Elliptic operators with unbounded drift coefficients and
Neumann boundary condition

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Abstract

We study the realization $A_N$ of the operator $A = \frac{1}{2} \Delta - \langle DU, D \cdot \rangle$ in $L^2(\Omega, \mu)$ with
Neumann boundary condition, where $\Omega$ is a possibly unbounded convex open set in
$\mathbb{R}^N$, $U$ is a convex unbounded function, $DU(x)$ is the element with minimal norm in
the subdifferential of $U$ at $x$, and $\mu(dx) = c \exp(-2U(x))dx$ is a probability measure,
infiniteisimally invariant for $A$. We show that $A_N$ is a dissipative self-adjoint operator
in $L^2(\Omega, \mu)$. Log-Sobolev and Poincaré inequalities allow then to study smoothing
properties and asymptotic behavior of the semigroup generated by $A_N$.

1 Introduction

Linear elliptic operators with regular and bounded coefficients in $\mathbb{R}^n$ is an old and well
studied subject. In particular, unique solvability and estimates for the solutions to equations of the type

$$\lambda u - Au = f$$

in suitable Banach spaces, such as $L^p$ spaces, Sobolev spaces, Hölder spaces etc. has been
the object of deep investigation and nowadays a satisfactory theory is available. See e.g.
the classical book [10].

If the coefficients of the elliptic operator $A$ are regular but unbounded one can prove
in general existence of a solution for large $\lambda$ but not uniqueness. For instance, for each
$\lambda > 0$ the one dimensional equation

$$\lambda \varphi - \frac{1}{2} \varphi_{xx} - x^3 \varphi_x = 0,$$

has a bounded non zero classical solution. See [9].

In the last few years an increasing interest has been devoted to elliptic operators with
unbounded coefficients in the whole $\mathbb{R}^N$, or in an unbounded open subset $\Omega$ of $\mathbb{R}^n$. In the
case of regular coefficients we refer to [4, 9, 11, 12, 14].

When the coefficients are not continuous one cannot expect in general existence of any
solution in spaces of continuous functions; also the choice of $L^p$ spaces with respect to the
Lebesgue measure is not appropriate and leads to several difficulties. See the book [7], where (1.1) is considered under minimal regularity conditions on the coefficients of $A$. A more natural choice is the space $L^p(\mathbb{R}^N, \mu)$ where $\mu$ is a probability measure such that

$$\int_{\Omega} Au(x) \mu(dx) = 0, \ u \in C_0^\infty(\mathbb{R}^N).$$

Several papers have been devoted to existence and uniqueness of invariant measures $\mu$ associated to elliptic operators, and to the properties of the realizations of such operators in $L^p(\mathbb{R}^N, \mu), 1 \leq p \leq \infty$, especially for $p = 2$. See e.g. [7, 5, 6, 16, 17].

In this paper we shall consider elliptic operators of the form

$$Au = \frac{1}{2} \Delta u - \langle DU, Du \rangle \quad (1.2)$$

This class of operators enjoys nice functional properties. As first realized by Kolmogorov [8], if $U$ is a $C^1$ function and $\exp(-2U)$ is in $L^1(\mathbb{R}^N)$, the probability measure

$$\nu(dx) = \left( \int_{\mathbb{R}^N} e^{-2U(x)} dx \right)^{-1} e^{-2U(x)} dx \quad (1.3)$$

is an infinitesimally invariant measure for $A$, and $A$ is symmetric in $L^2(\mathbb{R}^N, \nu)$ in the sense that

$$\int_{\mathbb{R}^N} Au(x) \nu(x) \nu(dx) = \int_{\mathbb{R}^N} A\nu(x) u(x) \nu(dx), \ u, \nu \in C_0^\infty(\mathbb{R}^N).$$

The condition that the drift $\langle F, Du \rangle$ has $F = -DU$ looks rather restrictive. However, it is well known (and easy to see) that if an operator $A$ of the type $Au = \frac{1}{2} \Delta u + \langle F, Du \rangle$ with regular $F$ admits an infinitesimally invariant probability measure $\nu = \rho(x) dx$, then $A$ is symmetric in $L^2(\mathbb{R}^N, \nu)$ if and only if $2F = D \log \rho$. In this case the corresponding diffusion process, described by the differential stochastic equation

$$dX = F(X) dt + dW(t),$$

is said to be **reversible**.

We shall assume that $U$ is a real valued convex function such that

$$\lim_{|x| \to +\infty} U(x) = +\infty. \quad (1.4)$$

No other growth assumption will be made. If $U$ is differentiable at $x$, $DU(x)$ is the gradient of $U$ at $x$; if $U$ is not differentiable at $x$, $DU(x)$ is meant as the element with minimal norm in the subdifferential of $U$ at $x$. Since $U$ is real valued and convex, it is continuous and $DU(x)$ is well defined for each $x \in \mathbb{R}^N$, but the function $x \mapsto DU(x)$ may be discontinuous.

After the study of the realization of $A$ in $L^2(\mathbb{R}^N, \nu)$, we shall consider an open convex set $\Omega \subset \mathbb{R}^N$ with $C^2$ boundary, and we shall study the realization $A_N$ of $A$ in $L^2(\Omega, \mu)$ with Neumann boundary condition, where

$$\mu(dx) = \left( \int_{\Omega} e^{-2U(x)} dx \right)^{-1} e^{-2U(x)} dx. \quad (1.5)$$

It is easy to see that if $u \in C^2(\overline{\Omega})$ has compact support and null normal derivative at the boundary, then

$$\int_{\Omega} Au(x) \mu(dx) = 0.$$

Therefore, the measure $\mu$ is infinitesimally invariant for $A$. Note that this is not true if $u$ satisfies the Dirichlet boundary condition.
The main result of this paper is that $A_N : D(A_N) = \{ u \in H^2(\Omega, \mu) : \partial u / \partial n = 0, \, Au \in L^2(\Omega, \mu) \}$ is a self-adjoint dissipative operator.

We use a penalization method, introducing the family of operators in $\mathbb{R}^N$

$$A_\varepsilon u(x) = \frac{1}{2} \Delta u(x) - \langle DU_\varepsilon(x), Du(x) \rangle, \quad x \in \mathbb{R}^N,$$

where

$$U_\varepsilon(x) = U(x) + \frac{1}{2\varepsilon}(\text{dist}(x, \Omega))^2.$$ (1.7)

