DIRICHLET BOUNDARY CONDITIONS
FOR ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT

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Abstract. We study the realisation $A$ of the operator $A = \Delta - \langle D\Phi, D\cdot \rangle$ in $L^2(\Omega, \mu)$ with Dirichlet boundary condition, where $\Omega$ is a possibly unbounded open set in $\mathbb{R}^N$, $\Phi$ is a semi-convex function and the measure $d\mu(x) = \exp(-\Phi(x)) \, dx$ lets $A$ be formally self-adjoint. The main result is that $A : D(A) = \{ u \in H^2(\Omega, \mu) : A u \in L^2(\Omega, \mu) \cap H^1_0(\Omega, \mu) \} \rightarrow L^2(\Omega, \mu)$ is a dissipative self-adjoint operator in $L^2(\Omega, \mu)$.

1. Introduction

Second-order elliptic operators with unbounded coefficients in $\mathbb{R}^N$ or in unbounded subsets of $\mathbb{R}^N$ have been the object of several recent papers; see e.g. [2, 3, 8, 1, 9]. Since the very first studies it was apparent that operators of the type $Au = \text{Tr} Q(x) D^2 u(x) + \langle F(x) Du(x) \rangle$, without potential terms, are not well settled in $L^p$ spaces with respect to the Lebesgue measure, unless the matrix $Q$ and the vector $F$ satisfy very severe restrictions, such as global Lipschitz continuity (see [9, 7]). It is much more natural and fruitful to work in suitably weighted $L^p$ spaces; see [3, 8]. This is what we do in this paper. We consider the operator $A$ defined by

$$ Au = \Delta u - \langle D\Phi, Du \rangle = e^{\Phi} \text{div} (e^{-\Phi} Du), $$

where $\Phi : \mathbb{R}^N \rightarrow \mathbb{R}$ is a $C^2$ semi-convex function, i.e., there is $\alpha \geq 0$ such that

$$ \Phi_\alpha(x) := \Phi(x) + \alpha|x|^2/2 \text{ is convex}, $$

or, equivalently, the matrix $D^2 \Phi(x) + \alpha I$ is nonnegative definite at each $x$. We emphasize that we do not assume any growth restriction on $\Phi$ or on its derivatives. The natural weight is then $\rho(x) = e^{-\Phi(x)}$ because, as it is easy to check, if $\Omega$ is any open set in $\mathbb{R}^N$,

$$ \int_\Omega A u v \, d\mu = - \int_\Omega \langle Du, Dv \rangle \, d\mu, \quad \forall u, v \in C_0^\infty(\Omega),$$

if $\mu(dx) = e^{-\Phi(x)} \, dx$, so that $A$ is associated to a nice Dirichlet form and it is formally self-adjoint in $L^2(\Omega, \mu)$. The aim of this paper is to study the realisation of $A$ in $L^2(\Omega, \mu)$ with Dirichlet boundary condition, i.e., the operator

$$ A : D(A) = \{ u \in H^2(\Omega, \mu) \cap H^1_0(\Omega, \mu) : Au \in L^2(\Omega, \mu) \} \rightarrow L^2(\Omega, \mu); \quad Au = A u. $$

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Note that for \( u \in H^2(\Omega, \mu) \), condition \( Au \in L^2(\Omega, \mu) \) is equivalent to requiring \( \langle D\Phi, Du \rangle \in L^2(\Omega, \mu) \). Our main result is that \( A \) is self-adjoint and dissipative, provided \( \partial \Omega \) is smooth enough and the normal derivative \( \partial \Phi/\partial n \) is bounded from above on \( \partial \Omega \). A lot of consequences then follow; see Section 3.

A natural approach to the study of \( A \) consists in defining an operator \( A_0 : C_c^\infty(\Omega) \to L^2(\Omega, \mu) \), \( A_0 u = Au \), in showing that \( A_0 \) is closable, and that its closure is self-adjoint and dissipative. But the problem of the characterisation of the domain of the closure still remains. So, we follow a more direct approach, solving the resolvent equation \( \lambda u - Au = f \) for all \( \lambda > 0 \) and \( f \in C_c^\infty(\Omega) \), which is dense in \( L^2(\Omega, \mu) \). Proving the existence of a solution to \( \lambda u - Au = f \) that vanishes on \( \partial \Omega \) is not hard, thanks to the regularity of the data. Estimates of its \( H^1(\Omega, \mu) \)-norm, and uniqueness of the solution in \( D(A) \), are easy consequences of the integration formula proved in Lemma 2.2 below. Estimating the second-order derivatives of \( u \) is much more delicate, and here the assumptions of semi-convexity and of upper boundedness of \( \partial \Phi/\partial n \) are used and play a fundamental role.

This paper is in some sense parallel to the paper \([3]\), where the operator \( A \) was studied in the whole space \( \mathbb{R}^N \) and in any convex regular open set \( \Omega \) with Neumann boundary condition. The conclusions of \([3]\) are similar to the ones of the present paper, but the assumptions on \( \Phi \) and \( \Omega \) are a bit different, i.e., \( \Phi \) is just convex, with no further regularity assumption, and \( \Omega \) is convex, too.

