{\mathbb{R}^n}f(y)D_i\varphi((x-y)/t)dy.$$

Since $\int_{\mathbb{R}^n}D_i\varphi((x-y)/t)dy = 0$, we get

$$D_ib_t(x) = \frac{1}{t^{n+1}}\int_{\mathbb{R}^n}(f(x-y) - f(x))D_i\varphi(y/t)dy, \tag{1.11}$$

which implies

$$\|D_ib_t\|_\infty \leq t^{\theta-1}[f]_{C^\theta}\int_{\mathbb{R}^n}|w|^\theta|D_i\varphi(w)|dw.$$

Therefore,

$$t^{-\theta}K(t,f) \leq t^{-\theta}(\|a_t\|_\infty + t\|b_t\|_{C^1}) \leq C\|f\|_{C^\theta}.$$

Hence $C^\theta(\mathbb{R}^n)$ is continuously embedded in $(C_b(\mathbb{R}^n), C^1_b(\mathbb{R}^n))_{\theta,\infty}$.

The proof of the second statement is similar. We recall that for every $b \in W^{1,p}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n \setminus \{0\}$ we have

$$\left(\int_{\mathbb{R}^n}\left(\frac{|b(x+h) - b(x)|}{|h|}\right)^p dx\right)^{1/p} \leq \|Db\|_{L^p}.$$

For every $f \in (L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p}$ let $f = a + b$, with $a \in L^p(\mathbb{R}^n)$, $b \in W^{1,p}(\mathbb{R}^n)$. For each $h \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{|f(x+h) - f(x)|^p}{|h|^{\theta p+n}}dx \\
&\leq \int_{\mathbb{R}^n}2^{p-1}\frac{|a(x+h) - a(x)|^p}{|h|^{\theta p+n}}dx + \int_{\mathbb{R}^n}2^{p-1}\frac{|b(x+h) - b(x)|^p}{|h|^{\theta p+n}}dx \\
&\leq 2^{2p-2}\frac{\|a\|^p_{L^p}}{|h|^{\theta p+n}} + 2^{p-1}\frac{|h|^p\|Db\|^p_{L^p}}{|h|^{\theta p+n}} \leq C_p|h|^{-\theta p-n}(\|a\|_{L^p} + |h|\,\|b\|_{W^{1,p}})^p,
\end{aligned}$$

so that, taking the infimum over all decompositions $f = a + b$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x+h) - f(x)|^p}{h^{\theta p+n}}\,dx \leq C_p|h|^{-\theta p-n}K(|h|,f)^p.$$

Integrating over $\mathbb{R}^n$ with respect to h we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x+h) - f(x)|^p}{h^{\theta p+n}}dx &\leq C_p\int_{\mathbb{R}^n}|h|^{-\theta p-n}K(|h|,f)^p dh \\
&= C_p\int_0^\infty\frac{K(r,f)^p}{r^{\theta p+1}}\,dr\int_{\partial B(0,1)}d\sigma_{n-1} = C_{p,n}\|f\|^p_{\theta,p}.
\end{aligned}$$

Therefore, $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p}$ is continuously embedded in $W^{\theta,p}(\mathbb{R}^n)$.

To prove the other embedding, for each $f \in W^{\theta,p}$ define a_t and b_t by (1.10). Then

$$\|a_t\|_{L^p}^p = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(y)-f(x)|\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\right)^p dx$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(y)-f(x)|^p\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dydx.$$

were for $p>1$ we applied the Jensen inequality to the probability measure $t^{-n}\varphi((x-y)/t)dy$. So we get

$$\int_0^\infty t^{-\theta p}\|a_t\|_{L^p}\frac{dt}{t} \leq \int_0^\infty\int_{\mathbb{R}^n\times\mathbb{R}^n}|f(y)-f(x)|^p\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)dy\,dx\,\frac{dt}{t}$$

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n}|f(y)-f(x)|^p\int_0^\infty t^{-\theta p}\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)\frac{dt}{t}\,dy\,dx$$

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n}|f(y)-f(x)|^p\int_{|x-y|}^\infty t^{-\theta p}\frac{1}{t^n}\varphi\left(\frac{x-y}{t}\right)\frac{dt}{t}\,dy\,dx$$

$$\leq \frac{\|\varphi\|_\infty}{\theta p+n}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(y)-f(x)|^p}{|y-x|^{\theta p+n}}\,dx\,dy = C[f]_{W^{\theta,p}}^p.$$

Using (1.11) and arguing similarly, we get also

$$\int_0^\infty t^{(1-\theta)p}\|D_ib_t\|_{L^p}\frac{dt}{t} \leq \frac{C_i^{p-1}\|D_i\varphi\|_\infty}{\theta p+n}[f]_{W^{\theta,p}}^p,$$

with $C_i = \int_{\mathbb{R}^n}|D_i\varphi(y)|dy$, while $\|b_t\|_{L^p} \leq \|f\|_{L^p}\|\varphi\|_{L^1} = \|f\|_{L^p}$.

Therefore, $t^{-\theta}K(t,f,L^p,W^{1,p}) \leq t^{-\theta}\|a_t\|_{L^p} + t^{1-\theta}\|b_t\|_{W^{1,p}} \in L_*^p(0,1)$, with norm estimated by $C\|f\|_{W^{\theta,p}}$, and the second part of the statement follows. □

Note that the proof of (1.8) yields also

$$(L^\infty(\mathbb{R}^n), Lip(\mathbb{R}^n))_{\theta,\infty} = (BUC(\mathbb{R}^n), BUC^1(\mathbb{R}^n))_{\theta,\infty} = C^\theta(\mathbb{R}^n).$$

Exercise. Following the method of example 1.0.6 show that

$$(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,q} = B^\theta_{p,q}(\mathbb{R}^n),$$

defined by $B^\theta_{p,q}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : [f]_{B^\theta_{p,q}} < \infty\}$, where

$$[f]_{B^\theta_{p,q}} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x)-f(x+h)|^p dx\right)^{q/p}\frac{1}{|h|^{\theta q+n}}dh\right)^{1/q},$$

and $\|f\|_{B^\theta_{p,q}} = \|f\|_{L^p} + [f]_{B^\theta_{p,q}}$.

Example 1.0.7 *Let $\Omega \in \mathbb{R}^n$ be an open set with the following property: there exists an extension operator E such that $E \in \mathcal{L}(C(\overline{\Omega}), C_b(\mathbb{R}^n)) \cap \mathcal{L}(C^\theta(\overline{\Omega}), C^\theta(\mathbb{R}^n)) \cap \mathcal{L}(C^1(\overline{\Omega}), C_b^1(\mathbb{R}^n))$, for some $\theta \in (0,1)$ (by extension operator we mean that $Ef_{|\overline{\Omega}} = f(x)$, for all $f \in C(\overline{\Omega})$). Then*

$$(C(\overline{\Omega}), C^1(\overline{\Omega}))_{\theta,\infty} = C^\theta(\overline{\Omega}).$$

Proof — Theorem 1.0.4 implies that

$$E \in \mathcal{L}((C(\overline{\Omega}), C^1(\overline{\Omega}))_{\theta,\infty}, (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}).$$

We know already that $(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty} = C^\theta(\mathbb{R}^n)$. So, for every $f \in (C(\overline{\Omega}), C^1(\overline{\Omega}))_{\theta,\infty}$, $Ef \in C^\theta(\mathbb{R}^n)$ and $\|Ef\|_{C^\theta(\mathbb{R}^n)} \leq C\|f\|_{(C(\overline{\Omega}),C^1(\overline{\Omega}))_{\theta,\infty}}$. Since $f = Ef_{|\overline{\Omega}}$, then $f \in C^\theta(\overline{\Omega})$ and $\|f\|_{C^\theta(\overline{\Omega})} \leq C\|f\|_{(C(\overline{\Omega}),C^1(\overline{\Omega}))_{\theta,\infty}}$.

Conversely, if $f \in C^\theta(\overline{\Omega})$ then $Ef \in C^\theta(\mathbb{R}^n) = (C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\theta,\infty}$. The retraction operator $Rg = g_{|\overline{\Omega}}$ belongs obviously to $\mathcal{L}(C_b(\mathbb{R}^n), C(\overline{\Omega})) \cap \mathcal{L}(C_b^1(\mathbb{R}^n), C^1(\overline{\Omega}))$. Again by theorem 1.0.4, $f = R(Ef) \in (C(\overline{\Omega}), C^1(\overline{\Omega}))_{\theta,\infty}$, with norm not exceeding $C\|Ef\|_{C^\theta(\mathbb{R}^n)} \leq C'\|f\|_{C^\theta(\overline{\Omega})}$. □

Such a good extension operator exists if Ω is an open set with uniformly C^1 boundary. $\partial\Omega$ is said to be uniformly C^1 if there are $N \in \mathbb{N}$ and a (at most) countable set of balls B_k whose interior parts cover $\partial\Omega$, such that the intersection of more than N of these balls is empty, and there exist diffeomorphisms $\varphi_k : B_k \mapsto B(0,1) \subset \mathbb{R}^n$ such that $\varphi_k(B_k \cap \overline{\Omega}) = \{y \in B(0,1) : y_n \geq 0\}$, and $\|\varphi_k\|_{C^1} + \|\varphi_k^{-1}\|_{C^1}$ are bounded by a constant independent of k. (In particular, each bounded Ω with C^1 boundary has uniformly C^1 boundary).

It is sufficient to construct E when $\Omega = \mathbb{R}^n_+$. The construction of E for any open set with uniformly C^1 boundary will follow by the usual method of local straightening the boundary.

If $\Omega = \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ we may use the reflection method: we set

$$Ef(x) = \begin{cases} f(x), & x_n \geq 0, \\ \alpha_1 f(x', -x_n) + \alpha_2 f(x', -2x_n), & x_n < 0, \end{cases}$$

where α_1, α_2 satisfy the continuity condition $\alpha_1 + \alpha_2 = 1$ and the differentiability condition $-\alpha_1 - 2\alpha_2 = 1$, that is $\alpha_1 = 3$, $\alpha_2 = -2$.

Then $E \in \mathcal{L}(C(\overline{\mathbb{R}^n_+}), C_b(\mathbb{R}^n)) \cap \mathcal{L}(C^\theta(\overline{\mathbb{R}^n_+}), C^\theta(\mathbb{R}^n)) \cap \mathcal{L}(C^1(\overline{\mathbb{R}^n_+}), C_b^1(\mathbb{R}^n))$, for every $\theta \in (0,1)$.

1.1 Intermediate spaces and reiteration

Let us introduce two classes of intermediate spaces for the interpolation couple (X, Y).

Definition 1.1.1 *Let $0 \leq \theta \leq 1$, and let E be any intermediate space between X and Y.*

(i) E is said to belong to the class J_θ between X and Y if there is a constant c such that

$$\|x\|_E \leq c\|x\|_X^{1-\theta}\|x\|_Y^\theta, \quad \forall x \in Y.$$

In this case we write $E \in J_\theta(X, Y)$.