Operators of this type in the whole $\mathbb{R}^N$ have been already considered, for instance in [5], under further assumptions on $U$. Here we prove that the realizations of the operators $A_\varepsilon$ in $L^2(\mathbb{R}^N, \nu_\varepsilon)$, with domain $D_\varepsilon = \{ u \in H^2(\mathbb{R}^N, \nu_\varepsilon) : \langle DU_\varepsilon, Du \rangle \in L^2(\mathbb{R}^N, \nu_\varepsilon) \}$

$$\nu_\varepsilon(dx) = \left( \int_{\mathbb{R}^N} e^{-2U_\varepsilon(x)} dx \right)^{-1} e^{-2U_\varepsilon(x)} dx$$

are self-adjoint and dissipative. Therefore the equation

$$\lambda u - A_\varepsilon u = \tilde{f},$$

where $\tilde{f}$ is the null extension of $f$ to the whole $\mathbb{R}^N$, has a unique solution $u_\varepsilon \in D_\varepsilon$. It is possible to see that $\| u_\varepsilon \|_{H^2(\mathbb{R}^N, \nu_\varepsilon)}$ is bounded by a constant independent of $\varepsilon$. It follows that the restrictions $u_\varepsilon|_{\Omega}$ are bounded in $H^2(\Omega, \mu)$ by a constant independent of $\varepsilon$. Therefore a sequence $u_{\varepsilon k}|_{\Omega}$ converges to a limiting function $u \in H^2(\Omega, \mu)$ which turns out to be a solution to

$$\begin{cases}
\lambda u - A u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{at } \partial \Omega.
\end{cases}$$

The solution is unique thanks to an a priori estimate, which comes from the integration by parts formula

$$\int_{\Omega} (Au)(x)v(x)\mu(dx) = \frac{-1}{2} \int_{\Omega} \langle Du(x), Dv(x) \rangle \mu(dx) + \frac{1}{2} \int_{\partial \Omega} \frac{\partial u}{\partial n}(x)v(x)\mu(dx).$$

(1.8)

Indeed, taking $v = u$, where $u$ is any solution, we get immediately

$$\| u \|_{L^2(\Omega, \mu)} \leq \frac{1}{\lambda} \| f \|_{L^2(\Omega, \mu)},$$

(1.9)

so that the solution is unique.

Formula (1.8) is crucial in our analysis. It implies also that $A_N$ is symmetric, and, through its consequence (1.9), that $A_N$ is dissipative. Since the resolvent set of $A_N$ is not empty, it follows that $A_N$ is self-adjoint.

The general theory of operators in Hilbert spaces yields that $A_N$ is the infinitesimal generator of an analytic contraction semigroup $T(t)$, which can be naturally extended to an analytic contraction semigroup in $L^p(\Omega, \mu)$ for each $p \in (1, \infty)$.

Under the further assumption that $U - \omega|x|^2/2$ is convex for some $\omega > 0$ we show that $\mu$ satisfies Poincaré and log-Sobolev inequalities, that is

$$\int_{\Omega} |u(x) - \overline{u}|^2 \nu(dx) \leq \frac{1}{2\omega} \int_{\Omega} |Du(x)|^2 \nu(dx), \quad u \in H^1(\Omega, \mu),$$

(1.10)

and

$$\int_{\Omega} u^2(x) \log(u^2(x)) \mu(dx) \leq \frac{1}{\omega} \int_{\Omega} |Du(x)|^2 \mu(dx) + \overline{u^2} \log(\overline{u^2}), \quad u \in H^1(\Omega, \mu).$$

(1.11)
Here $\overline{u} = \int_{\Omega} u(x)\mu(dx)$ is the mean value of $u$, and we adopt the convention $0 \log 0 = 0$.

As well known, (1.11) implies that $T(t)$ is hypercontractive, with

$$
\|T(t)\varphi\|_{L^p(\Omega, \mu)} \leq \|\varphi\|_{L^p(\Omega, \mu)}, \quad t > 0, \; p \geq 2, \; \varphi \in L^p(\Omega, \mu),
$$

and $q(t) = 1 + (p - 1) e^{2\alpha t}$.

A more general class of operators, namely $B_u = A u + \langle G, D u \rangle$ has been recently studied in [16], where suitable assumptions are made in order that the realization $B_p$ of $B$ in $L^p(\mathbb{R}^N, \nu)$, $1 < p < \infty$, generates an analytic semigroup in $L^p(\mathbb{R}^N, \nu)$. In the case $G = 0$, i.e. $B = A$, the assumptions of [16] are not comparable to ours; however they lead to the characterization of the domain of $B_p$ as the space $W^{2,p}(\mathbb{R}^N, \nu)$, for $1 < p < \infty$. Their method seems to be hardly extendable to the case of an unbounded $\Omega$ with Neumann boundary condition, and to the case of discontinuous coefficients. In fact, to our knowledge there are no papers about elliptic operators with unbounded coefficients in an unbounded domain and Neumann boundary condition. The paper [2] deals with elliptic operators with possibly unbounded discontinuous coefficients in an unbounded domain, but it is focused on existence and properties of infinitesimally invariant measures, the underlying boundary condition is the Dirichlet condition, and no effort is made towards the characterization of the domain of their realizations.

2 The realization of $A$ in $L^2(\mathbb{R}^N, \nu)$

In this section we describe the main properties of the realization of $A$ in $L^2(\mathbb{R}^N, \nu)$. We give alternative proofs of the results of [5] that we need here, avoiding some unnecessary assumptions made in [5].

Let $U : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex function, satisfying (1.4). Then $U$ is continuous, and there are $a \in \mathbb{R}$, $b > 0$ such that $U(x) \geq a + b|x|$, for each $x \in \mathbb{R}^N$. It follows that the probability measure (1.3) is well defined, and that $\int_{\mathbb{R}^N} |x|^k e^{-2U(x)} dx < \infty$ for each $k > 0$.

For each $x \in \mathbb{R}^N$, the subdifferential $\partial U(x)$ of $U$ at $x$ is the set $\{ y \in \mathbb{R}^N : U(\xi) \geq U(x) + \langle y, \xi - x \rangle, \forall \xi \in \mathbb{R}^N \}$. Since $U$ is convex, $\partial U(x)$ has a unique element with minimal norm, that we denote by $D U(x)$. Of course if $U$ is differentiable at $x$, $D U(x)$ is the usual gradient.

Since $D U$ is not continuous in general, and we made no growth assumptions, handling the operator $A$ is rather delicate. To overcome these difficulties we introduce the Moreau-Yosida approximations of $U$,

$$
U_\alpha(x) = \inf \left\{ U(y) + \frac{1}{2\alpha} |x - y|^2 : y \in \mathbb{R}^N \right\}, \quad x \in \mathbb{R}^N, \; \alpha > 0,
$$

which will be the main technical tool of this section. The functions $U_\alpha$ are convex, differentiable, and for each $x \in \mathbb{R}^N$ we have (see e.g. [3, Pr. 2.6, Pr. 2.11](1))

$$
U_\alpha(x) \leq U(x), |D U_\alpha(x)| \leq |D U(x)|, \quad \lim_{\alpha \rightarrow 0} U_\alpha(x) = U(x), \quad \lim_{\alpha \rightarrow 0} D U_\alpha(x) = D U(x).
$$

Moreover $D U_\alpha$ is Lipschitz continuous for each $\alpha$, with Lipschitz constant $1/\alpha$. This is of much help, because elliptic operators with Lipschitz continuous (although unbounded) coefficients have nice properties, and they are well studied. See e.g. [12, 13, 15].