2. The domain of \( A \) with Dirichlet boundary condition

Throughout the paper we assume that \( \Omega \) is an open set in \( \mathbb{R}^N \) with sufficiently smooth (at least \( C^2 \)) boundary. By \( L^2(\Omega) \) and \( H^k(\Omega) \), \( k \in \mathbb{N} \), we mean the usual \( L^2 \) and Sobolev spaces with respect to the Lebesgue measure. The spaces \( H^k(\Omega, \mu) \), \( k = 1, 2 \), are defined as the set of all \( u \in H^k_{\text{loc}}(\Omega) \) such that the function \( u \) and its partial derivatives up to the order \( k \) belong to \( L^2(\Omega, \mu) \). They are Hilbert spaces with the standard inner products \( \langle u, v \rangle = \int_{\Omega} (uv + \sum |\alpha|=1 D^\alpha u D^\alpha v \chi_{\Omega}) \, dx \).

\( H^k_0(\Omega, \mu) \) is the subspace of \( H^k(\Omega, \mu) \) consisting of the functions with null trace on the boundary. By \( C^k_0(\mathbb{R}^N) \) we denote the space of bounded functions with bounded derivatives up to order \( k \). We say that \( \partial \Omega \) is uniformly \( C^k \) if there exist \( r > 0 \), \( m \in \mathbb{N} \) and a (at most countable) family \( \{ B_j = B_r(x_j), j \in J \} \) of balls covering \( \partial \Omega \) with at most \( m \) overlapping and \( C^k \)-diffeomorphisms \( \phi_j : B_j \to B_1(0) \) such that \( \phi_j(B_j \cap \Omega) = B_1(0) \cap \{ y_N > 0 \} \) and \( \sup_j \| \phi_j \|_{C^k} + \| \phi_j^{-1} \|_{C^k} < \infty \).

Lemma 2.1. \( C^\infty_0(\Omega) \) is dense in \( L^2(\Omega, \mu) \) and in \( H^1_0(\Omega, \mu) \).

Proof. Let \( u \in L^2(\Omega, \mu) \), or \( u \in H^1_0(\Omega) \). Let \( \theta : \mathbb{R}^N \to \mathbb{R} \) be a smooth function such that \( 0 \leq \theta(x) \leq 1 \) for each \( x \), \( \theta \equiv 1 \) in \( B(0,1) \), \( \theta \equiv 0 \) outside \( B(0,2) \), and set \( u_n(x) = u(x)\theta(x/n) \). Then \( u_n \to u \) in \( L^2(\Omega, \mu) \). Indeed,

\[
\int_{\Omega} |u_n - u|^2 \, d\mu \leq \int_{\{ x \in \Omega : |x| \geq n \}} |u|^2 \, d\mu
\]

which goes to 0 as \( n \to \infty \). If \( u \in H^1_0(\Omega) \), then \( u_n \to u \) in \( H^1(\Omega, \mu) \), because \( Du_n(x) = \theta(x/n)Du(x) + D\theta(x/n)u(x)/n \). Since each \( u_n \) has bounded support, it may be approximated in \( L^2(\Omega) \) (respectively, in \( H^1(\Omega) \)) by a sequence of \( C^\infty_0(\Omega) \) functions. Such a sequence also approximates \( u_n \) in \( L^2(\Omega, \mu) \) (respectively, in \( H^1(\Omega, \mu) \)) because \( \mu \) is equivalent to the Lebesgue measure on each compact subset of \( \mathbb{R}^N \). \( \square \)
The realisation $A$ of $\mathcal{A}$ in $L^2(\Omega, \mu)$ with Dirichlet boundary condition is defined by (4). The following integration formulae will be very useful in what follows.

**Lemma 2.2.** Let $\psi \in H^1_0(\Omega, \mu)$, $u \in H^2(\Omega, \mu)$ be such that $Au \in L^2(\Omega, \mu)$. Then
\begin{equation}
\int_\Omega Au \psi \, d\mu = - \int_\Omega \langle Du, D\psi \rangle \, d\mu.
\end{equation}
More generally, if $\psi \in H^1(\Omega, \mu)$ and $u \in H^2(\Omega, \mu)$ is such that $Au \in L^2(\Omega, \mu)$, then
\begin{equation}
\int_\Omega Au \psi \, d\mu = - \int_\Omega \langle Du, D\psi \rangle \, d\mu + \int_{\partial\Omega} \frac{\partial u}{\partial n} \psi e^{-\Phi} \, d\sigma,
\end{equation}
where $d\sigma$ denotes the usual Lebesgue surface measure, the last integral is understood as $\lim_{R \to \infty} \int_{\partial\Omega} \frac{\partial u}{\partial n} \psi(\theta(x)/R) e^{-\Phi} \, d\sigma$, and $\theta$ is the function used in Lemma 2.1.

**Proof.** The proof of (4) is immediate if $\psi \in C^\infty_0(\Omega)$, and the statement follows by approximation in the general case. Equality (5) is obtained by approximating $\psi$ by $\psi(x)\theta(x/R)$. \qed

Let us state a consequence of Lemma 2.2.