(ii) E is said to belong to the class K_θ between X and Y if there is $k > 0$ such that

$$K(t, x) \leq kt^\theta\|x\|_E, \quad \forall x \in E, \ t > 0.$$

In this case we write $E \in K_\theta(X, Y)$.

Note that if $\theta \in (0,1)$ then $E \in K_\theta(X, Y)$ if and only if E is continuously embedded in $(X, Y)_{\theta,\infty}$. Moreover it is possible to prove that $E \in J_\theta(X, Y)$ if and only if $(X, Y)_{\theta,1}$ is continuously embedded in E.

The spaces $(X, Y)_{\theta,p}$ are in $K_\theta(X, Y) \cap J_\theta(X, Y)$, for every $p \in [1, \infty]$, respectively by definition and by corollary 1.0.5. But there are also intermediate spaces belonging to $K_\theta(X, Y) \cap J_\theta(X, Y)$ which are not interpolation spaces.

Example 1.1.2 $C^1_b(\mathbb{R}^n) \in J_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n)) \cap K_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$. *But* $C^1_b(\mathbb{R}^n)$ *is not an interpolation space between* $C_b(\mathbb{R}^n)$ *and* $C^2_b(\mathbb{R}^n)$.

Proof — From the inequalities ($i = 1, \dots, n$)

$$|f(x+he_i) - f(x) - D_i f(x)h| \leq \frac{1}{2}\|D_{ii}f\|_\infty h^2, \ \ \forall x \in \mathbb{R}^n, \ h > 0,$$

we get

$$|D_i f(x)| \leq \frac{|f(x+he_i) - f(x)|}{h} + \frac{1}{2}\|D_{ii}f\|_\infty h, \ \ \forall x \in \mathbb{R}^n, \ h > 0,$$

so that

$$\|D_i f\|_\infty \leq \frac{2\|f\|_\infty}{h} + \frac{1}{2}\|D_{ii}f\|_\infty h, \ \ \forall h > 0.$$

Taking the minimum on h over $(0, +\infty)$ we get

$$\|D_i f\|_\infty \leq 2(\|f\|_\infty)^{1/2}(\|D_{ii}f\|_\infty)^{1/2}, \ \ \forall f \in C^2(\mathbb{R}^n)$$

so that

$$\|f\|_{C^1} \leq (\|f\|_\infty)^{1/2}\left((\|f\|_\infty)^{1/2} + 2\textstyle\sum_{i=1}^n (\|D_{ii}f\|_\infty)^{1/2}\right)$$

$$\leq C(\|f\|_\infty)^{1/2}(\|f\|_{C^2})^{1/2}.$$

This implies that $C^1_b(\mathbb{R}^n)$ belongs to $J_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$. To prove that it belongs also to $K_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$, namely that it is continuously embedded in $(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))_{1/2,\infty}$, we argue as in example 1.0.6: for every $f \in C^1_b(\mathbb{R}^n)$ the functions a_t, b_t defined in (1.10) are easily seen to satisfy

$$\|a_t\|_\infty \leq Ct[f]_{Lip}, \ \ \|b_t\|_{C^1} \leq C\|f\|_{C^1}, \ \ \|D_{ij}b_t\|_\infty \leq Ct^{-1}[f]_{Lip}.$$

Therefore, $K(t, f, C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n)) \leq \|a_{t^{1/2}}\|_\infty + t\|b_{t^{1/2}}\|_{C^2} \leq Ct^{1/2}\|f\|_{C^1}$ so that $C^1_b(\mathbb{R}^n)$ is in $K_{1/2}(C_b(\mathbb{R}^n), C^2_b(\mathbb{R}^n))$.

But $C^1_b(\mathbb{R}^n)$ is not an interpolation space between $C_b(\mathbb{R}^n)$ and $C^2_b(\mathbb{R}^n)$, even for $n = 1$. Indeed, consider the family of operators

$$(T_\varepsilon f)(x) = \int_{-1}^1 \frac{x}{x^2 + y^2 + \varepsilon^2}(f(y) - f(0))\, dy, \ \ x \in \mathbb{R}.$$

It is easy to see that $\|T_\varepsilon\|_{\mathcal{L}(C_b(\mathbb{R}))}$ and $\|T_\varepsilon\|_{\mathcal{L}(C^2_b(\mathbb{R}))}$ are bounded by a constant independent of ε. Indeed, for every continuous and bounded f,

$$|(T_\varepsilon f)(x)| \leq 2\int_{-1}^1 \frac{|x|}{x^2 + y^2 + \varepsilon^2}\|f\|_\infty dy \leq 2\pi\|f\|_\infty,$$

$$(T_\varepsilon f)'(x) = \int_{-1}^1 \frac{-x^2 + y^2 + \varepsilon^2}{(x^2 + y^2 + \varepsilon^2)^2}(f(y) - f(0))dy,$$

and for every $f \in C^1_b(\mathbb{R})$,

$$(T_\varepsilon f)''(x) = \int_{-1}^1 \frac{-2x(-x^2 + 3y^2 + 3\varepsilon^2)}{(x^2 + y^2 + \varepsilon^2)^3}\int_0^y f'(s)ds\, dy$$

$$= \int_{-1}^1 \frac{-2x(-x^2 + 3y^2 + 3\varepsilon^2)}{(x^2 + y^2 + \varepsilon^2)^3}\int_0^y (f'(s) - f'(0))ds\, dy,$$

so that, if $f \in C^2(\mathbb{R})$,

$$|(T_\varepsilon f)''(x)| \leq |x| \int_{-1}^{1} \frac{x^2+3y^2+3\varepsilon^2}{(x^2+y^2+\varepsilon^2)^2}\, dy\, \|f''\|_\infty \leq 3\pi \|f''\|_\infty.$$

On the contrary, choosing $f_\varepsilon(x) = (x^2+\varepsilon^2)^{1/2}\eta(x)$, with $\eta \in C_0^\infty(\mathbb{R})$, $\eta \equiv 1$ in $[-1,1]$, we get

$$(T_\varepsilon f_\varepsilon)'(0) = \int_{-1}^{1} \frac{(y^2+\varepsilon^2)^{1/2}-\varepsilon}{y^2+\varepsilon^2}\, dy = \frac{1}{\varepsilon}\int_{-1/\varepsilon}^{1/\varepsilon} \frac{(s^2+1)^{1/2}-1}{s^2+1}\, ds$$

so that $\lim_{\varepsilon\to 0}(T_\varepsilon f_\varepsilon)'(0) = +\infty$, while the C^1 norm of f_ε is bounded by a constant independent of ε. Therefore $\|T_\varepsilon\|_{\mathcal{L}(C_b^1(\mathbb{R}))}$ blows up as $\varepsilon \to 0$. So, $C_b^1(\mathbb{R})$ cannot be an interpolation space between $C_b(\mathbb{R})$ and $C_b^2(\mathbb{R})$.

This counterexample is due to Mitjagin and Semenov, it shows also that $C^1([-1,1])$ is not an interpolation space between $C([-1,1])$ and $C^2([-1,1])$, and it may be obviously adapted to show that for any dimension n, $C_b^1(\mathbb{R}^n)$ is not an interpolation space between $C_b(\mathbb{R}^n)$ and $C_b^2(\mathbb{R}^n)$. $\square$

Remark 1.1.3 Arguing similarly one sees that $C_b^k(\mathbb{R}^n)$ is in the class $J_{1/2}(C_b^{k-1}(\mathbb{R}^n), C_b^{k+1}(\mathbb{R}^n)) \cap K_{1/2}(C_b^{k-1}(\mathbb{R}^n), C_b^{k+1}(\mathbb{R}^n))$, for every $k \in \mathbb{N}$. It follows easily that for $m_1 < k < m_2 \in \mathbb{N}$, $C_b^k(\mathbb{R}^n)$ belongs to the class $J_{(k-m_1)/(m_2-m_1)}(C_b^{m_1}(\mathbb{R}^n), C_b^{m_2}(\mathbb{R}^n))$. For instance, knowing that $C_b^1(\mathbb{R}^n)$ belongs to $J_{1/2}(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))$ and $C_b^2(\mathbb{R}^n)$ belongs to $J_{1/2}(C_b^1(\mathbb{R}^n), C_b^3(\mathbb{R}^n))$ one gets, for every $f \in C_b^3(\mathbb{R}^n)$,

$$\|f\|_{C^1} \leq C\|f\|_\infty^{1/2}\|f\|_{C^2}^{1/2} \leq C'\|f\|_\infty^{1/2}(\|f\|_{C^1}^{1/2}\|f\|_{C^3}^{1/2})^{1/2}$$

so that $\|f\|_{C^1}^{3/4} \leq C'\|f\|_\infty^{1/2}\|f\|_{C^3}^{1/4}$, which implies

$$\|f\|_{C^1} \leq C''\|f\|_\infty^{2/3}\|f\|_{C^3}^{1/3}$$

that is, $C_b^1(\mathbb{R}^n)$ belongs to $J_{1/3}(C_b(\mathbb{R}^n), C_b^3(\mathbb{R}^n))$.

Now we are able to state the Reiteration Theorem. It is one of the main tools of general interpolation theory.

Theorem 1.1.4 *Let $0 \leq \theta_0 < \theta_1 \leq 1$. Fix $\theta \in (0,1)$ and set $\omega = (1-\theta)\theta_0 + \theta\theta_1$. The following statements hold true.*

(i) If E_i belong to the class K_{θ_i} $(i = 0,1)$ between X and Y, then

$$(E_0, E_1)_{\theta,p} \subset (X,Y)_{\omega,p}, \quad \forall p \in [1,\infty].$$

(ii) If E_i belong to the class J_{θ_i} $(i = 0,1)$ between X and Y, then

$$(X,Y)_{\omega,p} \subset (E_0,E_1)_{\theta,p}, \quad \forall p \in [1,\infty].$$

Consequently, if E_i belong to $K_{\theta_i}(X,Y) \cap J_{\theta_i}(X,Y)$, then

$$(E_0,E_1)_{\theta,p} = (X,Y)_{\omega,p}, \quad \forall p \in [1,\infty],$$

with equivalence of the respective norms.

Remark 1.1.5 By proposition 1.2.3, the spaces $(X,Y)_{\theta,p}$ belong to $K_\theta(X,Y) \cap J_\theta(X,Y)$ for $0 < \theta < 1$ and $1 \le p \le \infty$. The Reiteration Theorem yields

$$((X,Y)_{\theta_0,q_0}, (X,Y)_{\theta_1,q_1})_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p},$$

for $0 < \theta_0, \theta_1 < 1$, $1 \le p$, $q \le \infty$. Moreover, since X belongs to $K_0(X,Y) \cap J_0(X,Y)$, and Y belongs to $K_1(X,Y) \cap J_1(X,Y)$ between X and Y, then

$$((X,Y)_{\theta_0,q}, Y)_{\theta,p} = (X,Y)_{(1-\theta)\theta_0+\theta,p},$$

and

$$(X, (X,Y)_{\theta_1,q})_{\theta,p} = (X,Y)_{\theta_1\theta,p},$$

for $0 < \theta_0, \theta_1 < 1$, $1 \le p$, $q \le \infty$.