The space $H^1(\mathbb{R}^N, \nu)$ is naturally defined as the set of all $u \in H^1_{loc}(\mathbb{R}^N)$ such that $u$, $D_i u \in L^2(\mathbb{R}^N, \nu)$, $i = 1, \ldots, N$. It is a Hilbert space with the standard scalar product

$$
\langle u, v \rangle = \left( \int_{\mathbb{R}^N} e^{-2U(x)} dx \right)^{-1} \int_{\mathbb{R}^N} \left( u v + \sum_{i=1}^{N} D_i u D_i v \right) e^{-2U(x)} dx.
$$

\(^{1}\)To be precise, what is called “Yosida approximations” in [3] are the functions $D U_\alpha = (I - (I + \alpha \partial U)^{-1})/\alpha$, which approximate $D U$.  

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Similarly, the space $H^2(\mathbb{R}^N, \nu)$ is defined as the set of all $u \in H^2_{\text{loc}}(\mathbb{R}^N)$ such that $u$, $D_i u$, $D_{ij} u \in L^2(\mathbb{R}^N, \nu)$, $i, j = 1, \ldots, N$. It is a Hilbert space with the scalar product

$$\langle u, v \rangle = \left( \int_{\mathbb{R}^N} e^{-2U(x)} \, dx \right)^{-1} \int_{\mathbb{R}^N} (u v + \sum_{i=1}^{N} D_i u D_i v + \sum_{i,j=1}^{N} D_{ij} u D_{ij} v) e^{-2U(x)} \, dx.$$ 

We shall use also the spaces $C^k_b(\mathbb{R}^N)$ ($k \in \mathbb{N} \cup \{\infty\}$), consisting of all $C^k$ functions $u : \mathbb{R}^N \to \mathbb{R}$ with bounded derivatives up to the order $k$, their subspace $C^\infty_b(\mathbb{R}^N)$ consisting of all smooth compactly supported functions, and the spaces $C^{k+\theta}_b(\mathbb{R}^N)$ ($k \in \mathbb{N} \cup \{0\}$, $\theta \in (0,1)$) consisting of functions in $C^k_b(\mathbb{R}^N)$ with uniformly $\theta$-Hölder continuous $k$-th order derivatives.

**Lemma 2.1** $C^\infty_b(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N, \nu)$, in $H^1(\mathbb{R}^N, \nu)$ and in $H^2(\mathbb{R}^N, \nu)$.

**Proof** It is well known that every function $u \in X$ with compact support (where $X = L^2(\mathbb{R}^N)$, $X = H^1(\mathbb{R}^N)$, or $X = H^2(\mathbb{R}^N)$) may be approximated in $X$ by a sequence of $C^\infty_b$ functions obtained by convolution with smooth mollifiers. Since $u$ has compact support, such a sequence approximates $u$ also in $L^2(\mathbb{R}^N, \nu)$, in $H^1(\mathbb{R}^N, \nu)$, or in $H^2(\mathbb{R}^N, \nu)$, respectively.

Therefore it is sufficient to show that every $u \in L^2(\mathbb{R}^N, \nu)$ (respectively, $u \in H^1(\mathbb{R}^N, \nu)$, $u \in H^2(\mathbb{R}^N, \nu)$) may be approximated by a sequence of $L^2$ (respectively, $H^1$, $H^2$) functions with compact support. Let $\theta : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that $0 \leq \theta(x) \leq 1$ for each $x$, $\theta \equiv 1$ in $B(0, 1)$, $\theta \equiv 0$ outside $B(0, 2)$, and set $u_n(x) = u(x)\theta(x/n)$. Then $u_n \to u$ in $L^2(\mathbb{R}^N, \nu)$. Indeed,

$$\int_{\mathbb{R}^N} |u_n - u|^2 \nu(dx) \leq \int_{|x| \geq n} |u|^2 \nu(dx)$$

which goes to 0 as $n \to \infty$. If $u \in H^1(\mathbb{R}^N, \nu)$, then

$$D_i u_n = D_i u(x)\theta(x/n) + u(x)D_i \theta(x/n)/n ,$$

where $D_i u\theta(\cdot/n)$ goes to $D_i u$ and $u D_i \theta(\cdot/n)/n$ goes to 0 as $n \to \infty$. If $u \in H^2(\mathbb{R}^N, \nu)$, then

$$D_{ij} u_n(x) = D_{ij} u(x) \theta(x/n) + D_i u(x) D_j \theta(x/n)/n + D_j u(x) D_i \theta(x/n)/n + u(x)D_{ij} \theta(x/n)/n^2,$$

and $D_{ij} u\theta(\cdot/n)$ goes to $D_{ij} u$ while the sequences $u D_i \theta(\cdot/n)/n$, $D_i u D_j \theta(\cdot/n)/n$, $D_j u D_i \theta(\cdot/n)/n$, $u D_{ij} \theta(\cdot/n)/n^2$ go to zero in $L^2(\mathbb{R}^N, \nu)$ as $n \to \infty$. The statement follows. \qed

Let us define now the realization $A$ of $\mathcal{A}$ in $L^2(\mathbb{R}^N, \nu)$ by

$$D(A) = \{ u \in H^2(\mathbb{R}^N, \nu) : \mathcal{A} u \in L^2(\mathbb{R}^N, \nu) \}$$

$$= \{ u \in H^2(\mathbb{R}^N, \nu) : \langle DU, Du \rangle \in L^2(\mathbb{R}^N, \nu) \},$$

\hspace{1cm} \text{(2.1)}

$$(Au)(x) = \mathcal{A} u(x), \quad x \in \mathbb{R}^N.$$

We shall show that $A$ is a self-adjoint dissipative operator. A first important step is next lemma, which yields that $A$ is symmetric.

**Lemma 2.2** For each $u \in D(A)$, $\psi \in H^1(\mathbb{R}^N, \nu)$ we have

$$\int_{\mathbb{R}^N} (Au)(x) \psi(x) \nu(dx) = -\frac{1}{2} \int_{\mathbb{R}^N} \langle Du(x), D\psi(x) \rangle \nu(dx).$$

\hspace{1cm} \text{(2.2)}
Proof — It is sufficient to prove that (2.2) holds for each \( \psi \in C^{\infty}_0(\mathbb{R}^N) \). Indeed, in this case, for \( \psi \in H^1(\mathbb{R}^N, \nu) \), (2.2) is obtained approximating \( \psi \) by a sequence of \( C^{\infty}_0 \) functions.

First we assume that \( U \) is continuously differentiable. If \( \psi \in C^{\infty}_0(\mathbb{R}^N) \), then the function \( \psi \exp(-2U) \) is continuously differentiable and it has compact support. Integrating by parts \( (\Delta u)(x)\psi(x)e^{-2U(x)} \) we get

\[
\frac{1}{2} \int_{\mathbb{R}^N} (\Delta u)(x)\psi(x)e^{-2U(x)} dx = -\frac{1}{2} \int_{\mathbb{R}^N} (Du(x), D(\psi(x)e^{-2U(x)}) dx
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^N} (Du(x), D\psi(x))e^{-2U(x)} dx + \frac{1}{2} \int_{\mathbb{R}^N} (Du(x), 2DU(x))\psi(x)e^{-2U(x)} dx
\]

so that (2.2) holds.