**Lemma 2.3.** If $\partial \Omega$ is uniformly $C^2$ and $u \in H^2(\Omega, \mu)$ is such that $Au \in L^2(\Omega, \mu)$, then $\partial u/\partial n$ is in $L^2(\partial\Omega, \exp(-\Phi) \, d\sigma)$. Moreover, there exists $C > 0$ such that for every $\varepsilon \in (0, 1)$ the following estimate holds:
\begin{equation}
\int_{\partial\Omega} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} \, d\sigma \leq \varepsilon \left( \| Au \|_{L^2(\Omega, \mu)}^2 + \| D^2 u \|_{L^2(\Omega, \mu)}^2 \right) + \frac{C}{\varepsilon} \| D u \|_{L^2(\Omega, \mu)}^2.
\end{equation}

**Proof.** It is sufficient to take $\psi = \langle Du, N \rangle$ in (5), where $N$ is any $C^1_0$ extension to $\mathbb{R}^N$ of the normal vector field $n$, and then to use the Hölder inequality. \qed

Lemma 2.2 implies that the operator $A$ is symmetric. In the next theorem we prove that it is self-adjoint if $\Phi$ is smooth enough, and
\begin{equation}
\frac{\partial \Phi}{\partial n} \leq 0 \quad \text{on} \quad \partial \Omega.
\end{equation}

**Theorem 2.4.** Assume that $\partial \Omega \in C^3$ and that $\Phi$ satisfies (2) and (6). Then $(A, D(A))$ is self-adjoint and dissipative in $L^2(\Omega, \mu)$. Moreover, the map $u \mapsto \langle (D^2 \Phi) Du, Du \rangle$ is continuous from $D(A)$ to $L^1(\Omega, \mu)$.

**Proof.** We have to show that, for $\lambda > 0$ and $f \in L^2(\Omega, \mu)$, the equation $\lambda u - Au = f$ has a unique solution $u \in D(A)$. Uniqueness is an immediate consequence of Lemma 2.2, taking $\psi = u$ in (5). Concerning existence, we first assume that $f \in C^\infty_0(\Omega)$ and we show that there is a solution $u \in D(A)$ satisfying
\begin{equation}
(7)\begin{cases}
(a) & \| u \|_{L^2(\Omega, \mu)} \leq \frac{1}{\lambda} \| f \|_{L^2(\Omega, \mu)}, \\
(b) & \| D u \|_{L^2(\Omega, \mu)} \leq \frac{1}{\sqrt{\lambda}} \| f \|_{L^2(\Omega, \mu)}, \\
(c) & \| D^2 u \|_{L^2(\Omega, \mu)} + \| \langle D^2 \Phi_u \rangle Du, Du \|_{L^1(\Omega, \mu)} \leq \left( 2 + \frac{\Omega}{\lambda} \right) \| f \|_{L^2(\Omega, \mu)},
\end{cases}
\end{equation}
where $\Phi_u$ is defined in (2). Using the Lax-Milgram lemma, we find $u \in H^1_0(\Omega, \mu)$ such that
\begin{equation}
\lambda \int_\Omega u \psi \, d\mu + \int_\Omega \langle Du, D\psi \rangle \, d\mu = \int_\Omega f \psi \, d\mu, \quad \forall \psi \in H^1_0(\Omega, \mu).
\end{equation}
By local elliptic regularity, \( u \in H^2_{loc}(\Omega) \) and \( \lambda u - Au = f \). In particular, \( Au \in L^2(\Omega, \mu) \). Again, by classical elliptic regularity,

\[
u \in C^{2,\beta}(\Omega \cap B(0, R)) \cap H^3(\Omega \cap B(0, R))
\]

for every \( R > 0 \) and \( \beta < 1 \).

Now we can prove (7). To prove estimates (a) and (b), we multiply the identity \( \lambda u - Au = f \) by \( u \), we integrate over \( \Omega \) and we use (4) to get

\[
\int_{\Omega} (\lambda u^2 + |Du|^2) \, d\mu = \int_{\Omega} fu \, d\mu \leq \|f\|_{L^2(\Omega, \mu)}\|u\|_{L^2(\Omega, \mu)}
\]

which implies that (a) and (b) hold. To prove (c) we differentiate the equation \( \lambda u - Au = f \) with respect to \( x_h, h + 1, \ldots, N \), and we get

\[
\lambda D_h u - \Delta(D_h u) + \langle D(D_h \Phi), Du \rangle + \langle D\Phi, D(D_h u) \rangle = D_h f,
\]

that is,

\[
\lambda D_h u - AD_h u + \sum_{k=1}^{N} D_{hk} \Phi D_k u = D_h f.
\]

Set \( \theta_R(x) = \theta(x/R) \). Multiplying by \( \theta_R^2 D_h u \), summing over \( h \), and integrating by parts, from (4) we get, since \( u \in H^3(\Omega \cap B(0, R)) \) for every \( R \),

\[
\int_{\Omega} \left\{ \theta_R^2 (\lambda |Du|^2 + |D^2u|^2 + \langle D^2\Phi Du, Du \rangle) + 2 \sum_{h=1}^{N} \theta_R \langle D(D_h u), D\theta_R \rangle D_h u \right\} \, d\mu = \int_{\partial \Omega} \theta_R^2 \sum_{h=1}^{N} \frac{\partial D_h u}{\partial n} D_h u \, e^{-\Phi} \, d\sigma + \int_{\Omega} \theta_R^2 \langle Df, Du \rangle \, d\mu.
\]