The following examples are immediate consequences of examples 1.0.6, 1.0.7 and of remark 1.1.5.

Example 1.1.6 *Let $0 \le \theta_1 < \theta_2 \le 1$, $0 < \theta < 1$, $1 \le p < \infty$. Then*

$$(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))_{\theta,\infty} = C^{(1-\theta)\theta_1+\theta\theta_2}(\mathbb{R}^n),$$
$$(W^{\theta_1,p}(\mathbb{R}^n), W^{\theta_2,p}(\mathbb{R}^n))_{\theta,p} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\mathbb{R}^n).$$

If Ω is an open set in $\mathbb{R}^n$ with uniformly C^1 boundary, then

$$(C^{\theta_1}(\overline{\Omega}), C^{\theta_2}(\overline{\Omega}))_{\theta,\infty} = C^{(1-\theta)\theta_1+\theta\theta_2}(\overline{\Omega}); \quad (W^{\theta_1,p}(\Omega), W^{\theta_2,p}(\Omega))_{\theta,\infty} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\Omega).$$

The results of Example 1.1.6 may be extended to any θ_1, θ_2, provided the boundary of Ω is smooth enough. More precisely, we have the following proposition:

Example 1.1.7 *Let $0 \le \theta_1 < \theta_2$, $0 < \theta < 1$. If $(1-\theta)\theta_1 + \theta\theta_2$ is not integer, then*

$$(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))_{\theta,\infty} = C^{(1-\theta)\theta_1+\theta\theta_2}(\mathbb{R}^n),$$

and for $1 \le p < \infty$,

$$(W^{\theta_1,p}(\mathbb{R}^n), W^{\theta_2,p}(\mathbb{R}^n))_{\theta,p} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\mathbb{R}^n).$$

If Ω is an open set in $\mathbb{R}^n$ with uniformly C^{θ_2} boundary, then

$$(C^{\theta_1}(\overline{\Omega}), C^{\theta_2}(\overline{\Omega}))_{\theta,\infty} = C^{(1-\theta)\theta_1+\theta\theta_2}(\overline{\Omega}), \quad (W^{\theta_1,p}(\Omega), W^{\theta_2,p}(\Omega))_{\theta,\infty} = W^{(1-\theta)\theta_1+\theta\theta_2,p}(\Omega).$$

Exercise. Show that for $0 < \theta < 1$, $\theta \ne 1/2$,

$$(C_b(\mathbb{R}^n), C_b^2(\mathbb{R}^n))_{\theta,\infty} = C^{2\theta}(\mathbb{R}^n).$$

Hint: prove that $(C_b^1(\mathbb{R}^n), C_b^2(\mathbb{R}^n))_{\alpha,\infty} = C^{1+\alpha}(\mathbb{R}^n)$ using example 1.0.6, then use the Reiteration Theorem with $E = C_b^1(\mathbb{R}^n)$.

What happens if $(1-\theta)\theta_1 + \theta\theta_2 = k$ is integer? We do not get C^k and $W^{k,p}$ spaces, but larger spaces i.e.

$$(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))_{\theta,\infty} = \mathcal{C}^{*,k}(\mathbb{R}^n),$$
$$(W^{\theta_1,p}(\mathbb{R}^n), W^{\theta_2,p}(\mathbb{R}^n))_{\theta,p} = B^k_{p,p}(\mathbb{R}^n),$$

where the Zygmund space $\mathcal{C}^{*,k}(\mathbb{R}^n)$ consists of the functions $f \in C_b^{k-1}(\mathbb{R}^n)$ such that

$$[f]_{\mathcal{C}^{*,k}} := \sum_{|\beta|=k-1} \sup_{x\in\mathbb{R}^n, h\ne 0} \frac{|D^\beta f(x+h) - 2D^\beta f(x) + D^\beta f(x-h)|}{|h|} < \infty,$$

and the Besov space $B^k_{p,p}(\mathbb{R}^n)$ consists of the functions $f \in W^{k-1,p}(\mathbb{R}^n)$ such that

$$[f]_{B^k_{p,p}} := \sum_{|\alpha|=k-1} \left(\int_{\mathbb{R}^n} \frac{dh}{|h|^{n+p}} \int_{\mathbb{R}^n} |D^\alpha f(x+h) - 2D^\alpha f(x) + D^\alpha f(x-h)|^p dx \right)^{1/p}$$

is finite.

1.2 Semigroups of smoothing linear operators

For the general theory of semigroups in Banach space we refer to [10]. For analytic semigroups, to [20]. Here we recall just the definitions and some properties which will be used in the sequel.

A *semigroup* (of linear operators) in a Banach space X is a family of linear operators in $\mathcal{L}(X)$ such that $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all t, $s \geq 0$. A semigroup is said to be *strongly continuous* if $T(\cdot)x : [0,\infty) \to X$ is continuous for each $x \in X$. If $T(t)$ is a strongly continuous semigroup, its *infinitesimal generator* A is defined by

$$D(A) = \left\{ x \in X : \exists \lim_{h \to 0^+} \frac{T(h)x - x}{h} \right\}, \quad Ax = \lim_{h \to 0^+} \frac{T(h)x - x}{h}.$$

An important class of semigroups are the *analytic semigroups*, generated by sectorial operators. A linear operator $A : D(A) \subset X \mapsto X$ is said to be a *sectorial operator* if there are constants $\omega \in \mathbb{R}$, $\beta \in (\pi/2, \pi)$, $M > 0$ such that

$$\begin{cases} (i) \quad \rho(A) \supset S_{\beta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \beta\}, \\[2ex] (ii) \quad \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \dfrac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\beta,\omega}. \end{cases} \tag{1.12}$$

(1.12) allows us to define a semigroup e^{tA} in X, by means of the Dunford integral

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0,$$

where $r > 0$, $\eta \in (\pi/2, \beta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta,\ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \eta,\ |\lambda| = r\}$, oriented counterclockwise. We also set $e^{0A} = I$.

It is possible to show that the function $t \mapsto e^{tA}$ is analytic in $(0, +\infty)$ with values in $\mathcal{L}(X)$ (in fact, with values in $\mathcal{L}(X, D(A^m))$ for every m), so that e^{tA} is called the analytic semigroup generated by A and A is called the generator of e^{tA}. One sees easily that for $x \in X$ there exists $\lim_{t \to 0} e^{tA}x$ if and only if $x \in \overline{D(A)}$ (and in this case the limit is x). Therefore e^{tA} is strongly continuous if and only if $D(A)$ is dense in X.

It is worth to remark that if a semigroup $T(t)$ is differentiable for $t > 0$ with values in $\mathcal{L}(X)$, with $\|T'(t)\| \leq M \exp(\omega t)$ for some $M \geq 0$, $\omega \in \mathbb{R}$, and either $T(t)$ is strongly continuous, or there exists $t_0 > 0$ such that $T(t_0)$ is one to one, then there exists a unique sectorial operator $A : D(A) \mapsto X$ such that $T(t) = e^{tA}$. If $T(t)$ is strongly continuous, then A coincides with the infinitesimal generator.

The following basic examples will be considered in the sequel.

Example 1.2.1 The *Gauss-Weierstrass* semigroup

$$T(t)f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} f(x-y) dy, \tag{1.13}$$

is well defined in the spaces $L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$, in $C_b^k(\mathbb{R}^n)$, in $C^{k+\alpha}(\mathbb{R}^n)$, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$, and in many other functional spaces. It is not hard to see that it is strongly continuous in $L^p(\mathbb{R}^n)$ for $1 \leq p < +\infty$, and in $BUC(\mathbb{R}^n)$, the space of the uniformly continuous and bounded functions in $\mathbb{R}^n$, and that it is not strongly continuous in $L^\infty(\mathbb{R}^n)$, in $C_b(\mathbb{R}^n)$ and in $C^\alpha(\mathbb{R}^n)$. $T(t)$ is called also *heat semigroup* because for a large class of initial data f, the function $u(t,x) = (T(t)f)(x)$ is a solution to the heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \\[2ex] u(0,x) = f(x), \quad x \in \mathbb{R}^n. \end{cases}$$

As easily seen, the derivatives D_i commute with $T(t)$ on $C_b^1(\mathbb{R}^n)$, and for every multi-index $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$, we have

$$\|D^\beta T(t) f\|_\infty \leq \frac{C}{t^{(|\beta|-h)/2}} \|f\|_{C_b^h(\mathbb{R}^n)}, \;\; t > 0, \; f \in C_b^h(\mathbb{R}^n), \tag{1.14}$$

provided $|\beta| = \sum_{i=1}^n \beta_i \geq h$. These estimates follow directly from the representation formula (1.13), and they imply that $T(t)$ is analytic in $L^\infty(\mathbb{R}^n)$ and in its subspaces $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$, because

$$\left\| \frac{d}{dt} T(t) f \right\|_\infty = \|\Delta T(t) f\|_\infty \leq \frac{C}{t} \|f\|_\infty, \;\; t > 0, \; f \in L^\infty(\mathbb{R}^n).$$

Estimates (1.14) imply also that for each $\varepsilon > 0$, and $k > h \in \mathbb{N} \cup \{0\}$, there is $C = C(\varepsilon, h, k)$ such that

$$\|T(t)\|_{\mathcal{L}(C_b^h(\mathbb{R}^n), C_b^k(\mathbb{R}^n))} \leq \frac{C e^{\varepsilon t}}{t^{(k-h)/2}}.$$

More generally, for $0 \leq \theta_1 \leq \theta_2$ we have

$$\|T(t)\|_{\mathcal{L}(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))} \leq \frac{C e^{\varepsilon t}}{t^{(\theta_2 - \theta_1)/2}}. \tag{1.15}$$

For θ_1 or θ_2 noninteger, this can be proved either directly, using formula (1.13), or by interpolation, recalling the characterization of example 1.1.7 and theorem 1.0.4. For instance, using the estimates $\|T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))} \leq C e^{\varepsilon t}/t$, $\|T(t)\|_{\mathcal{L}(C_b^1(\mathbb{R}^n), C_b^3(\mathbb{R}^n))} \leq C e^{\varepsilon t}/t$, and the characterizations $(C_b(\mathbb{R}^n), C_b^1(\mathbb{R}^n))_{\alpha,\infty} = C^\alpha(\mathbb{R}^n)$, $(C_b^2(\mathbb{R}^n), C_b^3(\mathbb{R}^n))_{\alpha,\infty} = C^{2+\alpha}(\mathbb{R}^n)$, we get $\|T(t)\|_{\mathcal{L}(C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))} \leq C' e^{\varepsilon t}/t$ for $t > 0$, which shows that $T(t)$ is analytic also in $C^\alpha(\mathbb{R}^n)$.