Let now \( U \) be merely continuous. Let \( U_\alpha \) be the Yosida approximations defined above, and define accordingly the differential operator \( A_\alpha \), the measure \( \nu_\alpha(dx) \), and the operator \( A_\alpha : D(A_\alpha) \to L^2(\mathbb{R}^N, \nu_\alpha) \). For each \( \psi \in C^{\infty}_0(\mathbb{R}^N) \) and for each \( \alpha > 0 \) we have

\[
\int_{\mathbb{R}^N} (A_\alpha u)(x)\psi(x) \exp(-2U_\alpha(x)) dx = -\frac{1}{2} \int_{\mathbb{R}^N} (Du(x), D\psi(x)) \exp(-2U_\alpha(x)) dx,
\]

where \( \exp(-2U_\alpha) \) goes to \( \exp(-2U) \) uniformly in \( \text{supp} \psi \), and since \( DU \) is locally bounded, \( A_\alpha u \exp(-2U_\alpha) \) goes to \( Au \exp(-2U) \) in \( L^2(\text{supp} \psi) \). Letting \( \alpha \to 0 \) we obtain (2.2). □

Taking \( \psi = u \) in (2.2) shows that \( A \) is symmetric.

In the next proposition we give good \( a \) priori estimates for regular functions, when \( U \) has Lipschitz continuous derivatives. This is enough for our aims because we will apply it to the operators \( A_\alpha \).

**Proposition 2.3** Let \( U : \mathbb{R}^N \to \mathbb{R} \) satisfy (1.4) and have Lipschitz continuous derivatives. For every \( u \in C^2_b(\mathbb{R}^N) \) we have

\[
\| Du \|^2_{L^2(\mathbb{R}^N, \nu)} \leq 2 \| Au \|_{L^2(\mathbb{R}^N, \nu)} \| u \|_{L^2(\mathbb{R}^N, \nu)},
\]

and

\[
\| D^2 u \|_{L^2(\mathbb{R}^N, \nu)} \leq 4 \| Au \|_{L^2(\mathbb{R}^N, \nu)}.
\]

**Proof** — Since \( DU \) is Lipschitz continuous, it has at most linear growth as \( |x| \to \infty \), so that \( |DU| \in L^2(\mathbb{R}^N, \nu) \). It follows that \( C^2_b(\mathbb{R}^N) \) is contained in \( D(A) \). Then estimate (2.3) follows immediately from (2.2) taking \( \psi = u \). To prove (2.4) we first consider a function \( u \in C^\infty_b(\mathbb{R}^N) \). In this case the functions \( f = Au \) and \( DhU, h = 1, \ldots, N \), are Lipschitz continuous; hence they belong to \( H^{1}_{loc}(\mathbb{R}^N) \) and we have

\[
AD_h u - \sum_{k=1}^{N} D_h U D_k u = D_h f,
\]

where the equality is meant in \( L^2_{loc}(\mathbb{R}^N) \). Multiplying by \( D_h u \) and summing up we get

\[
\sum_{h=1}^{N} AD_h u D_h u \geq \sum_{h=1}^{N} D_h f D_h u \text{ a.e.,}
\]

because \( D^2 U \geq 0 \) almost everywhere. Since \( D_h u \in D(A) \) and \( f \in H^1(\mathbb{R}^N, \nu) \), we may integrate both sides with respect to \( \nu(dx) \), and applying (2.2) to both sides we get

\[
-\frac{1}{2} \int_{\mathbb{R}^N} |D^2 u|^2 \nu(dx) \geq -2 \int_{\mathbb{R}^N} Au \cdot f \nu(dx),
\]

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which yields (2.4).

Let now \( u \in C_{\theta}^2(\mathbb{R}^N) \), and let \( u_k \) be a sequence in \( C_{\theta}^\infty(\mathbb{R}^N) \) that goes to \( u \) in \( H^2(\mathbb{R}^N, \nu) \) as \( k \to \infty \). We may assume that \( \|Du_k\|_{\infty} \) is bounded by a constant \( C \) independent of \( k \) (see the proof of lemma 2.1). Then \( D_{ij}u_k \) goes to \( D_{ij}u \) and \( \Delta u_k \) go to \( \Delta u \) in \( L^2(\mathbb{R}^N, \nu) \). Moreover, up to a subsequence \( \langle DU, Du_k - Du \rangle \) goes to 0 almost everywhere and it is bounded by \( (C + \|Du\|_{\infty})\|DU\|_{L^2} \leq L^2(\mathbb{R}^N, \nu) \). Therefore \( \langle DU, Du_k \rangle \) goes to \( \langle DU, Du \rangle \) in \( L^2(\mathbb{R}^N, \nu) \), so that \( \mathcal{A}u_k \) goes to \( \mathcal{A}u \) in \( L^2(\mathbb{R}^N, \nu) \). Since (2.4) holds for each \( u_k \), letting \( k \to \infty \) we obtain that (2.4) holds for \( u \). □

Now we are ready to solve the resolvent equation,

\[
\lambda u - Au = f
\]

for each \( \lambda > 0 \) and \( f \in L^2(\mathbb{R}^N, \nu) \).

**Theorem 2.4** Let \( U : \mathbb{R}^N \mapsto \mathbb{R} \) satisfy assumption (1.4). Then the resolvent set of \( \mathcal{A} \) contains \((0, +\infty)\) and

\[
\begin{align*}
(i) \quad & \|R(\lambda, \mathcal{A})f\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\
(ii) \quad & \|D\lambda(\lambda, \mathcal{A})f\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \\
(iii) \quad & \|D^2\lambda(\lambda, \mathcal{A})f\|_{L^2(\mathbb{R}^N, \nu)} \leq 4 \|f\|_{L^2(\mathbb{R}^N, \nu)}.
\end{align*}
\]

**Proof** — To begin with, we note that (2.5) has at most one solution in \( D(\mathcal{A}) \). Indeed, if \( u \in D(\mathcal{A}) \) satisfies \( \lambda u = Au \) then by (2.2) we have

\[
\int_{\mathbb{R}^N} \lambda(u(x))^2 \nu(dx) = \int_{\mathbb{R}^N} (Au)(x)u(x)\nu(dx) = -\frac{1}{2} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx) \leq 0,
\]

so that \( u = 0 \).

To find a solution to (2.5), first we consider the case where \( f \in C_{\theta}^\theta(\mathbb{R}^N) \) for some \( \theta \in (0, 1) \).