Since \( f \) has compact support, for \( R \) large enough \( \theta_R \equiv 1 \) on the support of \( f \). Using (4) again in the last integral, we write it as \( -\int_{\Omega} f(\lambda u - f) \, d\mu \). Thanks to the assumption \( D^2\Phi \geq -\alpha I \), we obtain

\[
\int_{\Omega} \theta_R^2 (\lambda |Du|^2 + |D^2u|^2 + \langle D^2\Phi u, Du \rangle) \, d\mu \leq \int_{\Omega} (\alpha \theta_R^2 |Du|^2 + CR^{-1} \theta_R |D^2u| |Du| + f(\lambda u - f)) \, d\mu
\]

\[
+ \int_{\partial \Omega} \theta_R^2 \langle (D^2 u)n, Du \rangle \, e^{-\Phi} \, d\sigma,
\]

for a suitable \( C > 0 \), independent of \( R \). Using (a) and (b) we get

\[
\int_{\Omega} (\alpha \theta_R^2 |Du|^2 + f(\lambda u - f)) \, d\mu \leq \left( 2 + \frac{\alpha}{\lambda} \right) \|f\|^2_{L^2(\Omega, \mu)}.
\]

Moreover,

\[
\int_{\Omega} CR^{-1} \theta_R |D^2u| |Du| \, d\mu \leq \frac{C}{2R} \int_{\Omega} \theta_R^2 |D^2u|^2 \, d\mu + \frac{C}{2R} \int_{\Omega} |Du|^2 \, d\mu.
\]

Let us now show that the boundary integral in (8) is negative. Since \( u = 0 \) on \( \partial \Omega \), we have \( \langle Du, \tau \rangle = 0 \) and \( \langle (D^2u)n, \tau \rangle = 0 \) for every tangent vector \( \tau \) to \( \partial \Omega \). Then \( Du = (\partial u/\partial n)n \) and \( \langle (D^2u)n, Du \rangle = \langle (D^2u)n, (\partial u/\partial n)n \rangle \) at \( \partial \Omega \). Therefore
Assume that Theorem 2.5. (a), (b), and (c) hold. Hence
\[
\int_\partial \frac{\partial^2}{\partial n^2} u(n, Du) e^{-\Phi} d\sigma = \int_\partial \frac{\partial^2}{\partial n^2} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma \leq 0,
\]
thanks to (9). Thus, we have proved that
\[
\int_\Omega \left( 1 - \frac{C}{2R} \right) \theta_R^2 |D^2 u|^2 + \theta_R^2 \langle (D^2 \Phi), Du, Du \rangle d\mu \leq \left( 2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} \right) \|f\|^2_{L^2(\Omega, \mu)},
\]
and statement (c) follows by letting \( R \to \infty \).

The general case \( f \in L^2(\Omega, \mu) \) is easily handled by approximation. Let \( (f_n) \subset C^\infty_0(\Omega) \) be such that \( f_n \to f \) in \( L^2(\Omega, \mu) \) and let \( u_n \in D(A) \) be such that \( \lambda u_n - Au_n = f_n \). The above estimates imply that the sequence \((u_n)\) converges to a function \( u \in H^2(\Omega, \mu) \) and it is readily seen that \( u \in D(A), \lambda u - Au = f \) and that (a), (b), and (c) hold.

Condition (9) can be relaxed assuming some more regularity on \( \partial \Omega \).

**Theorem 2.5.** Assume that \( \partial \Omega \subset C^3 \) and that it is uniformly \( C^2 \). Let \( \Phi \) be a \( C^2 \) function satisfying (2) and
\[
\frac{\partial \Phi}{\partial n} \leq k \text{ at } \partial \Omega,
\]
for some \( k \in \mathbb{R} \). Then \((A, D(A))\) is self-adjoint and dissipative in \( L^2(\Omega, \mu) \). Moreover, the map \( u \mapsto \langle (D^2 \Phi)Du, Du \rangle \) is continuous from \( D(A) \) to \( L^2(\Omega, \mu) \).

**Proof.** The proof is similar to the proof of Theorem 2.4. For \( f \in C^\infty_0(\Omega), \lambda > 0 \), let \( u \in H^1_0(\Omega, \mu) \) be the variational solution of the equation \( \lambda u - Au = f \). As in Theorem 2.4 we get estimates (7) (a), (b) and
\[
\frac{\partial}{\partial n} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma \leq \left( 2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} \right) \|f\|^2_{L^2(\Omega, \mu)} + \int_\partial \frac{\partial^2}{\partial n^2} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma.
\]
The boundary integral does not exceed
\[
k \int_\partial \frac{\partial^2}{\partial n^2} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma,
\]
and it can be estimated as follows (see also Lemma 2.3).

Let us take \( \psi = \theta_R^2 (Du, N) \) in (4), where \( N \) is any \( C^1_b \) extension to \( \mathbb{R}^N \) of the normal vector field \( n \), so that, using Hölder inequality, we obtain for every \( 0 < \varepsilon < 1 \)
\[
\int_\partial \frac{\partial^2}{\partial n^2} \left( \frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\sigma \leq \varepsilon (\|Au\|^2_{L^2(\Omega, \mu)} + \|\theta_R D^2 u\|^2_{L^2(\Omega, \mu)}) + \frac{C}{\varepsilon} \|Du\|^2_{L^2(\Omega, \mu)}.
\]
Since $Au = \lambda u - f$, writing the last inequality with $\varepsilon k \leq 1/2$ and combining it with (11) and with estimates (a), (b), we arrive at
\[
\int_\Omega \left( \frac{1}{2} - \frac{C}{2R} \right) \theta_R^2 |D^2 u|^2 \, d\mu + \int_\Omega \theta_R^2 \langle (D^2 \Phi_u) Du, Du \rangle \, d\mu \\
\leq \left( 2 + \frac{\alpha}{\lambda} + \frac{C}{2R\lambda} + C_1 \right) \|f\|_{L^2(\Omega, \mu)},
\]
with $C_1$ independent of $R$. Letting $R \to \infty$ we obtain estimate (c) of Theorem 2.4 (with different constants), and from now on the proof follows the same lines as in Theorem 2.3.

\[\square\]

**Remark 2.6.** If $D^2 \Phi$ is bounded from above, then the mapping $u \mapsto \langle (D^2 \Phi) Du, Du \rangle$ is bounded from $H^1(\Omega, \mu)$ to $L^1(\Omega, \mu)$ and the last statement of Theorems 2.4 and 2.5 is obvious. But, if $D^2 \Phi$ is not bounded, the statement is not obvious, and it will be used in the next section to obtain a quantitative Poincaré inequality.