The generator of $T(t)$ in each of the above mentioned functional spaces ($X = L^p(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$, $C^\alpha(\mathbb{R}^n)$) is the realization of the Laplacian in X.

Example 1.2.2 The *Ornstein-Uhlenbeck* semigroup

$$T(t) f(x) = \frac{1}{(4\pi t)^{n/2} (\det K_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\frac{|K_t^{-1/2} y|^2}{4t}} f(e^{tB} x - y) dy, \tag{1.16}$$

where Q is a symmetric and positive definite matrix, $B \neq 0$ is a $n \times n$ matrix, and

$$K_t = \frac{1}{t} \int_0^t e^{sB} Q e^{sB^*} ds, \tag{1.17}$$

with $e^{sB} = \sum_{n=0}^\infty s^n B^n / n!$. This semigroup is well defined in all the above functional spaces, it is strongly continuous in $L^p(\mathbb{R}^n)$ for $1 \leq p < +\infty$, if $B \neq 0$ it is not strongly continuous in $L^\infty(\mathbb{R}^n)$, in $C_b(\mathbb{R}^n)$, in $BUC(\mathbb{R}^n)$, and in $C^\alpha(\mathbb{R}^n)$.

For a large class of initial data f, the function $u(t,x) = (T(t)f)(x)$ is the solution to problem

$$\begin{cases} u_t(t,x) = \mathcal{A} u(t,x), \;\; t > 0, \; x \in \mathbb{R}^n, \\ \\ u(0,x) = f(x), \;\; x \in \mathbb{R}^n. \end{cases}$$

where $\mathcal{A}$ is the Ornstein-Uhlenbeck operator defined by

$$(\mathcal{A}) u(x) = \sum_{i,j=1}^n q_{ij} D_{ij} u(x) + \sum_{i,j=1}^n b_{ij} x_i D_j u(x) = \mathrm{Tr}(Q D^2 u)(x) + \langle Bx, Du(x) \rangle. \tag{1.18}$$

Even if the derivatives do not commute with $T(t)$, we have a simple commutation formula,

$$D(T(t)f) = e^{tB^*}T(t)(Df), \;\; t > 0, \; f \in C_b^1(\mathbb{R}^n).$$

that allows to prove estimates

$$\|D^\beta T(t)f\|_\infty \le \frac{Ce^{\varepsilon(|\beta|-h)t}}{t^{(|\beta|-h)/2}}\|f\|_{C_b^h(\mathbb{R}^n)}, \;\; t > 0, \; f \in C_b^h(\mathbb{R}^n), \tag{1.19}$$

for any multi-index β with $|\beta| \ge h$. Here ε depends on the matrix B, we may take $\varepsilon = 0$ if all the eigenvalues of B have negative real part. As in the case of the heat semigroup, (1.15) holds with suitable ε.

However, note that (1.19) does not imply that $T(t)$ is analytic in $L^\infty(\mathbb{R}^n)$ or in $C_b(\mathbb{R}^n)$, because $\mathcal{A}$ has unbounded coefficients, hence from estimate $\|T(t)f\|_{C_b^2(\mathbb{R}^n)} \le C\|f\|_\infty/t$ we cannot deduce that $\|\mathcal{A}T(t)f\|_\infty \le C\|f\|_\infty/t$. In fact, it is possible to show that $T(t)$ is not analytic in $L^p(\mathbb{R}^n)$, for $1 \le p \le \infty$.

In the sequel, we shall use two basic properties of strongly continuous and of analytic semigroups. First, the representation formula for the resolvent of the generator A of $T(t)$:

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt, \; f \in X, \;\; \lambda > \omega, \tag{1.20}$$

provided $\|T(t)\| \le Me^{\omega t}$ for some M, ω, and for all $t > 0$; and second, the variation of constants formula

$$u(t) = \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds, \tag{1.21}$$

that represents the (mild) solution to the forward Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \;\; 0 < t < T, \\[2mm] u(0) = x. \end{cases} \tag{1.22}$$

Precisely, we have the following results.

Proposition 1.2.3 *Let $T(t)$ be a strongly continuous or analytic semigroup such that $\|T(t)\| \le Me^{\omega t}$ for every $t \ge 0$, and let A be its generator. Then the resolvent set $\rho(A)$ of A contains the halfplane $\Sigma := \{\lambda \in \mathbb{C} : Re\ \lambda > \omega\}$, and (1.20) holds for $\lambda \in \Sigma$.*

Note that formula (1.20) is still meaningful in several cases in which $T(t)$ is not strongly continuous or analytic, and it may be used to define a generalized notion of infinitesimal generator, as follows.

Example 1.2.4 Let $T(t)$ be a semigroup in the Banach space $X = C_b(\overline{\Omega})$ such that $\|T(t)\| \le Me^{\omega t}$ and such that for each $f \in X$ and for each compact set $K \subset \overline{\Omega}$ the function $(t, x) \mapsto (T(t)f)(x)$ is continuous in $[0, +\infty) \times K$. For Re $\lambda > \omega$ the operator $R(\lambda)$ given by

$$(R(\lambda)f)(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt$$

belongs to $\mathcal{L}(X)$. Due to the semigroup property, it satisfies the resolvent identity

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu).$$

Moreover, $R(\lambda)$ is one to one for all $\lambda > \omega$: if $R(\lambda_0)f$ is the null function for some λ_0, then it vanishes for all λ because of the resolvent identity. Then for each $x \in \mathbb{R}^n$ the Laplace transform of the continuous function $t \mapsto (T(t)f)(x)$ is zero, so that $(T(t)f)(x) = 0$ for each $t \geq 0$, in particular taking $t = 0$ we get $f(x) = 0$.

The general spectral theory implies that there exists a unique closed operator A such that $\rho(A) \supset (0, \infty)$ and $R(\lambda) = R(\lambda, A)$, for every $\lambda > 0$.

Such operator A may still be called generator of $T(t)$.

Our main example is the Ornstein-Uhlenbeck operator defined above. For each $t > 0$ we have $\|T(t)\| \leq 1$, and it is easy to check that for each continuous and bounded f, for each compact set $K \subset \mathbb{R}^n$ the function $(t, x) \mapsto (T(t)f)(x)$ is continuous in $[0, +\infty) \times K$. In fact, $t \mapsto (T(t)f)(x)$ is analytic in $(0, +\infty)$ for each x.

Concerning formula (1.21), it is not hard to see that if $T(t)$ is a strongly continuous or analytic semigroup, $f \in C([0, T]; X)$, and (1.22) has a solution $u \in C^1([0, T]; X) \cap C([0, T]; D(A))$, then u is given by formula (1.21).

Whenever (1.21) makes sense, the function u defined there is called *mild solution* to (1.22).

Lecture 2

Schauder type theorems

We begin with an interpolation proof of the classical Schauder theorem.

Theorem 2.0.5 *Let $u \in C^\alpha(\mathbb{R}^n)$ be such that $\Delta u \in C^\alpha(\mathbb{R}^n)$, with $0 < \alpha < 1$. Then $u \in C^{2+\alpha}(\mathbb{R}^n)$, and there is $C > 0$, independent of u, such that*

$$\|u\|_{C^{2+\alpha}(\mathbb{R}^n)} \le C(\|u\|_\infty + \|\Delta u\|_{C^\alpha(\mathbb{R}^n)}).$$

Proof — Fix any $\lambda > 0$ and set $\lambda u - \Delta u = f$. Let $T(t)$ be the Gauss-Weierstrass semigroup defined in (1.13). Then for each $x \in \mathbb{R}^n$

$$u(x) = \int_0^{+\infty} e^{-\lambda t}(T(t)f)(x)dt. \tag{2.1}$$

If we use estimates (1.15) with $\theta_1 = \alpha$, $\theta_2 = 2+\alpha$ and $\varepsilon = \lambda/2$ we get $\|T(t)f\|_{C^{2+\alpha}(\mathbb{R}^n)} \le C\exp(\varepsilon t)t^{-1}$ and it is not obvious that that the integral defines a $C^{2+\alpha}$ function because of the singularity at $t = 0$. So, we use the following trick: we fix $\xi > 0$ and split $u = a(\xi) + b(\xi)$, where

$$a(\xi) = \int_0^\xi e^{-\lambda t}T(t)f\, dt, \qquad b(\xi) = \int_\xi^\infty e^{-\lambda t}T(t)f\, dt.$$

Then $a(\xi) \in C^{\alpha+\theta}(\mathbb{R}^n)$ for each $\theta \in (0,2)$, with

$$\|a(\xi)\|_{C^{\alpha+\theta}} \le C\int_0^\xi t^{-\theta/2}dt\, \|f\|_{C^\alpha} = C'\xi^{1-\theta/2}\|f\|_{C^\alpha},$$

and $b(\xi) \in C^{2+\alpha+\theta}(\mathbb{R}^n)$, with

$$\|b(\xi)\|_{C^{2+\alpha+\theta}} \le C\int_t^{+\infty} t^{-1-\theta/2}dt\, \|f\|_{C^\alpha} = C''\xi^{-\theta/2}\|f\|_{C^\alpha}.$$

Therefore, for each $\xi > 0$,

$$K(\xi, u, C^{\alpha+\theta}(\mathbb{R}^n), C^{2+\alpha+\theta}(\mathbb{R}^n)) \le \|a(\xi)\|_{C^{\alpha+\theta}} + \xi\|b(\xi)\|_{C^{2+\alpha+\theta}} \le C\xi^{1-\theta/2}\|f\|_{C^\alpha},$$

so that, using the definition of $(X, Y)_{\gamma,\infty}$ and then example 1.1.7,

$$u \in (C^{\alpha+\theta}(\mathbb{R}^n), C^{2+\alpha+\theta}(\mathbb{R}^n))_{1-\theta/2,\infty} = C^{2+\alpha}(\mathbb{R}^n),$$

and the statement follows. $\square$

It is well known that the domain of the Laplacian in $C_b(\mathbb{R}^n)$ is strictly bigger than $C_b^2(\mathbb{R}^n)$. Let us see where the above proof fails if we take $\alpha = 0$.

Estimates (1.15) are true also for $\alpha = 0$, so we still get

$$K(\xi, u, C^{\theta}(\mathbb{R}^n), C^{2+\theta}(\mathbb{R}^n)) \le C\xi^{1-\theta/2}\|f\|_\infty,$$

which implies that $u \in (C^{\theta}(\mathbb{R}^n), C^{2+\theta}(\mathbb{R}^n))_{1-\theta/2,\infty} = \mathcal{C}^{*2}(\mathbb{R}^n)$. So, what we miss in the integer case is the very last step.

Let us see some extension of theorem 2.0.5. First, we may replace α by $\alpha + k$, where k is any positive integer, and we get further Hölder regularity: u, $\Delta u \in C^{\alpha+k}(\mathbb{R}^n)$ implies that $u \in C^{\alpha+k+2}(\mathbb{R}^n)$.