Let \( U_\alpha \) be the Yosida approximations of \( U \) defined above, and let the differential operator \( \mathcal{A}_\alpha \), the measure \( \nu_\alpha(dx) \), and the operator \( \mathcal{A}_\alpha : D(\mathcal{A}_\alpha) \mapsto L^2(\mathbb{R}^N, \nu_\alpha) \) be defined accordingly. Since \( DU_\alpha \) is Lipschitz continuous, by [12, thm. 1] the problem

\[
\lambda u_\alpha - \mathcal{A}_\alpha u_\alpha = f
\]

has a unique solution \( u_\alpha \in C_{\theta}^{2+\theta}(\mathbb{R}^N) \). In fact [12, thm. 1] deals with large \( \lambda \)'s, but a standard application of the maximum principle (see e.g. [12, lemma 2.4]) and of the Schauder estimates of [12, thm. 1] show that (2.5) is uniquely solvable in \( C_{\theta}^{2+\theta}(\mathbb{R}^N) \) for each \( \lambda > 0 \). The integration by parts formula (2.2) gives

\[
\int_{\mathbb{R}^N} (\lambda u_\alpha - f)u_\alpha \nu_\alpha(dx) = -\frac{1}{2} \int_{\mathbb{R}^N} |Du_\alpha|^2 \nu_\alpha(dx) \leq 0,
\]

so that

\[
\|u_\alpha\|_{L^2(\mathbb{R}^N, \nu_\alpha)} \leq \frac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)}.
\]

Estimates (2.3) and (2.4) applied to \( u_\alpha \) give

\[
\|Du_\alpha\|_{L^2(\mathbb{R}^N, \nu_\alpha)} \leq \frac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu_\alpha)} \quad (2.8)
\]
and
\[ \|D^2 u_\alpha\|_{L^2(\mathbb{R}^N, \nu)} \leq 2\|\lambda u_\alpha - f\|_{L^2(\mathbb{R}^N, \nu)} \leq 4\|f\|_{L^2(\mathbb{R}^N, \nu)}. \]  \hspace{1cm} (2.9)

Since \( U_\alpha(x) \) goes to \( U(x) \) monotonically as \( \alpha \to 0 \), then \( \exp(-2U_\alpha(x)) \) goes to \( \exp(-2U(x)) \) monotonically, and \( (\int_{\mathbb{R}^N} e^{-2U_\alpha(x)} \, dx)^{-1} \) goes to \( (\int_{\mathbb{R}^N} e^{-2U(x)} \, dx)^{-1} \), \( \|f\|_{L^2(\mathbb{R}^N, \nu)} \) goes to \( \|f\|_{L^2(\mathbb{R}^N, \nu)} \) as \( \alpha \to 0 \). It follows that \( \|u_\alpha\|_{H^2(\mathbb{R}^N, \nu)} \) is bounded by a constant independent of \( \alpha \). Since \( \|\cdot\|_{H^2(\mathbb{R}^N, \nu)} \leq c(\alpha) \|\cdot\|_{H^2(\mathbb{R}^N, \nu_\alpha)} \), with \( \lim_{\alpha \to 0} c(\alpha) = 1 \), also \( \|u_\alpha\|_{H^2(\mathbb{R}^N, \nu)} \) is bounded by a constant independent of \( \alpha \), for \( \alpha \) small. Therefore there is a sequence \( u_{\alpha_n} \) that converges weakly in \( H^2(\mathbb{R}^N, \nu) \) to a function \( u \in H^2(\mathbb{R}^N, \nu) \), and converges to \( u \) pointwise a.e. and in \( H^1(B(0, R)) \) for each \( R > 0 \). This implies easily that \( u \) solves (2.5). Indeed, let \( \phi \in C_0^\infty(\mathbb{R}^N) \). For each \( n \in \mathbb{N} \) we have
\[ \int_{\mathbb{R}^N} (\lambda u_{\alpha_n} - \frac{1}{2} \Delta u_{\alpha_n} + \langle DU_{\alpha_n}, Du_{\alpha_n} \rangle - f) \phi e^{-2U} \, dx = 0. \]

Letting \( n \to \infty \), we get immediately that \( \int_{\mathbb{R}^N} (\lambda u_{\alpha_n} - \frac{1}{2} \Delta u_{\alpha_n}) \phi e^{-2U} \, dx \) goes to \( \int_{\mathbb{R}^N} (\lambda u - \frac{1}{2} \Delta u) \phi e^{-2U} \, dx \). Moreover \( \int_{\mathbb{R}^N} \langle DU_{\alpha_n}, Du_{\alpha_n} \rangle \phi e^{-2U} \, dx \) goes to \( \int_{\mathbb{R}^N} \langle DU, Du \rangle \phi e^{-2U} \, dx \) because \( DU_{\alpha_n} \) goes to \( DU \) in \( L^2(\text{supp} \phi) \). Therefore letting \( n \to \infty \) we get
\[ \int_{\mathbb{R}^N} (\lambda u - Au - f) \phi e^{-2U} \, dx = 0 \]
for each \( \phi \in C_0^\infty(\mathbb{R}^N) \), and hence \( \lambda u - Au = f \) almost everywhere. So, \( u \in D(A) \) is the solution of the resolvent equation, and letting \( \alpha \to 0 \) in (2.7), (2.8), (2.9), we get
\[ \|u\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \quad \|Du\|_{L^2(\mathbb{R}^N, \nu)} \leq \frac{2}{\sqrt{\lambda}} \|f\|_{L^2(\mathbb{R}^N, \nu)}, \quad \|D^2u\|_{L^2(\mathbb{R}^N, \nu)} \leq 4\|f\|_{L^2(\mathbb{R}^N, \nu)}. \]  \hspace{1cm} (2.10)

Let now \( f \in L^2(\mathbb{R}^N, \nu) \) and let \( f_n \) be a sequence of \( C_0^\infty(\mathbb{R}^N) \) functions going to \( f \) in \( L^2(\mathbb{R}^N, \nu) \) as \( n \to \infty \). Because of estimates (2.10), the solutions \( u_n \) of
\[ \lambda u_n - Au_n = f_n \]
are a Cauchy sequence in \( H^2(\mathbb{R}^N, \nu) \), and converge to a solution \( u \in H^2(\mathbb{R}^N, \nu) \) of (2.5). Due again to estimates (2.10), \( u \) satisfies (2.6).

\[ \Box \]

**Corollary 2.5** The operator \( A \) is self-adjoint and dissipative. \( H^1(\mathbb{R}^N, \nu) \) is the domain of \( \sqrt{-A} \).

**3 The operator \( A_N \)**

In this section we describe the main properties of the realization \( A_N \) of \( A \) in \( L^2(\Omega, \mu) \). We recall that \( \Omega \) is a convex open set in \( \mathbb{R}^N \) with \( C^2 \) boundary, and \( U : \mathbb{R}^N \to \mathbb{R} \) is a convex function satisfying (1.4). The functional spaces \( H^1(\Omega, \mu), H^2(\Omega, \mu), C_0^\infty(\overline{\Omega}, \mu), \) are defined as in the case \( \Omega = \mathbb{R}^N \).

Arguing as in the proof of lemma 2.1 we obtain a similar result.