We end this section by showing that $D(A)$ can be strictly contained in $H^2(\Omega, \mu) \cap H^1_0(\Omega, \mu)$.

**Example 2.7.** We construct a convex function $\phi : [0, \infty) \to \mathbb{R}$ such that $e^{-\phi}$ and $x^2e^{-\phi}$ are in $L^1(0, \infty)$ but $\phi^2 e^{-\phi} \notin L^1(0, \infty)$. Then $u(x) = x$ belongs to $H^2(\mu) \cap H^1_0(\mu)$ but not to $D(A)$. For simplicity, $\phi$ will be nonsmooth, however, smooth versions are easily obtained using straightforward arguments.

Let $0 = a_1 < b_1 < a_2 < b_2 < \cdots$ be points in $[0, \infty)$ such that $b_j - a_j = 1$. Set $1/l_j = a_{j+1} - b_j$, $l_1 = 1$ and define $\phi' = 1$ in $(a_1, b_1)$, $\phi' = l_j$ in $(b_j, a_{j+1})$ and $\phi' = l_{j-1}$ in $(a_j, b_j)$. We have to choose $1 = l_1 < l_2 < \cdots$ in such a way that $\phi$ satisfies the properties above. First observe that $\phi$ is convex, $\phi' \geq 1$, hence $\phi(x) \geq x$ and then $e^{-\phi}, x^2e^{-\phi} \in L^1(0, \infty)$. Moreover, if $x \in (b_j, a_{j+1})$, then $\phi(x) \leq j + 1 + \sum_{i=1}^{j-1} l_i$ and therefore
\[
\int_{b_j}^{a_{j+1}} \phi'^2 e^{-\phi} \, dx \geq l_j^2 \exp(-j + 1 + \sum_{i=1}^{j-1} l_i) \geq l_j \exp(-(j + 1 + \sum_{i=1}^{j-1} l_i)).
\]
Choosing (inductively) $l_j = e^{(j+1+\sum_{i=1}^{j-1} l_i)}$ the above integral is bigger than 1, hence, summing over $j$, $\phi'^2$ does not belong to $L^1(\mu)$.

3. Further properties of $A$

Under the assumptions of either Theorem 2.4 or Theorem 2.5, since the operator $A$ is self-adjoint and dissipative in $L^2(\Omega, \mu)$, it is the infinitesimal generator of an analytic contraction semigroup $T(t)$ in $L^2(\Omega, \mu)$. In this section we prove further properties of $T(t)$ and of $A$.

The characterisation of the domain of $(-A)^{1/2}$ is a standard consequence of the integration formula (11), as the following proposition shows. Recall that the norm in $H^1_0(\Omega, \mu)$ is given by $\|u\|_{H^1_0(\Omega, \mu)} = \|u\|_{L^2(\Omega, \mu)} + \|Du\|_{L^2(\Omega, \mu)}$.

**Proposition 3.1.** The domain of $(-A)^{1/2}$ is $H^1_0(\Omega, \mu)$. Therefore, the restriction of $T(t)$ to $H^1_0(\Omega, \mu)$ is an analytic semigroup in $H^1_0(\Omega, \mu)$.

**Proof.** Any $u \in D((-A)^{1/2})$ is the $L^2(\Omega, \mu)$-limit of a sequence of functions $u_n \in D(A) \subset H^1_0(\Omega, \mu)$ which is a Cauchy sequence with respect to the norm $\|u\|_{L^2} + \langle -Au, u \rangle_{L^2}$. From (11) it follows that $(Du_n)$ is a Cauchy sequence in $L^2(\Omega, \mu)$,
hence \( u \in H^1_0(\Omega, \mu) \). Conversely, let \( u \in H^1_0(\Omega, \mu) \) and let \( u_n \in C_0^\infty(\Omega) \subset D(A) \) converge to \( u \) in \( H^1(\Omega, \mu) \). Formula (\ref{eq:Cauchy}) implies that \((u_n)\) is a Cauchy sequence in \( D((-A)^{1/2}) \), hence \( u \in D((-A)^{1/2}) \).

**Corollary 3.2.** Under the assumptions of either Theorem 2.3 or Theorem 2.5, \( T(t) \) is a symmetric Markov semigroup, that is, a semigroup of self-adjoint positivity preserving operators in \( L^2(\Omega, \mu) \) that satisfy \( \|T(t)f\|_\infty \leq \|f\|_\infty \) for each \( f \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu) \) and \( t > 0 \).

**Proof.** Since \( A \) is self-adjoint, each \( T(t) \) is self-adjoint. To prove that each \( T(t) \) preserves positivity and that it is a contraction in \( L^\infty \), we use the Beurling-Deny criterion; see e.g. [3] Theorems 1.3.2, 1.3.3.