The same proof works if the Laplace operator is replaced by the Ornstein-Uhlenbeck operator. There is just a subtle difference, that however does not create big problems. In the case of the heat semigroup, $t \mapsto T(t)f$ is smooth (in fact, analytic) for $t > 0$ with values in $C^{\alpha+\theta}(\mathbb{R}^n)$ for each $\theta > 0$, but the same is not true in the case of the Ornstein-Uhlenbeck semigroup. In fact, it is possible to show that it is not measurable. Therefore, the integrals that define $a(\xi)$ and $b(\xi)$ are not meaningful as $C^{\alpha+\theta}$ and $C^{2+\alpha+\theta}$-valued integrals. They have to be understood pointwise:

$$a(\xi)(x) = \int_0^\xi e^{-\lambda t}(T(t)f)(x)dt, \;\; b(\xi)(x) = \int_\xi^\infty e^{-\lambda t}(T(t)f)(x)dt, \;\; x \in \mathbb{R}^n.$$

Also the Hölder estimates have to be done pointwise; for instance if $\alpha + \theta < 1$

$$|a(\xi)(x) - a(\xi)(y)| \le \int_0^\xi e^{-\lambda t}|(T(t)f)(x) - (T(t)f)(y)|dt$$

$$\le C\int_0^\xi t^{-\theta/2}dt\,\|f\|_{C^\alpha}|x-y|^{\alpha+\theta}.$$

From these examples it is clear that this method may be extended to a number of situations. What we need is three Banach spaces $Y^2 \subset Y^1 \subset X$, a smoothing semigroup, such that $\|T(t)\|_{\mathcal{L}(X,Y_i)} \le C\exp(\omega t)t^{-\gamma_i}$ for $t > 0$, with $0 \le \gamma_1 < 1 < \gamma_2$, and such that the resolvent $R(\lambda, A)$ of the generator is represented by formula (1.20) for some $\lambda > \omega$.

Usually the most difficult part is the proof of the estimates $\|T(t)\|_{\mathcal{L}(X,Y_i)} \le Ct^{-\gamma_i}$ for t small.

What we get is that the domain of A in X is continuously embedded in an interpolation space between Y_1 and Y_2, precisely in $(Y_1, Y_2)_{(1-\gamma_1)/(\gamma_2-\gamma_1),\infty}$.

We describe below some situations in which this procedure works. Estimates (1.15), as well as formula (1.20), are well known for semigroups generated by second order elliptic operators with regular bounded coefficients, and the procedure of theorem 2.0.5 gives an altrenative proof of Schauder estimates for such operators. However, the method may be applied to less standard situations.

Example 2.0.6 *Elliptic operators with unbounded regular coefficients in* $\mathbb{R}^n$.

Let $\mathcal{A}$ be a second order uniformly elliptic differential operator with regular, possibly unbounded, coefficients:

$$\begin{aligned}\mathcal{A}f(x) &= \sum_{i,j=1}^n q_{ij}(x)D_{ij}f(x) + \sum_{i=1}^n b_i(x)D_i f(x) + v(x)f(x),\\ &= \mathrm{Tr}(Q(x)D^2 f(x)) + \langle B(x), Df(x)\rangle + v(x)f(x).\end{aligned}$$

Under several assumptions, it is possible to show that the associated evolution problem

$$\begin{cases} u_t(t,x) = \mathcal{A}u(t,x), \;\; t > 0, \; x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \;\; x \in \mathbb{R}^n, \end{cases}$$

with $u_0 \in C_b(\mathbb{R}^n)$, has a unique solution u such that u is bounded on $[0,T] \times \mathbb{R}^n$ for each $T > 0$, and that setting $T(t)u_0 = u(t,\cdot)$, $T(t)$ enjoys nice smoothing properties, i.e. it satisfies (1.15) for each $0 \le \theta_1 \le \theta_2 \le 3$, and for suitable $\varepsilon > 0$. Moreover, for λ large and $f \in C_b(\mathbb{R}^n)$ the elliptic problem

$$\lambda u - \mathcal{A}u = f$$

has a unique bounded solution $u \in C_b(\mathbb{R}^n) \cap W^{2,p}_{loc}(\mathbb{R}^n)$ for each $p \ge 1$, and u is given by the representation formula (1.20). Then the above procedure gives a Schauder theorem for the operator $\mathcal{A}$.

The hard part of the job is to prove estimates (1.15). They may be obtained either by stochastic methods, see [7], and by deterministic methods, see [23] and the more recent and sharp [2, 3]. The coefficients may grow not more than polynomially in [7] and not more than exponentially in [23, 2, 3]. The common assumptions in all these papers are upper boundedness of v, dissipativity hypotheses on the drift, such as

$$\langle B(x) - B(y), x - y\rangle \le \omega |x-y|^2,$$

and some assumption guaranteeing uniqueness of the bounded solution in the parabolic and in the elliptic problem, such as

$$\exists \varphi \in C^2(\mathbb{R}^n) : \lim_{|x|\to+\infty} \varphi(x) = +\infty, \ \sup_{x\in\mathbb{R}^n} \mathcal{A}\varphi - \lambda_0\varphi < +\infty$$

for some $\lambda_0 \ge 0$.

In any case, these assumptions cover the Ornstein-Uhlenbeck operator, and more generally operators of the type $u \mapsto \Delta u + \langle F(x), Du\rangle$ with Lipschitz continuous F.

Example 2.0.7 *Elliptic operators in infinitely many variables.*

Cannarsa and Da Prato used the method of theorem 2.0.5 to get Schauder estimates for elliptic operators in separable Hilbert space H, such as

$$(\mathcal{A}u)(x) = \frac{1}{2}\mathrm{Tr}(QD^2u)(x) + \langle Bx, Du(x)\rangle$$

where $Q \in \mathcal{L}(H)$ is a symmetric nonnegative operator, and $B : D(B) \mapsto H$ is the infinitesimal generator of a strongly continuous semigroup.

The theory of Gaussian measures in Hilbert spaces allows to extend formulae (1.13) and (1.16) to this setting and to define infinite dimensional heat and Ornstein-Uhlenbeck semigroups. However, there are additional difficulties that do not exist in finite dimensions. In the case $B = 0$, one needs that the trace of Q (defined as the sum $\sum_{i=1}^{\infty}\langle Qe_i, e_i\rangle$ for each orthonormal basis $\{e_i : i \in \mathbb{N}\}$ of H) is finite. If B is of negative type, i.e. $\|e^{tB}\|_{\mathcal{L}(H)} \le Ce^{-\omega t}$ with C, $\omega > 0$, we can allow also $Q = I$, but we need that the operator K_t defined in (1.17) has finite trace for each $t > 0$. In this second case, estimates (1.15) and the conclusion of theorem 2.0.5 hold, with $\mathbb{R}^n$ replaced by H. See the papers [5, 6], and the book [8] as a general reference.

Example 2.0.8 *The degenerate (hypoelliptic) Ornstein-Uhlenbeck operator.*

Let us consider again the Ornstein-Uhlenbeck operator (1.18). Now we allow the matrix $Q \ge 0$ to be noninvertible, but we assume that

$$\det K_t > 0, \quad t > 0. \tag{2.2}$$

This condition is equivalent to the fact that the operator $\mathcal{A}$ is hypoelliptic in the sense of Hörmander ([13]). So, if $f \in C^\infty(\mathbb{R}^n)$ and u is a distributional solution of

$$\mathcal{A}u = f, \tag{2.3}$$

then $u \in C^\infty(\mathbb{R}^n)$. The hypoellipticity condition is also equivalent to

$$\text{Rank}\,[Q^{1/2}, BQ^{1/2}, \dots, B^{n-1}Q^{1/2}] = n, \tag{2.4}$$

which is well known in control theory and is called *Kalman rank condition.* See e.g. [32, Ch. 1] for a proof of the equivalence.

Since $\mathcal{A}$ is a degenerate elliptic operator, the usual Hölder spaces are not the right setting for $\mathcal{A}$. It is more convenient to replace them by suitable Hölder spaces, adapted to $\mathcal{A}$, defined as follows.

Let $k \in \{0, \dots, n-1\}$ be the smallest integer such that

$$\text{Rank}\,[Q^{1/2}, BQ^{1/2}, \dots, B^{k}Q^{1/2}] = n.$$

Note that the matrix Q is nonsingular if and only if $k = 0$. Set $V_0 = \text{Range } Q^{1/2}$, $V_h = \text{Range } Q^{1/2} + \text{Range } BQ^{1/2} + \dots + \text{Range } B^hQ^{1/2}$. Of course, $V_h \subset V_{h+1}$ for every h, and $V_k = \mathbb{R}^n$. Define the orthogonal projections E_h as

$$\begin{cases} E_0 = \text{projection on } V_0, \\ \\ E_{h+1} = \text{projection on } (V_h)^\perp \text{ in } V_{h+1}, \quad h = 1, \dots, k-1. \end{cases} \tag{2.5}$$

Then $\mathbb{R}^n = \bigoplus_{h=0}^k E_h(\mathbb{R}^n)$. By possibly changing coordinates in $\mathbb{R}^n$ we will consider an orthogonal basis $\{e_1, \dots, e_n\}$ consisting of generators of the subspaces $E_h(\mathbb{R}^n)$. For every $h = 0, \dots, k$ we define I_h as the set of indices i such that the vectors e_i with $i \in I_h$ span $E_h(\mathbb{R}^n)$. After the changement of coordinates the second order derivatives which appear in (1.18) are only the $D_{ij}u$ with $i, j \in I_0$.

The distance d associated to $\mathcal{A}$ is defined by

$$d(x,y) = \sum_{h=0}^{k} |E_h(x-y)|^{1/(2h+1)},$$

where $|\cdot|$ is the usual Euclidean norm. For $\alpha > 0$ such that $\alpha/(2h+1) \notin \mathbb{N}$ for $h = 0, \dots, k$, the space $C^\alpha_d(\mathbb{R}^n)$ is the set of all the bounded functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that for every $x_0 \in \mathbb{R}^n$, $0 \le h \le k$, the mapping $E_h(\mathbb{R}^n) \mapsto \mathbb{R}$, $x \mapsto f(x_0 + x)$ belongs to $C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))$, with norm bounded by a constant independent of x_0. It is endowed with the norm

$$\|f\|_{C^\alpha_d(\mathbb{R}^n)} = \sum_{h=0}^{k} \sup_{x_0 \in \mathbb{R}^n} \|f(x_0 + \cdot)\|_{C^{\alpha/(2h+1)}(E_h(\mathbb{R}^n))}.$$

In particular, for $0 < \alpha < 1$ it is the space of the bounded functions f such that

$$[f]_{C^\alpha_d(\mathbb{R}^n)} = \sup_{x,y \in \mathbb{R}^n,\, x \neq y} |f(x) - f(y)| \left(\sum_{h=0}^{k} |E_h(x-y)|^{\alpha/(2h+1)} \right)^{-1} < \infty.$$

Using the representation formula (1.16) it is possible to show that (1.15) holds provided the usual Hölder spaces are replaced by the spaces $C^\theta_d(\mathbb{R}^n)$.