**Lemma 3.1** Let \( \mu \) be the measure defined in (1.5). Then the functions in \( C_0^\infty(\overline{\Omega}) \) with compact support are dense in \( L^2(\Omega, \mu) \), in \( H^1(\Omega, \mu) \), and in \( H^2(\Omega, \mu) \).
**Proof** — First we consider a compactly supported function \( u \), belonging to \( L^2(\Omega, \mu) \), to \( H^1(\Omega, \mu) \), or to \( H^2(\Omega, \mu) \). Since the boundary of \( \Omega \) is \( C^2 \), then \( u \) may be extended to the whole \( \mathbb{R}^N \) in such a way that the extension is in \( L^2(\mathbb{R}^N) \), in \( H^1(\mathbb{R}^N) \), or in \( H^2(\mathbb{R}^N) \), respectively. The extension is then approximated by a sequence \( u_n \) of \( C_0^\infty \) functions obtained by convolution with smooth mollifiers. The restrictions of \( u_n \) to \( \Omega \) converge to \( u \) in \( L^2(\Omega, \mu) \), in \( H^1(\Omega, \mu) \), or in \( H^2(\Omega, \mu) \), respectively.

If \( u \) has not compact support, let \( \theta \) be the function defined in the proof of lemma 2.1. The sequence \( u_n(x) = u(x)\theta(x/n) \) consists of \( L^2 \) (respectively, \( H^1 \), \( H^2 \)) functions with compact support, and converges to \( u \) in \( L^2(\Omega, \mu) \), in \( H^1(\Omega, \mu) \), or in \( H^2(\Omega, \mu) \), respectively (the proof is the same of lemma 2.1). The statement follows. \( \square \)

The realization \( A_N \) of \( \mathcal{A} \) in \( L^2(\Omega, \mu) \) with Neumann boundary condition is defined by

\[
D(A_N) = \{ u \in H^2(\Omega, \mu) : \mathcal{A}u \in L^2(\Omega, \mu), \frac{\partial u}{\partial n} = 0 \text{ at } \partial \Omega \}
\]

\[= \{ u \in H^2(\Omega, \mu) : \langle Du, Du \rangle \in L^2(\Omega, \mu), \frac{\partial u}{\partial n} = 0 \text{ at } \partial \Omega \}, \quad (3.1)\]

\[(A_N u)(x) = \mathcal{A}u(x), \quad x \in \Omega.\]

A formula similar to (2.2) holds if \( \mathbb{R}^N \) is replaced by \( \Omega \). The proof is the same as in lemma 2.2, and it is omitted.

**Lemma 3.2** Let \( \psi \in H^1(\Omega, \mu) \) and let \( u \in H^2(\Omega, \mu) \) be such that \( \mathcal{A}u \in L^2(\Omega, \mu) \). Then we have

\[
\int_{\Omega} (Au(x)\psi(x))\mu(dx) = -\frac{1}{2} \int_{\Omega} \langle Du(x), Du(x) \rangle \mu(dx) + \frac{1}{2} \int_{\partial \Omega} \frac{\partial u}{\partial n}(x)\psi(x)\mu(dx). \quad (3.2)
\]

**Theorem 3.3** The resolvent set of \( A_N \) contains \( (0, +\infty) \). For every \( \lambda > 0 \) we have

\[
(i) \quad \|R(\lambda, A_N)f\|_{L^2(\Omega, \mu)} \leq \frac{1}{\lambda}\|f\|_{L^2(\Omega, \mu)},
\]

\[\left\{ (ii) \quad \|DR(\lambda, A_N)f\|_{L^2(\Omega, \mu)} \leq \frac{2}{\sqrt{\lambda}}\|f\|_{L^2(\Omega, \mu)}, \quad (3.3)\right.\]

\[\left. (iii) \quad \|D^2R(\lambda, A_N)f\|_{L^2(\Omega, \mu)} \leq 4\|f\|_{L^2(\Omega, \mu)}.\right.\]

**Proof** — Let \( \lambda > 0 \), let \( f \in L^2(\Omega, \mu) \), and consider the resolvent equation

\[
\left\{ \begin{array}{l}
\lambda u - \mathcal{A}u = f \quad \text{in } \Omega,
\\
\frac{\partial u}{\partial n} = 0 \quad \text{at } \partial \Omega.
\end{array} \right. \quad (3.4)
\]

Uniqueness of the solution to (3.4) in \( H^2(\Omega, \mu) \) is easy. Indeed, if \( u \in H^2(\Omega, \mu) \), \( \frac{\partial u}{\partial n} = 0 \) and \( \lambda u - \mathcal{A}u = 0 \), then taking \( \psi = u \) in (3.2) we get \( \lambda \|u\|_{L^2(\Omega, \mu)}^2 = 0 \), and hence \( u = 0 \).

Now we show that (3.4) has a solution \( u \in D(A_N) \). For each \( \epsilon > 0 \) let

\[
V_\epsilon(x) = U(x) + \frac{1}{2\epsilon}(\text{dist}(x, \Omega))^2, \quad x \in \mathbb{R}^N,
\]

and let the differential operator \( \mathcal{L}_\epsilon \) be defined by

\[
(\mathcal{L}_\epsilon u)(x) = \frac{1}{2} \Delta u(x) - \langle Du_\epsilon(x), Du(x) \rangle, \quad x \in \mathbb{R}^N.
\]
The function $V_\varepsilon$ satisfies obviously (1.4); moreover since $\Omega$ is convex, then $V_\varepsilon$ is convex. Set
\[ Z_\varepsilon = \int_{\mathbb{R}^N} \exp(-2V_\varepsilon(x))dx, \quad \nu_\varepsilon(dx) = \frac{1}{Z_\varepsilon} \exp(-2V_\varepsilon(x))dx, \quad (3.5) \]
and let $A_\varepsilon$ be the realization of $L_\varepsilon$ in $L^2(\mathbb{R}^N, \nu_\varepsilon)$ defined by
\[ D(A_\varepsilon) = \{ u \in H^2(\mathbb{R}^N, \nu_\varepsilon) : \langle DV_\varepsilon, Du \rangle \in L^2(\mathbb{R}^N, \nu_\varepsilon) \}. \]
Let $\tilde{f}$ be defined by $\tilde{f}(x) = f(x)$ for $x \in \Omega$, $\tilde{f}(x) = 0$ for $x$ outside $\Omega$. By theorem 2.4, the problem
\[ \lambda u - \mathcal{L}_\varepsilon u = \tilde{f}, \quad x \in \mathbb{R}^N, \quad (3.6) \]
has a unique solution $u_\varepsilon \in D(A_\varepsilon)$, which satisfies the estimates
\[ \left\{ \begin{array}{l}
\| u_\varepsilon \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)} \leq \frac{1}{\lambda} \| \tilde{f} \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)}, \\
\| Du_\varepsilon \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)} \leq \frac{2}{\sqrt{\lambda}} \| \tilde{f} \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)}, \\
\| D^2 u_\varepsilon \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)} \leq 4 \| \tilde{f} \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)} 
\end{array} \right. \quad (3.7) \]
due to (2.6). If in addition $f(x) \geq 0$ a.e., then $u_\varepsilon(x) \geq 0$ for each $x$. Since
\[ \| \tilde{f} \|_{L^2(\mathbb{R}^N, \nu_\varepsilon)} = \left( \frac{1}{Z_\varepsilon} \int_\Omega f^2 e^{-2V_\varepsilon} dx \right)^{1/2} = \left( \frac{1}{Z_\varepsilon} \int_\mathbb{R} e^{-2V_\varepsilon} dx \right)^{1/2} \| f \|_{L^2(\Omega, \mu)} \]
remains bounded as $\varepsilon \to 0$, then $u_\varepsilon$ is bounded in $H^2(\mathbb{R}^N, \nu_\varepsilon)$ and the restriction $u_\varepsilon|_\Omega$ is bounded in $H^2(\Omega, \mu)$. Up to a sequence, $u_\varepsilon|_\Omega$ converges weakly in $H^2(\Omega, \mu)$ to a function $u \in H^2(\Omega, \mu)$ and it converges to $u$ pointwise a.e. and in $H^{3/2}(\Omega \cap B(0, R), dx)$ for every $R > 0$. Since $\lambda u_\varepsilon - A u_\varepsilon = f$ in $\Omega$, then $u$ satisfies $\lambda u - A u = f$ almost everywhere in $\Omega$. Since $\lambda u, \Delta u, f$ belong to $L^2(\Omega, \mu)$, then also $\langle DU, Du \rangle$ does.