As \( D((-A)^{1/2}) = H^1_0(\Omega, \mu) \), then \( u \in D((-A)^{1/2}) \) implies \( |u| \in D((-A)^{1/2}) \), and

\[
\|(-A)^{1/2}(|u|)^2\| = \int_\Omega |D(|u|)|^2 d\mu \leq \int_\Omega |Du|^2 d\mu = \|(-A)^{1/2}u\|^2,
\]

so that \( T(t) \) is positivity-preserving for all \( t > 0 \). Again, since \( D((-A)^{1/2}) = H^1_0(\Omega, \mu) \), if \( 0 \leq u \in D((-A)^{1/2}) \), then \( u \wedge 1 \in D((-A)^{1/2}) \), and

\[
\|(-A)^{1/2}(u \wedge 1)^2\| = \int_\Omega |D(u \wedge 1)|^2 d\mu \leq \int_\Omega |Du|^2 d\mu = \|(-A)^{1/2}u\|^2.
\]

This implies that \( \|T(t)f\|_\infty \leq \|f\|_\infty \) for each \( f \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu) \). \( \square \)

Another immediate consequence of the integration formula (\ref{eq:Cauchy}) is that \( A \) is injective: if \( u \in D(A) \) and \( Au = 0 \), then \( Au \cdot u = 0 \), and integrating over \( \Omega \) we obtain \( Du = 0 \) so that \( u \) is constant on each connected component of \( \Omega \); since \( u \) vanishes at \( \partial \Omega \), then \( u = 0 \).

A natural question is now whether 0 is in the resolvent set of \( A \). This is true if \( D(A) \) is compactly embedded in \( L^2(\Omega, \mu) \), because in this case the spectrum of \( A \) consists of a sequence of isolated eigenvalues. But in general \( D(A) \) is not compactly embedded in \( L^2(\Omega, \mu) \), as the following counterexample shows.

**Example 3.3.** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be any convex \( C^2 \) function such that \( \varphi(x) = x \) for \( x \geq 0 \). Set \( \Phi(x, y) = \varphi(x) + y^2 \), and let \( \Omega \) be the half-plane \( \{(x, y) \in \mathbb{R}^2 : y > 0\} \). Then \( D(A) \) is not compactly embedded in \( L^2(\Omega, \mu) \).

**Proof.** Let \( \theta \in C_0^\infty(0, \infty) \) be such that \( \int_0^\infty (\theta(y))^2 \exp(-y^2) dy = 1 \), and set for each \( n \in \mathbb{N}, n \geq 3 \),

\[
u_n(x, y) = \frac{x^n}{\sqrt{(2n)!}} \theta(y), \quad x, y \geq 0, \quad u_n(x, y) = 0 \text{ otherwise.}
\]

Since \( d\mu = \exp(-\varphi(x) - y^2) dx dy \), then \( \|u_n\|_{L^2(\Omega, \mu)} = 1 \) for each \( n \). Moreover,

\[
D_x u_n(x, y) = \frac{n x^{n-1}}{\sqrt{(2n)!}} \theta(y), \quad D_y u_n(x, y) = \frac{x^n}{\sqrt{(2n)!}} \theta'(y), \quad x > 0,
\]

\[
D_{xx} u_n(x, y) = \frac{n(n-1)x^{n-2}}{\sqrt{(2n)!}} \theta(y), \quad D_{yy} u_n(x, y) = \frac{x^n}{\sqrt{(2n)!}} \theta''(y), \quad x > 0,
\]

and every derivative vanishes for \( x \leq 0 \). Therefore, \( u_n \in D(A) \) and \( \|Au_n\|_{L^2(\Omega, \mu)} \) is bounded by a constant independent of \( n \). But no subsequence may converge in
Assume that \( s \) satisfies an additional (mild) nonoscillation condition. In the next proposition we show that the answer is positive if \( \Phi \) satisfies an additional (mild) nonoscillation condition.

**Proposition 3.4.** Assume that \( \Phi \in C^2(\mathbb{R}^N) \) satisfies \( \Delta \Phi \leq a|D\Phi|^2 + b \) for some \( a < 1, b \in \mathbb{R} \). Then the map \( u \mapsto |D\Phi|u \) is bounded from \( H^1_0(\Omega, \mu) \) to \( L^2(\Omega, \mu) \). If, in addition, \( |D\Phi| \to \infty \) at infinity, the embedding of \( H^1_0(\Omega, \mu) \) (hence that of \( D(A) \)) in \( L^2(\Omega, \mu) \) is compact.

**Proof.** Since \( C_0^\infty(\Omega) \) is dense in \( H^1_0(\Omega, \mu) \) it is sufficient to show that

\[
|||D\Phi|||_{L^2(\Omega, \mu)} \leq C||u||_{H^1(\Omega, \mu)}
\]

for some \( C > 0 \) and every \( u \in C_0^\infty(\Omega, \mu) \). Integrating by parts and using Young’s inequality we get for every \( \varepsilon > 0 \) and for a suitable \( C_\varepsilon \)

\[
\int_{\Omega} |u|^2 |D\Phi|^2 \, d\mu = - \int_{\Omega} |u|^2 (D\Phi, De^{-\Phi}) \, dx
\]

\[
= \int_{\Omega} |u|^2 \Delta \Phi e^{-\Phi} \, dx + 2 \int_{\Omega} u(D\Phi, Du) e^{-\Phi} \, dx
\]

\[
\leq (a + \varepsilon) \int_{\Omega} |u|^2 |D\Phi|^2 \, d\mu + C_\varepsilon \int_{\Omega} |Du|^2 \, d\mu + b \int_{\Omega} |u|^2 \, d\mu.
\]