Using the above arguments we arrive at the following theorem: Let $0 < \alpha < 1$, $\lambda > 0$. Then for every $f \in C^\alpha_d(\mathbb{R}^n)$ the problem

$$\lambda u - \mathcal{A}u = f$$

has a unique distributional solution u in the space of the uniformly continuous and bounded functions. Moreover u is a strong solution, it belongs to $\mathcal{C}^{2+\alpha}_d(\mathbb{R}^n)$, and there is $C > 0$, independent of f, such that

$$\|u\|_{\mathcal{C}^{2+\alpha}_d(\mathbb{R}^n)} \le C\|f\|_{\mathcal{C}^\alpha_d(\mathbb{R}^n)}. \tag{2.6}$$

See [22].

The simplest significant example of this class of operators is the Kolmogorov operator in $\mathbb{R}^2$,

$$\mathcal{A}u(x,y) = u_{xx} - xu_y,$$

in which case $k = 1$, V_0 is the x-axis, V_1 is the y-axis, and $\mathcal{C}_d^\alpha(\mathbb{R}^2) = C_{x\ y}^{\alpha,\alpha/3}(\mathbb{R}^2)$, $C_d^{2+\alpha}(\mathbb{R}^2) = C_{x\ y}^{2+\alpha,2/3+\alpha/3}(\mathbb{R}^2)$.

Example 2.0.9 *The Heisenberg Laplacian.*

The Heisenberg Laplacian is the operator in $\mathbb{R}^3$

$$\Delta_H f(x,y,z) = f_{xx} + f_{yy} + 4yf_{xz} - 4xf_{yz} + 4(x^2+y^2)f_{zz}. \tag{2.7}$$

It is the simplest example of a sublaplacian in a stratified Lie group. See [11, 29] for the general theory, and [14] for a number of problems where the Heisenberg Laplacian arises. Note that Δ_H is elliptic degenerate at each point of R^3; it is possible to show that it is hypoelliptic.

The sum

$$(x,y,z) \oplus (x',y',z') = (x+x', y+y', z+z'+2(yx'-xy')),$$

gives $\mathbb{R}^3$ the structure of noncommutative Lie group, called the Heisenberg group. The distance in $\mathbb{R}^3$ associated to the Heisenberg group and to the Heisenberg Laplacian is $d(p,\widetilde{p}) = |(-\widetilde{p})\oplus p|$, where $|\cdot|$ is the pseudonorm defined by $|(x,y,z)| = ((x^2+y^2)^2+z^2)^{1/4}$.

Several mathematical structures have to be adapted to the sum $\oplus$ and to the distance d. For instance, the space of the "uniformly continuous" functions is defined by

$$BUC_H(\mathbb{R}^3) = \{f \in L^\infty(\mathbb{R}^3) :\ \lim_{p\to 0} \|f(\cdot\oplus p) - f\|_\infty = 0\}.$$

The "derivatives" D_j, $j = 1,2,3$ are defined by

$$D_j f(p) = \lim_{h\to 0} \frac{f(p\oplus he_j) - f(p)}{h}, \tag{2.8}$$

for all functions $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that such limits exist. If f is smooth, then

$$D_1 f = f_x + 2yf_z,\ \ D_2 f = f_x - 2xf_z,\ \ D_3 f = f_z,$$

and

$$\Delta_H f = (D_1^2 + D_2^2)f.$$

Note that, unlike in the euclidean setting, D_1 and D_2 do not commute, and we have

$$[D_1, D_2] = -4D_3.$$

Let $k \in \mathbb{N}$, let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ be a multi-index with $\alpha_i \in \{1,2\}$, and set $|\alpha| = k$, $D_\alpha = D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_k}$. D_α is said to be a derivative of order k. The spaces $C_H^k(\mathbb{R}^3)$ are defined as the sets of the continuous and bounded functions f such that there exists $D_\alpha f \in C_b(\mathbb{R}^3)$ for each derivative of order k. They are endowed with the norm $\|f\|_{C_H^k} = \sum_{0\le|\alpha|\le k} \|D_\alpha f\|_\infty$.

For $0 < \theta < 1$ the Hölder space with respect to the distance d and the sum $\oplus$ is

$$\begin{cases} C_H^\theta(\mathbb{R}^3) = \left\{ f \in L^\infty(\mathbb{R}^3) :\ [f]_{C_H^\theta} = \displaystyle\sup_{p\in\mathbb{R}^3,\, h\neq 0} \frac{|f(p\oplus h) - f(p)|}{|h|^\theta} < +\infty \right\}, \\ \\ \|f\|_{C_H^\theta} = \|f\|_\infty + [f]_{C_H^\theta}. \end{cases} \tag{2.9}$$

For $0<\theta<1$, $k\in\mathbb{N}$ we set (see [11, 15, 29])

$$\begin{cases} C_H^{\theta+k}(\mathbb{R}^3) = \{f\in C_H^k(\mathbb{R}^3):\ D_\alpha f\in C_H^\theta(\mathbb{R}^3),\ |\alpha|\le k\}, \\ \|f\|_{C_H^{\theta+k}} = \|f\|_{C_H^k} + [f]_{C_H^{\theta+k}} = \|f\|_{C_H^k} + \sum_{|\alpha|=k}[D_\alpha f]_{C_H^\theta}. \end{cases} \tag{2.10}$$

The realization L of the Heisenberg Laplacian in $X = L^p(\mathbb{R}^3)$, $1\le p<\infty$, or in $X = BUC_H(\mathbb{R}^3)$ generates a contraction analytic semigroup $T(t)$ thanks to [11, Thm 3.1]. An explicit representation formula for $T(t)$ is in [12].

Again, estimates (1.15) hold provided the Hölder norms are replaced by the C_H^θ norms. Therefore, Schauder theorems in such Hölder spaces hold. See [25, 27].

Example 2.0.10 *Elliptic operators in $L^1(\mathbb{R}^n)$.*

Instead of taking as reference space a space of continuous and bounded, or Hölder continuous functions, we consider now $X = L^1(\mathbb{R}^n)$. The heat semigroup $T(t)$ is easily seen to satisfy

$$(i)\ \|DT(t)f\|_{L^1}\le \frac{C}{t^{1/2}}\|f\|_{L^1},\ \ (ii)\ \|DT(t)f\|_\infty \le \frac{C}{t^{(n+1)/2}}\|f\|_{L^1},\ \ t>0,\ f\in L^1(\mathbb{R}^n). \tag{2.11}$$

The procedure of 2.0.5, applied to the derivatives D_iu instead of u, give that if u, Δu are in $L^1(\mathbb{R}^n)$ then

$$D_iu\in (L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1/n,\infty} = L^{n/(n-1),\infty}(\mathbb{R}^n), \tag{2.12}$$

where for $p\ge 1$, $L^{p,\infty}(\mathbb{R}^n)$ is the Lorentz space defined by

$$L^{p,\infty}(\mathbb{R}^n) = \{f\in L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n) : \sup_{\sigma>0}\sigma^p|\{x\in\mathbb{R}^n : |f(x)|>\sigma\}<\infty\}.$$

It is well known that the domain of the realization of the Laplacian in $L^1(\mathbb{R}^n)$ is not contained in $W^{2,1}(\mathbb{R}^n)$ if $N>1$, i.e. the derivatives of the functions u in the domain do not belong to $W^{1,1}(\mathbb{R}^n)$. One may ask whether a weaker inclusion holds, i.e. $D_iu\in L^{n/(n-1)}(\mathbb{R}^n)$ (this is because, by Sobolev embedding, $W^{1,1}(\mathbb{R}^n)\subset L^{n/(n-1)}(\mathbb{R}^n)$). But easy counterexamples show that this is not true, and the most we can get is (2.12).

Estimates (2.11), and consequently embedding (2.12), are true for a number of semigroups generated by elliptic operators in L^1 spaces, including elliptic operators with unbounded Lipschitz continuous coefficients in $\mathbb{R}^n$. See [28].

[To be honest, here I cheated a bit. I mentioned the interpolation space $(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1/n,\infty}$, but up to now I considered only interpolation spaces between couples of Banach spaces such that $Y\subset X$, while $L^\infty(\mathbb{R}^n)$ is not contained in $L^1(\mathbb{R}^n)$. However, the general interpolation theory and the procedure of theorem 2.0.5 work as well.]

Example 2.0.11 *Elliptic operators in fractional Sobolev spaces.*

Let us come back to theorem 2.0.5 for the Laplacian, taking as reference space $X = L^p(\mathbb{R}^n)$ or $X = W^{\alpha,p}(\mathbb{R}^n)$, $1\le p<\infty$. It is natural to work in the interpolation spaces $(X,Y)_{\theta,p}$ instead of $(X,Y)_{\theta,\infty}$, but in this case estimates similar to (1.15),

$$\|T(t)\|_{\mathcal{L}(L^p(\mathbb{R}^n),W^{\theta,p}(\mathbb{R}^n))}\le \frac{Ce^{\varepsilon t}}{t^{\theta/2}},\ \ \|T(t)\|_{\mathcal{L}(W^{\theta_1,p}(\mathbb{R}^n),W^{\theta_2,p}(\mathbb{R}^n))}\le \frac{Ce^{\varepsilon t}}{t^{(\theta_2-\theta_1)/2}},$$

(that are easily seen to be true) are not enough. What we need is

$$\begin{cases} t^{(\theta_2-\theta_1)/2}e^{-\epsilon t}\|T(t)\|_{\mathcal{L}(W^{\theta_1,p}(\mathbb{R}^n),W^{\theta_2,p}(\mathbb{R}^n))} \leq c(t), \quad t>0, \\ \\ c \in L^p_*(0,+\infty), \quad i=1,2. \end{cases} \tag{2.13}$$

which is still true if θ_1 and θ_2 are not integers, but the proof is not immediate. Once (2.13) is established, we use it to prove that for $f \in W^{\alpha,p}(\mathbb{R}^n)$, the function u given by (2.1) satisfies

$$\xi \mapsto \xi^{\theta/2-1}K(\xi, u, W^{\alpha+\theta,p}(\mathbb{R}^n), W^{2+\alpha+\theta,p}(\mathbb{R}^n)) \in L^p_*(0,+\infty),$$

and hence, by the definition of $(X,Y)_{\gamma,p}$ and then example 1.1.7,

$$u \in (W^{\alpha+\theta,p}(\mathbb{R}^n), W^{2+\alpha+\theta,p}(\mathbb{R}^n))_{1-\theta/2,p} = W^{2+\alpha,p}(\mathbb{R}^n),$$

if α is not integer. This implies that the domain of the Laplacian in the space $W^{\alpha,p}(\mathbb{R}^n)$ is $W^{2+\alpha,p}(\mathbb{R}^n)$, if α is not integer.

Estimates (2.13), as well as the conclusion, are still true for the Ornstein-Uhlenbeck semigroup. A proof is in [26].