Let us prove that $\partial u/\partial n = 0$ at the boundary. For each $\psi \in C_0^\infty(\mathbb{R}^N)$ we have, by (2.2)
\[ \int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx) = -\frac{1}{2} \int_{\mathbb{R}^N} \langle D u_\varepsilon, D \psi \rangle \nu_\varepsilon(dx). \]
On the other hand,
\[ \int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx) = \int_{\mathbb{R}^N \setminus \Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx) + \int_{\Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx), \]
where
\[ \int_{\mathbb{R}^N \setminus \Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx) = \lambda \int_{\mathbb{R}^N \setminus \Omega} u_\varepsilon \psi \nu_\varepsilon(dx), \]
because $\lambda u_\varepsilon - \mathcal{L}_\varepsilon u_\varepsilon = 0$ in $\mathbb{R}^N \setminus \Omega$, and
\[ \int_{\Omega} \mathcal{L}_\varepsilon u_\varepsilon \psi \nu_\varepsilon(dx) = -\frac{1}{2} \int_{\Omega} \langle D u_\varepsilon, D \psi \rangle \nu_\varepsilon(dx) + \frac{1}{2} \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial n}(x) \psi(x) \nu_\varepsilon(dx) \]
because of (3.2). It follows that
\[ -\frac{1}{2} \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial n}(x) \psi(x) \nu_\varepsilon(dx) = -\frac{1}{2} \int_{\mathbb{R}^N \setminus \Omega} \langle D u_\varepsilon, D \psi \rangle \nu_\varepsilon(dx) - \lambda \int_{\mathbb{R}^N \setminus \Omega} u_\varepsilon \psi \nu_\varepsilon(dx). \]
Note that \( \| \psi \|_{H^1(\mathbb{R}^N \setminus \Omega, \nu_\varepsilon)} \) goes to 0 as \( \varepsilon \to 0 \). Since both \( u_\varepsilon \) and \( |Du_\varepsilon| \) are bounded in \( L^2(\mathbb{R}^N, \nu_\varepsilon) \), the right hand side goes to 0 as \( \varepsilon \to 0 \). On the other hand, since \( u_\varepsilon \) goes to \( u \) in \( H^{3/2}(\Omega \cap \text{supp} \psi) \), then \( \partial u_\varepsilon / \partial n \to \partial u / \partial n \) in \( L^2(\partial \Omega \cap \text{supp} \psi) \). So we have

\[
\int_{\partial \Omega} \frac{\partial \psi(x)}{\partial n} e^{-2V(x)} dx = \lim_{\varepsilon \to 0} \int_{\partial \Omega} \frac{\partial u_\varepsilon(x)}{\partial n} e^{-2V_\varepsilon(x)} \nu_\varepsilon(dx) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} e^{-2V(x)} dx \int_{\partial \Omega} \frac{\partial u_\varepsilon(x)}{\partial n} \psi(x) \nu_\varepsilon(dx) = 0.
\]

This implies that \( \partial u / \partial n = 0 \) at \( \partial \Omega \), and \( u \in D(A_N) \). So, \( u \) is the unique solution in \( D(A_N) \) of problem (3.4). Therefore \( \rho(A_N) \supset (0, +\infty) \). Estimates (3.3) follow letting \( \varepsilon \) to 0 in (3.7).

**Corollary 3.4** The operator \( A_N \) is self-adjoint and dissipative in \( L^2(\Omega, \mu) \). The measure \( \mu \) is an infinitesimally invariant measure for \( A_N \). The space \( H^1(\Omega, \mu) \) is the domain of \( (-A_N)^{1/2} \).

**Proof** — The first statement is an immediate consequence of theorem 3.3, and of formula (3.2) which yields that \( A_N \) is symmetric. Taking \( \psi \equiv 1 \), formula (3.2) shows that \( \int_{\Omega} A_N u \mu(dx) = 0 \) for each \( u \in D(A_N) \), and therefore \( \mu \) is an infinitesimally invariant measure for \( A_N \). It proves also that

\[
\int_{\Omega} |(-A_N)^{1/2} u|^2 \mu(dx) = \frac{1}{2} \int_{\Omega} |Du|^2 \mu(dx)
\]

for each \( u \in D(A_N) \), and this implies that \( D((-A_N)^{1/2}) = H^1(\Omega, \mu) \).

## 4 Poincaré and Log–Sobolev inequalities

In this section we prove the Poincaré and Log–Sobolev inequalities for the measure \( \mu \). In addition to (1.4) we assume that

\[
\exists \omega > 0 \text{ such that } x \mapsto U(x) - \omega |x|^2 / 2 \text{ is convex.} \tag{4.1}
\]

In the case of a twice continuously differentiable \( U \) this means just \( D^2 U(x) \geq \omega I \) for each \( x \).

If \( (\Lambda, m) \) is any measure space and \( u \in L^1(\Lambda, m) \) we set

\[
\overline{u}_m = \int_{\Lambda} u(x)m(dx). \tag{4.2}
\]

Poincaré and Log–Sobolev inequalities in the whole \( \mathbb{R}^N \) were proved in [1] under the further assumptions that \( U \) is continuously differentiable. In our case such assumptions may be avoided.