Choosing \( \varepsilon \) such that \( a + \varepsilon < 1 \), the first statement follows. Concerning the second one, we observe that for each \( \varepsilon > 0 \) there is \( R > 0 \) such that \( |D\Phi| \geq 1/\varepsilon \) in \( \Omega \setminus B(0, R) \). Hence for every \( u \) in the unit ball \( B \) of \( H^1_0(\Omega) \) we have

\[
\frac{1}{\varepsilon^2} \int_{\Omega \setminus B(0, R)} |u|^2 \, d\mu \leq \int_{\Omega \setminus B(0, R)} |u|^2 |D\Phi|^2 \, d\mu \leq C^2.
\]

Since the embedding of \( H^1(\Omega \cap B(0, R)) \) into \( L^2(\Omega \cap B(0, R)) \) is compact, we can find \( \{f_1, \ldots, f_k\} \subset L^2(\Omega \cap B(0, R)) \) such that the balls \( B(f_i, \varepsilon) \subset L^2(\Omega \cap B(0, R)) \) cover the restrictions of the functions of \( B \) to \( \Omega \cap B(0, R) \). Denoting by \( \tilde{f}_i \) the zero-extension of \( f_i \) to the whole of \( \Omega \), it follows that \( B \subset \bigcup_{i=1}^k B(\tilde{f}_i, (C + 1)\varepsilon) \), and the proof is complete.

The compactness of the resolvent is a consequence of the logarithmic Sobolev inequality

\[
\int_{\Omega} u^2 \log(|u|) \, d\mu \leq \frac{1}{\omega} \int_{\Omega} |Du|^2 \, d\mu + ||u||^2_{L^2(\Omega, \mu)} \log(||u||_{L^2(\Omega, \mu)}),
\]

for all \( u \in H^1_0(\Omega, \mu) \) and some \( \omega > 0 \) (where we set \( 0 \log 0 = 0 \)).

In what follows we give sufficient conditions for the validity of (12).
Proposition 3.5. Let us denote by $\lambda(x)$ the smallest eigenvalue of the matrix $D^2\Phi(x)$. Then:

(i) if $\lambda(x) \geq \omega_0$ for all $x \in \mathbb{R}^N$ then (12) holds with $\omega = \omega_0$;

(ii) if $\liminf_{|x| \to \infty} \lambda(x) > 0$, then (12) holds for some $\omega > 0$.

Proof. (i) Let $u \in H_0^1(\Omega, \mu)$ and extend $u$ outside $\Omega$ by setting $u(x) = 0$ for $x \notin \Omega$. Then the extension is in $H^1(\mathbb{R}^N, \nu)$, where $d\nu(x) = c \exp(-\Phi(x)) \, dx$, $c^{-1} = \int_{\mathbb{R}^N} \exp(-\Phi) \, dx \geq 1$. By [3], for each $u \in H^1(\mathbb{R}^N, \nu)$ we have

\[
\int_{\mathbb{R}^N} |u|^2 \log |u| \, d\nu \leq \frac{1}{\omega_0} \int_{\mathbb{R}^N} |Du|^2 \, d\nu + \|u\|^2_{L^2(\mathbb{R}^N, \nu)} \log(\|u\|_{L^2(\mathbb{R}^N, \nu)}).
\]

Since $u$ vanishes outside $\Omega$ we easily get

\[
\int_{\Omega} |u|^2 \log |u| \, d\mu \leq \frac{1}{\omega_0} \int_{\Omega} |Du|^2 \, d\mu + \|u\|^2_{L^2(\Omega, \mu)} \left( \frac{1}{2} \log c + \log(\|u\|_{L^2(\Omega, \mu)}) \right)
\]

and (12) follows since $c \leq 1$.

(ii) The proof is similar to (i), using [11] Theorem 1.3 instead of [3].

\[\square\]

Corollary 3.6. Under the assumptions of Proposition 3.5, $H_0^1(\Omega, \mu)$ is compactly embedded in $L^2(\Omega, \mu)$. Therefore, $\sup \sigma(A) < 0$. Moreover $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$ with $q(t) = 1 + e^{\omega t}$, and

\[(13) \quad \|T(t)f\|_{L^{q(t)}(\Omega, \mu)} \leq \|f\|_{L^2(\Omega, \mu)}, \quad t > 0, \quad f \in L^2(\Omega, \mu).
\]

Proof. Let $B$ be the unit ball of $H_0^1(\Omega, \mu)$. Inequality (12) yields the existence of a positive constant $C$ such that $\int_{\Omega} |u|^2 \, d\mu \leq C$ for every $u \in B$. Given $t \geq 1$, let $E = \{u \leq t\}$. Then for $R > 0$

\[
\int_{\Omega \setminus B(0, R)} |u|^2 \, d\mu \leq \int_{\Omega \setminus B(0, R)} t^2 \, d\mu + \frac{1}{\log t} \int_{\Omega \setminus B(0, R)} |u|^2 \log |u| \, d\mu
\]

\[
\leq t^2 \mu(\Omega \setminus B(0, R)) + \frac{C}{\log t}
\]

hence, given $\epsilon > 0$, there exists $R > 0$ such that $\int_{\Omega \setminus B(0, R)} |u|^2 \, d\mu \leq \epsilon$ for every $u \in B$. As in Proposition 3.4 this proves that $H_0^1(\Omega, \mu)$ is compactly embedded in $L^2(\Omega, \mu)$. The fact that $T(t)$ maps $L^2(\Omega, \mu)$ into $L^{q(t)}(\Omega, \mu)$, as well as estimate (13), follow from [3] 6.