Lecture 3

Hölder regularity in evolution problems

We try to follow the procedure of Lecture 2 for evolution problems, of the type

$$\begin{cases} u'(t) = Au(t) + f(t), \;\; 0 < t < T, \\ \\ u(0) = 0. \end{cases}$$

Also for evolution problems, we start with the case where A is the realization of the Laplacian in a space of Hölder continuous functions, and then we see to which extent it is possible to generalize the procedure.

As in the stationary case, the main ingredients are a representation formula for u that involves the semigroup $T(t)$ generated by A, i.e. the variation of constants formula

$$u(t) = \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds,$$

and the smoothing properties of the semigroup.

The parallel of the Schauder theorem of Lecture 2 is the following result which was proved for the first time by Kružkhov, Castro and Lopes in [17, 18] with different methods.

The space $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$ consists of the continuous and bounded functions $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ which are α-Hölder continuous with respect to the space variables, uniformly in t. It is endowed with the norm

$$\|f\|_{C^{0,\alpha}} = \|f\|_\infty + \sup_{0\le t\le T} [f(t,\cdot)]_{C^\alpha(\mathbb{R}^n)}.$$

Theorem 3.0.12 *Let $f \in C^{0,\alpha}([0,T]\times\mathbb{R}^n)$. Then problem*

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x), \;\; 0 \le t \le T, \; x \in \mathbb{R}^n, \\ \\ u(0,x) = 0, \end{cases}$$

has a unique classical solution u such that u_t and $D_{ij}u$ belong to $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$. There is $C > 0$ independent of f such that

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\alpha}(\mathbb{R}^n)} \le C \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^\alpha(\mathbb{R}^n)}.$$

Proof — We use the variation of constants formula above, that implies easily that u is continuous and bounded in $[0,T]\times\mathbb{R}^n$.

Let us show that $u(t,\cdot)$ is bounded with values in $C^{2+\alpha}(\mathbb{R}^n)$. Fix $t\in[0,T]$. For $\xi>0$ split again $u(t)=a(\xi)+b(\xi)$, where

$$a(\xi)=\int_0^{\min(\xi,t)}(T(s)f(t-s))(x)ds,$$

$$b(\xi)=\int_{\min(\xi,t)}^t(T(s)f(t-s))(x)ds.$$

Then $a(\xi)\in C^{\alpha+\theta}(\mathbb{R}^n)$, and $b(\xi)\in C^{2+\alpha+\theta}(\mathbb{R}^n)$, for each $\theta\in(0,2)$. Precisely, using (1.15) we get

$$\|a(\xi)\|_{C^{\alpha+\theta}(\mathbb{R}^n)}\le\int_0^{\min(\xi,t)}\frac{Ce^{\varepsilon T}}{s^{\theta/2}}ds\sup_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)}$$

$$\le\frac{Ce^{\varepsilon T}}{1-\theta/2}\xi^{1-\theta/2}\sup\nolimits_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)},$$

$$\|b(\xi)\|_{C^{2+\alpha+\theta}}\le\int_{\min(\xi,t)}^t\frac{Ce^{\varepsilon T}}{s^{1+\theta/2}}ds\sup_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)}$$

$$\le\frac{Ce^{\varepsilon T}}{\theta/2}\xi^{-\theta/2}\sup\nolimits_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)}.$$

Therefore,

$$\xi^{-(1-\theta/2)}K(\xi,u(t),C^{\alpha+\theta}(\mathbb{R}^n),C^{2+\alpha+\theta}(\mathbb{R}^n))$$

$$\le Ce^{\varepsilon T}\left(\frac{1}{1-\theta/2}+\tfrac{2}{\theta}\right)\sup\nolimits_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)},\quad\xi>0,$$

so that $u(t)$ belongs to $(C^{\alpha+\theta}(\mathbb{R}^n),C^{2+\alpha+\theta}(\mathbb{R}^n))_{1-\theta/2,\infty}=C^{2+\alpha}(\mathbb{R}^n)$, and $\|u(t)\|_{C^{2+\alpha}(\mathbb{R}^n)}$ $\le C\sup_{0\le s\le T}\|f(s)\|_{C^\alpha(\mathbb{R}^n)}$, for every $t\in[0,T]$. To show that u is differentiable with respect to t it is sufficient to examine the variation of constants formula. For every $s\in[0,T]$ and $x\in\mathbb{R}^n$ the function $t\mapsto(T(t-s)f(s,\cdot))(x)$ is continuously differentiable in $(s,T]$, and $\partial/\partial t\ (T(t-s)f(s,\cdot))(x)=(\Delta T(t-s)f(s,\cdot))(x)$. Moreover, by (1.15) we have

$$|(\Delta T(t-s)f(s,\cdot))(x)|\le\frac{C}{(t-s)^{1-\alpha/2}}\sup_{0\le\sigma\le T}\|f(\sigma,\cdot)\|_{C^\alpha(\mathbb{R}^n)},$$

so that u is differentiable with respect to time and $u_t=\Delta u+f$ in $[0,T]\times\mathbb{R}^n$. The statement follows. □

Corollary 3.0.13 *Let $f\in C^{0,\alpha}([0,T]\times\mathbb{R}^n)$ and $u_0\in C^{2+\alpha}(\mathbb{R}^n)$. Then problem*

$$\begin{cases}u_t(t,x)=\Delta u(t,x)+f(t,x), & 0\le t\le T,\ x\in\mathbb{R}^n,\\ \\ u(0,x)=u_0(x), & x\in\mathbb{R}^n,\end{cases}$$

has a unique classical solution u such that u_t and $D_{ij}u$ belong to $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$. There is $C>0$ independent of f such that

$$\sup_{0\le t\le T}\|u(t)\|_{C^{2+\alpha}(\mathbb{R}^n)}\le C(\|u_0\|_{C^{2+\alpha}(\mathbb{R}^n)}+\sup_{0\le t\le T}\|f(t)\|_{C^\alpha(\mathbb{R}^n)}).$$

Theorem 3.0.12 and its corollary are immediately extended to the Ornstein-Uhlenbeck operator, with the same proof. Since the ingredients are estimates (1.15) and the variation of constants formula for the solution, it can be extended to a number of other situations, including all the examples of Lecture 2. They can be extended also to fractional Sobolev spaces, using $(X,Y)_{\theta,p}$ spaces such as in example 2.0.11, see [26]. We do not enter into further details.

Lecture 4

Regularity in parabolic Hölder spaces

The most popular Hölder regularity theorem for parabolic evolution problems is not theorem 3.0.12, but the Ladyzhenskaja–Solonnikov–Ural'ceva theorem (see e.g. the book [19]) that deals with regularity in the so called parabolic Hölder spaces. For $0 < \alpha < 1$ the space $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)$ consists of the bounded functions $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ which are $\alpha/2$-Hölder continuous with respect to the time variable, uniformly in x, and α-Hölder continuous with respect to the space variables, uniformly in t. It is endowed with the norm

$$\|f\|_{C^{\alpha/2,\alpha}} = \|f\|_\infty + \sup_{x\in\mathbb{R}^n}[f(\cdot,x)]_{C^{\alpha/2}([0,T])} + \sup_{0\le t\le T}[f(t,\cdot)]_{C^\alpha(\mathbb{R}^n)}.$$

It is clear that $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)$ coincides with the space of functions $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ such that $t \mapsto f(t,\cdot)$ is bounded with values in $C^\alpha(\mathbb{R}^n)$ and belongs to $C^{\alpha/2}([0,T]; C_b(\mathbb{R}^n))$.

The space $C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^n)$ consists of the bounded differentiable functions $f : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}$ such that f_t, $D_{ij}f$ belong to $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)$, and D_if belong to $C^{1/2+\alpha/2,1+\alpha}([0,T]\times\mathbb{R}^n)$, for $i,j = 1\ldots,n$.

Consider the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x), \;\; 0\le t\le T,\; x\in\mathbb{R}^n, \\ \\ u(0,x) = u_0(x), \;\; x\in\mathbb{R}^n, \end{cases} \tag{4.1}$$

Theorem 4.0.14 *Let $0 < \alpha < 1$ and let $f \in C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)$, $u_0 \in C^{2+\alpha}(\mathbb{R}^n)$. Then problem (4.1) has a unique solution $u \in C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^n)$, and there is C independent of f such that*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\mathbb{R}^n)} \le C(\|u_0\|_{C^{2+\alpha}(\mathbb{R}^n)} + \|f\|_{C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)}).$$

Proof — We already know that the function u defined by the variation of constants formula is the unique solution to problem (4.1), and that u_t, $D_{ij}u$ are in $C^{0,\alpha}([0,T]\times\mathbb{R}^n)$, due to corollary 3.0.13. To prove that they belong to $C^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n)$ we need to show that they are $\alpha/2$-Hölder continuous with respect to t.

To estimate the Hölder seminorm of u_t it is sufficient to estimate the Hölder seminorm of Δu. To this aim we use again estimates (1.15) with $\theta_1 = 0$ and $\theta_2 = 0, 2, 4$, which imply

$$\|T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \le M_0, \;\; \|\Delta T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \le \frac{M_1}{t}, \;\; \|\Delta^2 T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \le \frac{M_2}{t^2}, \;\; 0 < t \le T,$$

and with $\theta_1 = \alpha$, $\theta_2 = 2$, which imply

$$\|\Delta T(t)\|_{\mathcal{L}(C^\alpha(\mathbb{R}^n),C_b(\mathbb{R}^n))} \le \frac{M_{1,\alpha}}{t^{1-\alpha/2}}, \;\; 0 < t \le T.$$

Let us split $u = u_1 + u_2$, where

$$\begin{cases} u_1(t,\cdot) = \displaystyle\int_0^t T(t-\sigma)(f(\sigma,\cdot) - f(t,\cdot))d\sigma, \ \ 0 \le t \le T, \\ \\ u_2(t,\cdot) = T(t)u_0 + \displaystyle\int_0^t T(t-\sigma)f(t,\cdot)d\sigma, \ \ 0 \le t \le T. \end{cases} \tag{4.2}$$

Then

$$\begin{cases} (i) \quad \Delta u_1(t,\cdot) = \displaystyle\int_0^t \Delta T(t-s)(f(s,\cdot) - f(t,\cdot))ds, \ \ 0 \le t \le T, \\ \\ (ii) \quad \Delta u_2(t,\cdot) = \Delta T(t)u_0 + (T(t)-1)f(t,\cdot), \ \ \le t \le T. \end{cases} \tag{4.3}$$