**Proposition 4.1** Let \( U \) satisfy (1.4) and (4.1), and let \( \nu \) be defined by (1.3). Then

\[
\int_{\mathbb{R}^N} |u(x) - \overline{u}_\nu|^2 \nu(dx) \leq \frac{1}{4\omega} \int_{\mathbb{R}^N} |Du(x)|^2 d\nu(dx), \quad u \in H^1(\mathbb{R}^N, \nu), \tag{4.3}
\]

and

\[
\int_{\mathbb{R}^N} u^2(x) \log(u^2(x)) \nu(dx) \leq \frac{1}{\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu(dx) + \overline{u}^2 \nu \log(\overline{u}^2 \nu), \quad u \in H^1(\mathbb{R}^N, \nu). \tag{4.4}
\]
Proof — Let us consider the Moreau-Yosida approximations $U_\alpha$ of $U$. For each $\alpha$, $U_\alpha$ satisfies (1.4) with constant $\omega_\alpha$ which goes to $\omega$ as $\alpha$ goes to 0, and $DU_\alpha$ is Lipschitz continuous. Therefore (see e.g. [1]) Poincaré and Log-Sobolev inequalities hold for the measures $\nu_\alpha$, in the form

$$\int_{\mathbb{R}^N} |u(x) - \overline{\nu}_\alpha|^2 \nu_\alpha(dx) \leq \frac{1}{2\omega_\alpha} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx), \quad u \in H^1(\mathbb{R}^N, \nu_\alpha),$$

(where $\overline{\nu}_\alpha$ stands for $\overline{\nu}_{\nu_\alpha}$) and

$$\int_{\mathbb{R}^N} u^2(x) \log(u^2(x)) \nu_\alpha(dx) \leq \frac{1}{\omega_\alpha} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\alpha(dx) + \overline{u}_{\alpha}^2 \log(\overline{u}_\alpha^2), \quad u \in H^1(\mathbb{R}^N, \nu_\alpha).$$

Taking $u \in C_0^\infty(\mathbb{R}^N)$ and letting $\alpha \to 0$ we see that $\overline{\nu}_\alpha$ goes to $\overline{\nu}_\nu$, $\overline{u}_{\alpha}$ goes to $\overline{u}_\nu$, and $u$ satisfies (4.3) and (4.4). Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N, \nu)$ the statement follows. □

Having estimates (4.3) and (4.4) as a tool, it is not hard to prove Poincaré and Log-Sobolev inequalities in our situation.

**Proposition 4.2** Let $U$ satisfy (1.4) and (4.1), and let $\mu$ be defined by (1.5). Then

$$\int_{\Omega} |u(x) - \overline{\mu}_\nu|^2 \mu(dx) \leq \frac{1}{2\omega} \int_{\Omega} |Du(x)|^2 d\mu(dx), \quad u \in H^1(\Omega, \mu),$$

(4.5)

and

$$\int_{\Omega} u^2(x) \log(u^2(x)) \mu(dx) \leq \frac{1}{\omega} \int_{\Omega} |Du(x)|^2 \mu(dx) + \overline{u}_{\mu}^2 \log(\overline{u}_\mu^2), \quad u \in H^1(\Omega, \mu).$$

(4.6)

**Proof** — Let $u \in H^1(\Omega, \mu)$ have compact support, and extend $u$ to an $H^1(\mathbb{R}^N)$ function with compact support, still denoted by $u$. Let

$$\nu_\varepsilon(dx) = \left( \int_{\mathbb{R}^N} e^{-2V_\varepsilon(x)} dx \right)^{-1} e^{-2V_\varepsilon(x)} dx,$$

where $V_\varepsilon$ is defined by (1.7). By proposition 4.2, for each $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^N} |u(x) - \overline{\nu}_\varepsilon|^2 \nu_\varepsilon(dx) \leq \frac{1}{2\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\varepsilon(dx),$$

(4.7)

(where $\overline{\nu}_\varepsilon$ stands for $\overline{\nu}_{\nu_\varepsilon}$) and

$$\int_{\mathbb{R}^N} u^2(x) \log(u^2(x)) \nu_\varepsilon(dx) \leq \frac{1}{\omega} \int_{\mathbb{R}^N} |Du(x)|^2 \nu_\varepsilon(dx) + \overline{u}_\varepsilon^2 \log(\overline{u}_\varepsilon^2).$$

(4.8)

Since

$$\lim_{\varepsilon \to 0} V_\varepsilon(x) = \begin{cases} U(x) & \text{if } x \in \Omega \\ +\infty & \text{if } x \notin \Omega, \end{cases}$$

then $\overline{\nu}_\varepsilon$ goes to $\overline{\nu}_\mu = \int_{\Omega} u(x) \mu(dx)$, $\overline{u}_{\varepsilon}$ goes to $\overline{u}_\mu$ as $\varepsilon$ goes to 0, and letting $\varepsilon$ go to 0 in (4.7), (4.8) we obtain that $u$ satisfies (4.5) and (4.6). Since the compactly supported functions are dense in $H^1(\Omega, \mu)$ the statement follows. □

Proposition 4.2 yields important properties of the semigroup $T(t)$ generated by $A_N$.

**Corollary 4.3** Under assumptions (1.4) and (4.1), 0 is a simple isolated eigenvalue of $A_N$. The rest of the spectrum, $\sigma(A_N) \setminus \{0\}$ is contained in $(-\infty, -\omega)$, and

$$\|T(t)u - \overline{\nu}_\mu\|_{L^2(\Omega, \mu)} \leq e^{-\omega t} \|u - \overline{\nu}_\mu\|_{L^2(\Omega, \mu)}, \quad u \in L^2(\Omega, \mu), \quad t > 0.$$  

(4.9)
Moreover we have
\[ \|T(t)\varphi\|_{L^q(t)(\Omega, \mu)} \leq \|\varphi\|_{L^P(\Omega, \mu)}, \quad p \geq 2, \quad \varphi \in L^P(\Omega, \mu), \] (4.10)
where
\[ q(t) = 1 + (p - 1)e^{2\omega t}, \quad t > 0. \] (4.11)

**Proof** — 0 is obviously an eigenvalue of \( A_N \), whose kernel contains the constant functions. In fact, the kernel of \( A_N \) consists only of constant functions: if \( Au = 0 \), then \( T(t)u = u \) and hence \( DT(t)u = Du \) for each \( t > 0 \), but since \( \|DT(t)u\|_{L^2(\Omega, \mu)} \leq \mathcal{C}t^{-1/2}\|u\|_{L^2(\Omega, \mu)} \) (this is a consequence of the equality \( D((-A_N)^{1/2}) = H^1(\Omega, \mu) \)) letting \( t \to +\infty \) we get \( Du = 0 \) so that \( u \) is constant.

A standard argument shows now that 0 is isolated. Let \( u \in H^1(\Omega, \mu) \) and set \( g(t) = \|T(t)u - \bar{u}\|_{L^2(\Omega, \mu)}^2 \). Using (1.8) and (4.5) we get easily \( g'(t) \leq -2\omega g(t) \), for each \( t > 0 \), so that \( u \) satisfies (4.9). Since \( H^1(\Omega, \mu) \) is dense in \( L^2(\Omega, \mu) \), then (4.9) holds for every \( u \in L^2(\Omega, \mu) \). From the general theory of strongly continuous semigroups it follows that 0 is isolated in \( \sigma(A) \), and, since \( \sigma(A_N) \subset \mathbb{R} \), that \( \sigma(A) \setminus \{0\} \subset (-\infty, -\omega] \).

Since \( A_N \) is self-adjoint, all its isolated eigenvalues are semisimple. The kernel of \( A_N \) is one dimensional, and hence 0 is a simple eigenvalue.

The hypercontractivity estimate (4.10) is a consequence of the Log– Sobolev inequality (4.6). See e.g. [1]. □

**References**