\[\square\]

A necessary and sufficient condition in order that 0 be in the resolvent of $A$ is that the Poincaré inequality holds, i.e.,

\[(14) \quad \int_{\Omega} |u|^2 \, d\mu \leq \frac{1}{\omega} \int_{\Omega} |Du|^2 \, d\mu, \quad u \in H_0^1(\Omega, \mu),
\]

for some $\omega > 0$. More precisely, since $A$ is self-adjoint, then $\langle (-A - \omega I)u, u \rangle \geq 0$ for each $u \in D(A)$ if and only if $\sigma(A + \omega I) \subset (-\infty, 0]$. In other words, (14) holds for each $u \in D(A)$ (or, equivalently, for each $u \in H_0^1(\Omega, \mu) = D((-A)^{1/2})$) if and only if $\sigma(A) \subset (-\infty, -\omega]$. In this case we have

\[(15) \quad \|T(t)f\|_{L^2(\Omega, \mu)} \leq e^{-\omega t} \|f\|_{L^2(\Omega, \mu)}, \quad t > 0, \quad f \in L^2(\Omega, \mu).
\]

Indeed, for each $t > 0$ and $f \in L^2(\Omega, \mu)$,

\[\frac{d}{dt} \|T(t)f\|^2 = \int_{\Omega} 2AT(t)f \cdot T(t)f \, d\mu = -2\|DT(t)f\|^2 \leq -2\omega\|T(t)f\|^2.
\]
If $\Omega = \mathbb{R}^N$, the Poincaré inequality for functions having zero mean is a consequence of the logarithmic Sobolev inequality (in which case $D(A)$ is compactly embedded in $L^2(\Omega, \mu)$) and the constant $\omega$ in (14) is the same as in (12); see [10]. This is not true in our setting; see Example 3.9 below. However, in the next proposition we show how to get an explicit estimate of $\omega$ in (14) when (6) holds.

**Proposition 3.7.** Assume that (6) holds and that there exists $\omega_0 > 0$ such that the map $x \mapsto \Phi(x) - \omega_0|x|^2/2$ is convex. Then (14) holds with $\omega = \omega_0$.

**Proof.** We have only to show that $\sigma(A) \subset (-\infty, -\omega_0]$. Corollary 3.6 yields that the resolvent of $A$ is compact, hence $\sigma(A)$ consists of eigenvalues. If $\lambda u - Au = 0$ for some $\lambda \in \mathbb{R}$ and $0 \neq u \in D(A)$, we write (8) with $f = 0$ and let $R \to \infty$. Since the boundary integral is nonpositive and $D^2\Phi \geq \omega_0 I$ we get $(\lambda + \omega_0) \int_{\Omega} |Du|^2 \leq 0$. Since $u$ is not a constant, then $Du \neq 0$ and $\lambda \leq -\omega_0$. This concludes the proof. \qed

Let us again consider Example 3.3 and show that, in general, the Poincaré inequality does imply that the embedding $D(A) \subset L^2(\Omega, \mu)$ is compact.

**Example 3.8.** We use the same notation as in Example 3.3. Proposition 3.7 applied to the one-dimensional function $y \mapsto y^2$, $y > 0$, yields

$$\int_0^\infty |u(x,y)|^2 e^{-y^2} dy \leq \frac{1}{2} \int_0^\infty |D_y u(x,y)|^2 e^{-y^2} dy, \quad \text{a.e. } x \in \mathbb{R}, \; u \in H^1(\Omega, \mu).$$

Multiplying by $e^{-\phi(x)}$ and integrating with respect to $x \in \mathbb{R}$, we deduce

$$\int_{\Omega} |u(x,y)|^2 d\mu \leq \frac{1}{2} \int_{\Omega} |D_y u(x,y)|^2 d\mu$$

so that the Poincaré inequality holds, even if $D(A)$ is not compactly embedded in $L^2(\Omega, \mu)$, as we have shown in Example 3.3.

If assumption (6) is replaced by the boundedness of $\partial \Phi/\partial n$ at $\partial \Omega$ and still $\Phi(x) - \omega^2 |x|^2$ is convex, the constant $\omega$ in (14) may also depend on the constant $k$ in (10), as we show in the following example.

**Example 3.9.** Let $N = 1$ and let $Au = u'' - xu'$ be the Ornstein-Uhlenbeck operator. Here $\Phi(x) = x^2/2$, hence $D^2\Phi \equiv 1$ and (12) holds with $\omega = 1$. Let $\Omega_a = (-\infty, a)$ and set $u(x) = a - x$. Then $u \in D(A)$ and

$$\int_{-\infty}^a |u'|^2 d\mu \left(\int_{-\infty}^a |u|^2 d\mu\right)^{-1} \to 0$$

as $a \to \infty$. This shows that the spectrum of $A$ in $L^2(\Omega_\alpha, \mu)$ is not contained in $(-\infty, -1]$ for large $\alpha$, hence the constant $\omega$ in (14) is smaller than 1.

**References**