For $0 \le s \le t \le T$ we have

$$\begin{array}{l} \Delta u_1(t,\cdot) - \Delta u_1(s,\cdot) = \displaystyle\int_0^s \Delta\left(T(t-\sigma) - T(s-\sigma)\right)(f(\sigma,\cdot) - f(s,\cdot))d\sigma \\ \\ +(T(t) - T(t-s))(f(s,\cdot) - f(t,\cdot)) + \displaystyle\int_s^t \Delta e^{(t-\sigma)A}(f(\sigma,\cdot) - f(t,\cdot))d\sigma, \end{array} \tag{4.4}$$

so that, since $\Delta(T(t-\sigma) - T(s-\sigma)) = \int_{s-\sigma}^{t-\sigma} \Delta^2 T(\tau)d\tau$,

$$\begin{array}{l} \|\Delta u_1(t,\cdot) - \Delta u_1(s,\cdot)\| \le M_2 \displaystyle\int_0^s (s-\sigma)^{\alpha/2} \int_{s-\sigma}^{t-\sigma} \tau^{-2} d\tau \, d\sigma \sup_{x\in\mathbb{R}^n} [f(\cdot,x)]_{C^{\alpha/2}([0,T])} \\ \\ +2M_0(t-s)^{\alpha/2}[f]_{C^\alpha} + M_1 \displaystyle\int_s^t (t-\sigma)^{\alpha/2-1} d\sigma \sup_{x\in\mathbb{R}^n} [f(\cdot,x)]_{C^{\alpha/2}([0,T])} \\ \\ \le M_2 \displaystyle\int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha/2-2} d\tau \sup_{x\in\mathbb{R}^n} [f(\cdot,x)]_{C^{\alpha/2}([0,T])} \\ \\ +(2M_0 + 2M_1\alpha^{-1})(t-s)^{\alpha/2} \sup_{x\in\mathbb{R}^n} [f(\cdot,x)]_{C^{\alpha/2}([0,T])} \\ \\ \le \left(\dfrac{M_2}{\alpha(1-\alpha/2)} + 2M_0 + \dfrac{2M_1}{\alpha} \right)(t-s)^{\alpha/2} \sup_{x\in\mathbb{R}^n} [f(\cdot,x)]_{C^{\alpha/2}([0,T])}. \end{array} \tag{4.5}$$

Therefore, Δu_1 is $\alpha/2$-Hölder continuous with respect to time. Concerning Δu_2, we add and subtract $(T(t) - T(s))f(0,\cdot)$ and we get

$$\|\Delta u_2(t,\cdot) - \Delta u_2(s,\cdot)\| \leq \|(T(t) - T(s))(\Delta u_0 + f(0,\cdot))\|$$

$$+\|(T(t) - T(s))(f(s,\cdot) - f(0,\cdot))\| + \|(T(t) - 1)(f(t,\cdot) - f(s,\cdot))\|$$

$$\leq \int_s^t \|\Delta T(\sigma)\|_{\mathcal{L}(C^\alpha(\mathbb{R}^n),C_b(\mathbb{R}^n))} d\sigma \, \|\Delta u_0 + f(0,\cdot)\|_{C^\alpha(\mathbb{R}^n)}$$

$$+s^{\alpha/2} \left\| \Delta \int_s^t T(\sigma) d\sigma \right\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \sup_{x\in\mathbb{R}^n}[f(\cdot,x)]_{C^{\alpha/2}([0,T])} \tag{4.6}$$

$$+(M_0 + 1)(t-s)^{\alpha/2} \sup_{x\in\mathbb{R}^n}[f(\cdot,x)]_{C^{\alpha/2}([0,T])}$$

$$\leq \frac{2M_{1,\alpha/2}}{\alpha} \|\Delta u_0 + f(0,\cdot)\|_{C^\alpha(\mathbb{R}^n)} (t-s)^{\alpha/2}$$

$$+\left(\frac{2M_1}{\alpha} + M_0 + 1\right)(t-s)^{\alpha/2} \sup_{x\in\mathbb{R}^n}[f(\cdot,x)]_{C^{\alpha/2}([0,T])},$$

so that also Δu_2 is Hölder continuous with respect to time, and the estimate

$$\sup_{x\in\mathbb{R}^n}[u_t(\cdot,x)]_{C^{\alpha/2}([0,T])} + \sup_{x\in\mathbb{R}^n}[\Delta u(\cdot,x)]_{C^{\alpha/2}([0,T])}$$

$$\leq C(\sup_{x\in\mathbb{R}^n}[f(\cdot,x)]_{C^{\alpha/2}([0,T])} + \|\Delta u_0 + f(0,\cdot)\|_{C^\alpha(\mathbb{R}^n)})$$

follows.

The rest of the statement is a consequence of this first part.

To find time Hölder regularity of all the second order derivatives $D_{ij}u$ we may argue as follows: we use the fact that $C^2(\mathbb{R}^n)$ belongs to the class $J_{1-\alpha/2}$ between $C^\alpha(\mathbb{R}^n)$ and $C^{2+\alpha}(\mathbb{R}^n)$. Moreover, since $\|u_t(t,\cdot)\|_{C^\alpha(\mathbb{R}^n)}$ is bounded in $[0,T]$, then $t \mapsto u(t,\cdot)$ is Lipschitz continuous with values in $C^\alpha(\mathbb{R}^n)$, with Lipschitz constant $\sup_{0\leq\sigma\leq T}\|u_t(\sigma,\cdot)\|_{C^\alpha}$. So, for $0 \leq s \leq t \leq T$ we have

$$\|u(t,\cdot) - u(s,\cdot)\|_{C^2} \leq C\|u(t,\cdot) - u(s,\cdot)\|_{C^\alpha}^{\alpha/2} \|u(t,\cdot) - u(s,\cdot)\|_{C^{2+\alpha}}^{1-\alpha/2}$$

$$\leq C((t-s)\sup_{0\leq\sigma\leq T}\|u_t(\sigma,\cdot)\|_{C^\alpha})^{\alpha/2}(2\sup_{\sigma\in[0,T]}\|u(\sigma,\cdot)\|_{C^{2+\alpha}})^{1-\alpha/2}$$

$$\leq C'(t-s)^{\alpha/2}$$

and the statement follows.

The same procedure works to prove that all the first order derivatives $D_i u$ are $(1/2 + \alpha/2)$- Hölder continuous with respect to time. In this case we have to use the fact that $C_b^1(\mathbb{R}^n)$ belongs to the class $J_{1/2+\alpha/2}$ between $C^\alpha(\mathbb{R}^n)$ and $C^{2+\alpha}(\mathbb{R}^n)$. □

In the proof we have used in a crucial way the analyticity of the heat semigroup, in particular estimate

$$\|\Delta T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \leq \frac{M_1}{t}, \;\; 0 < t \leq T, \tag{4.7}$$

and its consequence

$$\|\Delta^2 T(t)\|_{\mathcal{L}(C_b(\mathbb{R}^n))} \leq \frac{M_2}{t^2}, \;\; 0 < t \leq T,$$

which follows from (4.7) recalling that $\Delta^2 T(t) = \Delta T(t/2)\Delta T(t/2)$. Both estimates are immediately deduced from (1.15) because the Laplacian and its square have bounded coefficients. Of course this is not true in the case of the Ornstein-Uhlenbeck operator and, more generally, of elliptic operators with unbounded coefficients, for which the above proof fails. (In fact, one can prove that the statement of theorem 4.0.14 is false for the Ornstein-Uhlenbeck operator).

The part of the proof about the time Hölder regularity of Δu can be extended to evolution equations in general Banach spaces X, as follows. This result is due to Sinestrari [30].

Theorem 4.0.15 *Let $A : D(A) \mapsto X$ be the generator of an analytic semigroup in a Banach space X, and consider the Cauchy problem*

$$\begin{cases} u'(t) = Au(t) + f(t), \;\; 0 < t < T, \\ \\ u(0) = u_0 \end{cases} \tag{4.8}$$

where $f \in C^\theta([0,T], X)$, $0 < \theta < 1$, and $u_0 \in D(A)$ are such that $Au_0 + f(0) \in (X, D(A))_{\theta,\infty}$. Then problem (4.8) has a unique solution $u \in C^{1+\theta}([0,T], X) \cap C^\theta([0,T], D(A))$, and there is C such that

$$\begin{aligned} &\|u\|_{C^{1+\theta}([0,T],X)} + \|Au\|_{C^\theta([0,T],X)} + \sup_{0\leq t\leq T} \|u'(t)\|_{(X,D(A))_{\theta,\infty}} \\ &\leq C(\|f\|_{C^\theta([0,T],X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{(X,D(A))_{\theta,\infty}}). \end{aligned} \tag{4.9}$$

In our case, we had $X = C_b(\mathbb{R}^n)$, $A =$ the realization of the Laplacian in X, $\theta = \alpha/2$, and $(X, D(A))_{\alpha/2,\infty} = C^\alpha(\mathbb{R}^n)$.

Note that to treat the term $(T(t) - T(s))(\Delta u_0 + f(0,\cdot))$, which appears in $\Delta u_2(t,\cdot) - \Delta u_2(s,\cdot)$, we used the estimate $\|\Delta T(\sigma)\|_{\mathcal{L}(C^\alpha(\mathbb{R}^n), C_b(\mathbb{R}^n))} \leq M_{1,\alpha}\sigma^{\alpha/2-1}$. Such estimate is what characterizes the interpolation space $(X, D(A))_{\alpha/2,\infty}$ when A generates an analytic semigroup, in the sense of the following proposition.

Proposition 4.0.16 *Let A be a sectorial operator, and let e^{tA} be the analytic semigroup generated by A. Then for $0 < \theta < 1$, $1 \leq p \leq \infty$, $T > 0$ we have*

$$(X, D(A))_{\theta,p} = \{x \in X : \varphi(t) = t^{1-\theta}\|Ae^{tA}x\| \in L^p_*(0,T)\},$$

and the norms $\|\cdot\|_{(X,D(A))_{\theta,p}}$ and

$$x \mapsto \|x\| + \|\varphi\|_{L^p_*(0,T)}$$

are equivalent.

The proof of theorem 4.0.14 may be generalized to those elliptic operators that generate analytic semigroups in spaces of continuous and bounded functions, and such that $(X, D(A))_{\alpha/2,\infty}$ is a Hölder space.

These facts are true for second order uniformly elliptic operators with regular and bounded coefficients in $\mathbb{R}^n$ or in open subsets of $\mathbb{R}^n$ with regular boundary and with the usual boundary conditions (Dirichlet, Neumann, or Robin), as well as for $2m$-order uniformly elliptic operators with regular and bounded coefficients and with suitable boundary conditions. But the proofs of these properties – in particular, generation of analytic semigroups – are not trivial. See [20, Ch. 3].

Exercise. Show that if $X = C_b(\mathbb{R}^n)$ and A is the realization of the Laplacian in X, then $(X, D(A))_{\alpha/2,\infty} = C^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$. Hint: the inclusion $C^\alpha(\mathbb{R}^n) \subset (X, D(A))_{\alpha/2,\infty}$ follows from $C^2_b(\mathbb{R}^n) \subset D(A)$ and from example 1.1.7; for the opposite inclusion show that $C^1_b(\mathbb{R}^n)$ is in the class $J_{1/2}$ between $C_b(\mathbb{R}^n)$ and $D(A)$ using (1.20) and estimate $\|DT(t)f\|_\infty \leq Ct^{-1/2}\|f\|_\infty$, and then use the Reiteration Theorem.

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